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NONLINEAR RITZ APPROXIMATION FOR FREDHOLM FUNCTIONALS

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Abstract. In this article we use the modify Lyapunov-Schmidt reduction to find nonlinear Ritz approximation for a Fredholm functional. This functional corresponds to a nonlinear Fredholm operator defined by a nonlinear fourth-order differential equation.

1. Introduction

Many of the nonlinear problems that appear in Mathematics and Physics can be written in the operator equation form

$$f(u,\lambda)=b, \quad u\in O\subset X,\ b\in Y,\ \lambda\in\mathbb{R}^n, \tag{1.1}$$

where f is a smooth Fredholm map of index zero and X, Y are Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\theta(\xi,\lambda)=\beta, \quad \xi\in M,\ \beta\in N, \tag{1.2}$$

can be used, where M and N are smooth finite dimensional manifolds.

A passage from (1.1) into (1.2) (variant local scheme of Lyapunov -Schmidt) with the conditions that equation (1.2) has all the topological and analytical properties of (1.1) (multiplicity, bifurcation diagram, etc) can be found in [10, 12, 13, 16].

Suppose that $f:\Omega\subset E\to F$ is a nonlinear Fredholm map of index zero. A smooth map $f:\Omega\subset E\to F$ has variational property, if there exists a functional $V:\Omega\subset E\to R$ such that $f=\operatorname{grad}_H V$ or equivalently,

$$\frac{\partial V}{\partial u}(u,\lambda)h=\langle f(u,\lambda),h\rangle_H, \quad \forall u\in\Omega,\ h\in E,$$

where $\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space H. In this case, the solutions of equation $f(u,\lambda)=0$ are the critical points of functional $V(u,\lambda)$. Suppose that $f:E\to F$ is a smooth Fredholm map of index zero, E, F are Banach spaces and

$$\frac{\partial V}{\partial u}(u,\lambda)h=\langle f(u,\lambda),h\rangle_H, \quad h\in E.$$

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where V is a smooth functional on E. Also it is assumed that $E \subset F \subset H$, where $H$is a Hilbert space. By using a method of finite dimensional reduction (Local scheme of Lyapunov-Schmidt) the problem

$$V(u, \lambda) \to \text{extr} \quad u \in E,\ \lambda \in \mathbb{R}^n$$

can be reduced into equivalent problem

$$W(\xi, \lambda) \to \text{extr} \quad \xi \in \mathbb{R}^n$$

The function $W(\xi, \lambda)$ is called key function.

If $N = \operatorname{span}\{e_1, \dots, e_n\}$ is a subspace of E, where $e_1, \dots, e_n$ is an orthonormal set in H, then the key function $W(\xi, \lambda)$ can be defined in the form of

$$W(\xi, \lambda) = \inf_{u:\langle u, e_i\rangle = \xi_i\ \forall i} V(u, \lambda), \quad \xi = (\xi_1, \dots, \xi_n).$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, etc.) [12]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function. If f has a variational property, then the equation

$$\theta(\xi, \lambda) = \operatorname{grad} W(\xi, \lambda) = 0$$

is called bifurcation equation.

Now we formulate one of the most important theorem of bifurcation analysis [9].

Theorem 1.1 ([9]). *If a mapping $\tilde{f}(\cdot, \xi) : E \cap N^{\perp} \to F \cap N^{\perp}$ is proper and the condition $\langle \frac{\partial f}{\partial x}(x)h, h\rangle > 0$ is satisfied for every (x, h) in $E \times ((E \cap N^{\perp}) \setminus 0)$, then the marginal mapping $\varphi : \xi \to \sum_{i=1}^{n} \xi_i e_i + h(\xi)$, (where $h(\xi)$ is defined by equation $\tilde{f}(h, \xi) = 0$), establishes a one-to-one correspondence between critical points of key function $W(\xi, \lambda)$ and critical points of the (given) functional $V(u, \lambda)$. Moreover, the local singularity rings of the corresponding functions at the points ξ and $\varphi(\xi)$ are isomorphic to each other and, if two simple critical points correspond to each other, then their Morse indices are equal to each other.*

Definition 1.2 ([9]). The set of all λ for which the function $W(\xi, \lambda)$ has degenerate critical points is called Caustic and denoted by Σ.

$$\Sigma = \{\lambda \in R : \frac{\partial W}{\partial \xi} = 0,\ \frac{\partial^2 W}{\partial \xi^2} = 0\}.$$

It is well known that in the Lyapunov-Schmidt method, the space E is decomposed into two orthogonal subspaces and then every element $u \in E$ can be written in the unique form as a sum of two elements such that the solution of the equation (1.1) consists of the homogeneous solution and the particular solution. Sapronov and his group [9, 17] used the complement solution to find the function $W(\xi, \lambda)$ which denotes the linear Ritz approximation of the functional $V(u, \lambda)$. The study of boundary value problems by using Lyapunov-Schmidt reduction can be found in [1, 2, 3, 4, 5, 9]. Most of the authors that work this way have studied the linear Ritz approximation of Fredholm functional. A review for the finite dimensional reduction can be found in [9, 12, 13, 14, 15, 17]. In [5] the author introduced an example to find nonlinear approximation of bifurcation solutions of the fourth-order differential equation

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = 0$$

In [6] the author introduce a general method for finding nonlinear Ritz approximation of Fredholm functionals. To the best of our information the method is new. In this paper we find the nonlinear Ritz approximation of a functional $V(u,\lambda)$ which denotes the potential of the nonlinear operator

$$f(u,\lambda) = \frac{d^4u}{dx^4} + \lambda\frac{d^2u}{dx^2} + u + u^2 + u^3.$$

2. Modified Lyapunov-Schmidt reduction

Consider the nonlinear Fredholm operator of index zero $f : E \to F$ defined by

$$f(u,\lambda) = 0, \quad \lambda \in \mathbb{R}^n,\ u \in \Omega \subset E \tag{2.1}$$

where E, F are real Banach spaces and Ω is an open subset of E. Assume that the operator f has a variational property, i.e, there exists a functional $V : \Omega \subset E \to R$ such that $f = \operatorname{grad}_H V$ where Ω is a bounded domain. The operator f can be written as

$$f(u,\lambda) = Au + Nu = 0,$$

where $A = \frac{\partial f}{\partial u}(u_0,\lambda)$ is a linear continuous Fredholm operator, $\frac{\partial f}{\partial u}(u_0,\lambda)$ the Frechet derivative of the operator f at the point u_0 and N the nonlinear operator. In this article we consider the operator A as a differential operator. By using Lyapunov-Schmidt reduction, the decomposition is obtained below

$$E = M \oplus M^\perp, \quad F = \tilde{M} \oplus \tilde{M}^\perp$$

where $M = \ker A$ is the null space of the operator A, $\dim M = \dim \tilde{M} = n$ and $M^\perp$, $\tilde{M}^\perp$ are the orthogonal complements of the subspaces M and $\tilde{M}$ respectively. If $e_1, e_2, \dots, e_n$ is an orthonormal set in H such that $Ae_i = \alpha_i(\lambda)e_i$, $\alpha_i(\lambda)$ is continuous function, $i = 1, \dots, n$, then every element $u \in E$ can be represented in the unique form of

$$u = w + v, \quad w = \sum_{i=1}^{n} \xi_i e_i \in M, \quad M \perp v \in M^\perp,\ \xi_i = \langle u, e_i\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in Hilbert space H. There exist projections $p : E \to M$ and $I - p : E \to M^\perp$ such that $w = pu$ and $(I-p)u = v$. Similarly, there exist projections $Q : F \to \tilde{M}$ and $I - Q : F \to \tilde{M}^\perp$ such that

$$f(u,\lambda) = Qf(u,\lambda) + (I-Q)f(u,\lambda) \tag{2.2}$$

or

$$f(w+v,\lambda) = Qf(w+v,\lambda) + (I-Q)f(w+v,\lambda)$$

It follows that

$$Qf(w+v,\lambda) + (I-Q)f(w+v,\lambda) = 0$$

and hence the result becomes

$$Qf(w+v,\lambda) = 0,$$
$$(I-Q)f(w+v,\lambda) = 0.$$

The implicit function theorem implies that

$$W(\xi,\delta) = V(\Phi(\xi,\delta),\delta), \quad \xi = (\xi_1,\xi_2,\dots,\xi_n)^\top$$

where $\deg W \geq 2$, then the linear Ritz approximation of the functional V is a function W defined by

$$W(\xi,\delta) = V\Big(\sum_{i=1}^{n} \xi_i e_i, \delta\Big) = W_0(\xi) + W_1(\xi,\delta) \tag{2.3}$$

where $W_0(\xi)$ is a homogenous polynomial of order $n \geq 3$ such that $W_0(0) = 0$ and $W_1(\xi,\delta)$ is a polynomial function of degree less than n. Let $q_1, q_2, \dots, q_m$ be the coefficients of the quadratic terms of the function $W_1(\xi,\delta)$, then the function $W_1(\xi,\delta)$ can be written in the form of

$$W_1(\xi,\delta) = W_2(\xi,\delta) + \sum_{k=1}^{m} q_k \xi_k^2$$

where $\deg W_2 = d$, $2 < d < n$.

The nonlinear Ritz approximation of the functional V is a function W defined by

$$W(\xi,\delta) = V\Big(\sum_{i=1}^{n} \xi_i e_i + \Phi(\sum_{i=1}^{n} \xi_i e_i, \delta), \delta\Big)$$

where $\Phi(w,\delta) = v(x,\xi,\delta)$, $v \in N^{\perp}$. To determine the nonlinear Ritz approximation of the functional V, Taylor's expansion of the functions $\mu_k(\xi)$ and $v(x,\xi,\delta)$ is used by assuming the following:

$$q_k = \hat{q}_k + \mu_k(\xi) = \hat{q}_k + \sum_{j=2}^{r} D_k^{(j)}(\xi), \quad k = 1, \dots, m,$$

$$v(x,\xi,\delta) = \sum_{j=2}^{r} B^{(j)}(\xi).$$

where $D_k^{(j)}(\xi)$ and $B^{(j)}(\xi)$ are homogenous polynomials of degree j with coefficients μ_{ki} and $v_{ji}(x,\delta)$ respectively, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. Since

$$Qf(u,\lambda) = \sum_{i=1}^{n} \langle f(u,\lambda), e_i \rangle e_i = 0$$

it follows that

$$\sum_{i=1}^{n} \langle Au + Nu, e_i \rangle e_i = 0$$

Hence

$$\sum_{i=1}^{n} q_i \xi_i e_i + \sum_{i=1}^{n} \langle Nu, e_i \rangle e_i = 0, \qquad q_i = \alpha_i(\lambda)$$

or

$$\sum_{i=1}^{n} q_i \xi_i e_i + \sum_{i=1}^{n} \Big[\int_{\Omega} N(w+v) e_i \Big] e_i = 0. \tag{2.4}$$

From (2.2) it follows that

$$(I-Q)f(u,\lambda) = f(u,\lambda) - Qf(u,\lambda)\,.$$

From $A(w+v) + N(w+v) = 0$ it follows that

$$Av + N(w+v) + \sum_{i=1}^{n} q_i \xi_i e_i = 0 \tag{2.5}$$

Substituting the values of q_i, $\mu_i(\xi)$ and $v(x,\xi,\delta)$ in (2.4) and (2.5) yields

$$\sum_{i=1}^{n}\Big[\hat{q}_i+\sum_{j=2}^{r}D_i^{(j)}(\xi)\Big]\xi_i e_i+\sum_{i=1}^{n}\Big[\int_\Omega N\Big(\sum_{i=1}^{n}\xi_i e_i+\sum_{j=2}^{r}B^{(j)}(\xi)\Big)e_i\Big]e_i=0, \tag{2.6}$$

$$A\Big(\sum_{j=2}^{r}B^{(j)}(\xi)\Big)+N\Big(\sum_{i=1}^{n}\xi_i e_i+\sum_{j=2}^{r}B^{(j)}(\xi)\Big)+\sum_{i=1}^{n}\Big(\hat{q}_i+\sum_{j=2}^{r}D_i^{(j)}(\xi)\Big)\xi_i e_i=0. \tag{2.7}$$

To determine the functions $v(x,\xi,\lambda)$ and $\mu_k(\xi)$ we equate the coefficients of $\hat{\xi}=\xi_1\xi_2\dots\xi_n$ in (2.6) to find the value of μ_{ki} and after some calculations from (2.7) it is obtained a linear ordinary differential equation in the variable $v_{ji}(x,\lambda)$. Solving the resulting equation one can find the value of $v_{ji}(x,\lambda)$.

3. Applications

In this section we introduced an example to study the bifurcation of periodic solutions of the nonlinear fourth-order differential equation

$$\frac{d^4u}{dx^4}+\lambda\frac{d^2u}{dx^2}+u+u^2+u^3=0, \tag{3.1}$$

by finding the nonlinear Ritz approximation of the energy functional $V(u,\lambda)$ given by

$$V(u,\lambda)=\int_0^{2\pi}\Big(\frac{(u'')^2}{2}-\lambda\frac{(u')^2}{2}+\frac{u^2}{2}+\frac{u^3}{3}+\frac{u^4}{4}\Big)dx.$$

To do this suppose that $f:E\to F$ is a nonlinear Fredholm operator of index zero defined by

$$f(u,\lambda)=\frac{d^4u}{dx^4}+\lambda\frac{d^2u}{dx^2}+u+u^2+u^3 \tag{3.2}$$

where $E=\Pi^4([0,2\pi],R)$ is the space of all periodic continuous functions that have derivative of order at most four, $F=\Pi_0([0,2\pi],R)$ is the space of all periodic continuous functions, $u=u(x)$ and $x\in[0,2\pi]$. Since the operator f is variational, then there exists a functional V such that f is the gradient of V, *i.e.*

$$f(u,\lambda)=\operatorname{grad}_H V(u,\lambda)$$

hence every solution of equation (3.1) is a critical point of the functional V [9]. Thus the study of the solutions of equation (3.1) is equivalent to the study of an extreme problem

$$V(u,\lambda)\to\text{extr},\qquad u\in E.$$

Analysis of bifurcation can be found by using the local method of Lyapunov-Schmidt, so by localizing the parameter

$$\lambda=\lambda_1+\mu(\xi),\quad \mu:\mathbb{R}\to\mathbb{R}\text{ is a continuous function}$$

the reduction leads to the function of one variable

$$W(\xi,\delta)=\inf_{\langle u,e\rangle=\xi}V(u,\delta).$$

It is well known that in the reduction of Lyapunov-Schmidt the function $W(\xi,\delta)$ is smooth. This function has all the topological and analytical properties of functional V [6]. In particular, for small δ there is one-to-one corresponding between the critical points of functional V and smooth function W, preserving the type of critical points (multiplicity, bifurcation diagram, index Morse, etc.) [6]. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (3.1) is given by

$$h''' + \lambda h'' + h = 0, \quad h \in E$$

Let $N = \ker(A) = \operatorname{span}\{e\}$, $e = \sin(x)/\sqrt{\pi}$ and $A = f_u(0, \lambda) = \frac{d^4}{dx^4} + \lambda \frac{d^2}{dx^2} + 1$, then every element $u \in E$ can be written in the form

$$u = w + v, \quad w = \xi e \in N, \quad \xi \in \mathbb{R},\ v \in \hat{E} = N^{\perp} \cap E.$$

By the implicit function theorem, there exists a smooth map $\Phi : N \to \hat{E}$ such that

$$W(\xi, \delta) = V(\Phi(\xi, \delta), \delta),$$

and then the linear Ritz approximation of the functional V is a function W given by

$$W(\xi, \delta) = V(\xi e, \delta) = \xi^4 + q\xi^2.$$

the nonlinear Ritz approximation of the functional V is a function W given by

$$W(\xi, \delta) = V(\xi e + \Phi(\xi, \delta), \delta), \quad v(x, \xi) = \Phi(\xi, \delta).$$

We will apply the method in section 2 to find the nonlinear Ritz approximation of the functional V. So from the Lyapunov-Schmidt method we note that the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E = N \oplus \hat{E}, \quad \hat{E} = N^{\perp} \cap E = \{v \in E : v \perp N\}.$$

Similarly, the space F decomposed in direct sum of two subspaces, N and orthogonal complement to N,

$$F = N \oplus \hat{F}, \quad \hat{F} = N^{\perp} \cap F = \{v \in F : v \perp N\}.$$

There exist projections $p : E \to N$ and $I - p : E \to \hat{E}$ such that $pu = w$ and $(I - p)u = v$, (I is the identity operator). Hence every vector $u \in E$ can be written in the form

$$u = w + v, \quad w \in N, \quad N \perp v \in \hat{E}.$$

Similarly, there exists projections $Q : F \to N$ and $I - Q : F \to \hat{F}$ such that

$$f(u, \lambda) = Qf(u, \lambda) + (I - Q)f(u, \lambda) \tag{3.3}$$

Accordingly, (3.2) can be written in the form

$$\begin{gathered} Qf(w + v, \lambda) = 0, \\ (I - Q)f(w + v, \lambda) = 0. \end{gathered}$$

To determine the nonlinear Ritz approximation of the functional V, the functions $v(x, \xi, \lambda) = O(\xi^3)$ and $\mu(\xi) = O(\xi^2)$ must be found in the form of power series in terms of ξ, as follows:

$$\begin{gathered} v(x, \xi) = v_0(x)\xi^3 + v_1(x)\xi^4 + v_2(x)\xi^5 + \dots, \\ \mu(\xi) = \mu_0\xi^2 + \mu_1\xi^3 + \mu_2\xi^4 + \dots, \end{gathered} \tag{3.4}$$

and (3.2) can be written in the form

$$f(u, \lambda) = Au + Tu = 0, \quad Tu = u^2 + u^3.$$

Since

$$Qf(u, \lambda) = \langle f(u, \lambda), \sin(x)\rangle \sin(x) = 0,$$

we have $\langle Au + Tu, \sin(x)\rangle \sin(x) = 0$ and hence

$$-\pi\xi\mu(\xi) + \int_0^{2\pi} (v + \xi\sin(x))^2 \sin(x)\,dx + \int_0^{2\pi} (v + \xi\sin(x))^3 \sin(x)\,dx = 0. \tag{3.5}$$

From (3.3) and (3.5) we have

$$v^{iv} + (\lambda_1 + \mu(\xi))v'' + v + (v + \xi\sin(x))^2 + (v + \xi\sin(x))^3 - \xi\mu(\xi)\sin(x) = 0 \tag{3.6}$$

As a consequence

$$\begin{gathered}
-\pi\xi\mu(\xi) + \int_0^{2\pi} v^2\sin(x)\,dx + 2\xi\int_0^{2\pi} v(\sin(x))^2\,dx + \frac{3\pi}{4}\xi^3 \\
+ 3\xi^2\int_0^{2\pi} v(\sin(x))^3\,dx + 3\xi\int_0^{2\pi} v^2(\sin(x))^2\,dx + \int_0^{2\pi} v^3\sin(x)\,dx = 0, \\
v^{iv} + (\lambda_1 + \mu(\xi))v'' + v + v^2 + 2v\xi\sin(x) + \xi^2(\sin(x))^2 \\
+ v^3 + 3v^2\xi\sin(x) + 3v\xi^2(\sin(x))^2 + \xi^3(\sin(x))^3 - \xi\mu(\xi)\sin(x) = 0.
\end{gathered} \tag{3.7}$$

To determine the functions $v(x,\xi)$ and $\mu(\xi)$ first we substitute (3.4) in (3.7) and then we find the coefficients $\mu_0, \mu_1, \mu_2, v_0, v_1$ and v_2 by equating the terms of ξ as follows: Equating the coefficients of ξ^3 we have the following two equations,

$$\begin{gathered}
-\pi\mu_0 + \frac{3\pi}{4} = 0, \\
v_0^{(4)} + \lambda_1 v_0'' + v_0 + (\sin(x))^3 - \mu_0\sin(x) = 0
\end{gathered} \tag{3.8}$$

From the first equation in (3.8) we have $\mu_0 = 3/4$. Substituting this value in the second equation of (3.8), we have the linear differential equation

$$v_0^{(4)} + \lambda_1 v_0'' + v_0 + (\sin(x))^3 - \frac{3}{4}\sin(x) = 0,$$

and then we have

$$v_0^{(4)} + \lambda_1 v_0'' + v_0 - \frac{1}{4}\sin(3x) = 0. \tag{3.9}$$

Then

$$v_0(x) = \frac{\sin(3x)}{256}.$$

Similarly, equating the coefficients of ξ^4 we have

$$\begin{gathered}
-\pi\mu_1 + 2\int_0^{2\pi} v_0(x)(\sin(x))^2 dx = 0, \\
v_1^{(4)} + \lambda_1 v_1'' + v_1 + 2v_0\sin(x) - \mu_1 = 0.
\end{gathered} \tag{3.10}$$

From the first equation in (3.10) we have $\mu_1 = 0$. Substituting this value in the second equation of (3.10) we have

$$v_1^{(4)} + \lambda_1 v_1'' + v_1 + \frac{\sin(x)\sin(3x)}{128} = 0. \tag{3.11}$$

Then

$$v_1(x) = \frac{-1}{256}\Big[\frac{\cos(2x)}{9} - \frac{\cos(4x)}{225}\Big].$$

Equating the coefficients of ξ^5 we have

$$\begin{gathered}
-\pi\mu_2 + 2\int_0^{2\pi} v_1(x)(\sin(x))^2 dx + 3\int_0^{2\pi} v_0(x)(\sin(x))^3 dx = 0, \\
v_2^{(4)} + \lambda_1 v_2'' + v_2 + \mu_0 v_0'' + 2v_1 \sin(x) + 3v_0(\sin(x))^2 - \mu_2 \sin(x) = 0
\end{gathered} \tag{3.12}$$

substituting the values of v_0 and v_1 in the first equation of (3.12) and then solving this equation we find that

$$\mu_2 = -\frac{23}{9216}.$$

Also, substituting the values of μ_0, v_0 and v_1 and in the second equation of (3.12) we have the linear differential equation

$$v_2^{(4)} + \lambda_1 v_2'' + v_2 + \frac{3}{1024}[7\sin(3x) + \sin(5x)] - \frac{6\sin(3x) - \sin(x)}{11520} = 0 \tag{3.13}$$

Solving (3.13) we have

$$v_2(x) = \Big[\frac{21}{65536} - \frac{1}{122880}\Big]\sin(3x) + \frac{1}{24}\Big[\frac{1}{8192} + \frac{1}{276480}\Big]\sin(5x).$$

Now substituting the values of $\mu_0, \mu_1, \mu_2, v_0, v_1$ and v_2 in (3.4) we have the bifurcation equation

$$\begin{aligned}
u(x,\xi) &= \frac{\xi\sin(x)}{\sqrt{\pi}} + \frac{\xi^3}{256\pi\sqrt{\pi}}\sin(3x) - \frac{\xi^4}{57600\pi}\Big[25\cos(2x) - \cos(4x)\Big] \\
&\quad + \xi^5\Big(\Big[\frac{21}{65536\pi^2\sqrt{\pi}} - \frac{1}{122880\pi\sqrt{\pi}}\Big]\sin(3x) \\
&\quad + \frac{1}{24}\Big[\frac{1}{8192\pi^2\sqrt{\pi}} + \frac{1}{276480\pi\sqrt{\pi}}\Big]\sin(5x)\Big) + O(\xi^7) \\
\lambda &= \lambda_1 + \frac{3}{4}\xi^2 - \frac{23}{9216}\xi^4 + O(\xi^6)
\end{aligned} \tag{3.14}$$

From the above result we deduced the following theorem.

Theorem 3.1. *The key function of the functional V has the form*

$$\begin{aligned}
\hat{W}(\xi,\delta) &= U(\xi,\delta) + O(|\xi|^{20}) + O(|\xi|^{20})O(|\delta|) \\
&= c_1\xi^{20} + c_2\xi^{18} + c_3\xi^{16} + c_4\xi^{14} + c_5\xi^{12} + \alpha_1\xi^{10} + \alpha_2\xi^8 \\
&\quad + \alpha_3\xi^6 + c_6\xi^4 + \alpha_4\xi^2 + O(|\xi|^{20}) + O(|\xi|^{20})O(|\delta|),
\end{aligned} \tag{3.15}$$

where

$$\begin{gathered}
c_1 = 0.11742\times 10^{19}, \quad c_2 = 0.52310\times 10^{21}, \quad c_3 = 0.47769\times 10^{23}, \\
c_4 = 0.23733\times 10^{27}, \quad c_5 = -0.41660\times 10^{29}, \\
\alpha_1 = -(0.63868\times 10^{31} + 0.99142\times 10^{29}\lambda_1), \\
\alpha_2 = -(0.11220\times 10^{32}\lambda_1 + 0.94599\times 10^{32}), \\
\alpha_3 = 0.31596\times 10^{35} - 0.17154\times 10^{33}\lambda_1, \\
c_6 = -0.77749\times 10^{37}, \quad \alpha_4 = -(0.24658\times 10^{38} + 0.12329\times 10^{38}\lambda_1)
\end{gathered}$$

The prove of Theorem 3.1 follows directly from the formula

$$\hat{W}(\xi,\delta) = V(\xi e + \Phi(\xi,\delta),\delta).$$

We note that $c_1, c_2, c_3, c_4, c_5, c_6$ are constants and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are parameters.The key function $\hat{W}(\xi, \delta)$ in Theorem 3.1 is the required nonlinear Ritz approximation of the functional $V(\xi e + \Phi(\xi, \delta), \delta)$. The geometry of the bifurcation of critical points and the principal asymptotic of the branches of bifurcating points for the function $\hat{W}(\xi, \delta)$ are entirely determined by its principal part $U(\xi, \delta)$. The function has all the topological and analytical properties of functional V , also the function have 19 critical points. The point $u(x) = \xi e + v(x, \xi)$ is a critical point of the functional $V(u, \lambda)$ if and only if the point ξ is a critical point of the function $\hat{W}(\xi, \delta)$, (Theorem 1.1). This means that the existence of the solutions of equation (3.2) depend on the existence of the critical points of the functional $V(u, \lambda)$ and then on the existence of the critical points of the function $\hat{W}(\xi, \delta)$. From this notation we can find a nonlinear approximation of the solutions of equation (3.2) corresponding to each critical point of the function $\hat{W}(\xi, \delta)$. To avoid the singularities of the function $U(\xi, \delta)$ we must find the caustic, so from definition 1.2 the caustic of the function $U(\xi, \delta)$ is the set of all λ_1 satisfying the equation

$$\begin{aligned}
&(\lambda_1 + 6.168595401)(\lambda_1 + 6.168557117)(\lambda_1 + 1.997966599)\\
&\times (\lambda_1 - 7.420188558)(\lambda_1 - 7.420220242)(\lambda_1^2 + 10.79759396\lambda_1 + 39.30282312)\\
&\times (\lambda_1^2 + 10.79751137\lambda_1 + 39.30229176)(\lambda_1^2 + 6.369291114\lambda_1 + 42.39826243)\\
&\times (\lambda_1^2 + 6.369244386\lambda_1 + 42.39775611)(\lambda_1^2 + 4.002037322\lambda_1 + 4.004078787)\\
&\times (\lambda_1^2 - 0.1636805324\lambda_1 + 46.42060955)(\lambda_1^2 - 0.1636921686\lambda_1 + 46.42102879)\\
&\times (\lambda_1^2 - 7.233265906\lambda_1 + 50.57130598)(\lambda_1^2 - 7.233315078\lambda_1 + 50.57170600)\\
&\times (\lambda_1^2 - 12.74882579\lambda_1 + 53.82009714)(\lambda_1^2 - 12.74888749\lambda_1 + 53.82053975)\\
&= 0.
\end{aligned} \tag{3.16}$$

The only real values satisfying the above equation are

$$\Sigma = \{-6.168595401, -6.168557117, -1.997966599, 7.420188558, 7.420220242\}.$$

Hence the caustic dividing the real lines into following six sets

$$\begin{gathered}
(-\infty, -6.168595401),\ (-6.168595401, -6.168557117),\\
(-6.168557117, -1.997966599),\ (-1.997966599, 7.420188558),\\
(7.420188558, 7.420220242),\ (7.420220242, \infty)
\end{gathered}$$

every set has a fixed number of nondegenerate critical points. The spreading of real critical points of the function $U(\xi, \delta)$ is given below:

If $\lambda_1 \in (-\infty, -6.168595401)$, then we have five nondegenerate critical points (three minima and two maxima).

If $\lambda_1 \in (-6.168595401, -6.168557117)$, then we have five nondegenerate critical points (three minima and two maxima).

If $\lambda_1 \in (-6.168557117, -1.997966599)$, then we have five nondegenerate critical points (three minima and two maxima).

If $\lambda_1 \in (-1.997966599, 7.420188558)$, then we have three nondegenerate critical points (two minima and one maximum).

If $\lambda_1 \in (7.420188558, 7.420220242)$, then we have three nondegenerate critical points (two minima and one maximum).

If $\lambda_1 \in (7.420220242, \infty)$, then we have three nondegenerate critical points (two minima and one maximum).

To explain our results we have the following: We found that the linear Ritz approximation of the functional $V(u, \lambda)$ is the function

$$W(\xi, \delta) = \xi^4 + q\xi^2;$$

the critical points of this function are degenerate when $q = 0$, so for every $q \neq 0$ we have three nondegenerate critical points of the function $W(\xi, \delta)$. Corresponding to each nondegenerate critical point we have a linear approximation solution of (3.1) in the form of $w = \xi \sin(x)/\sqrt{\pi}$. These solutions have only the two geometric representations shown in Figure 1.

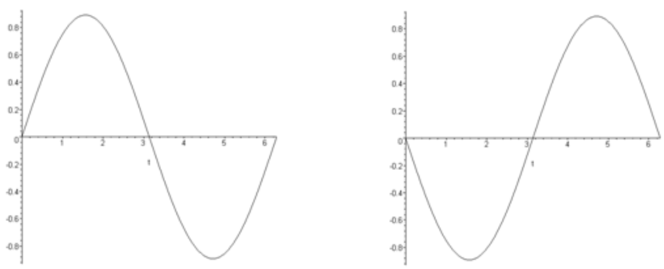

FIGURE 1. Graphs of the function $w = \xi \sin(x)/\sqrt{\pi}$.

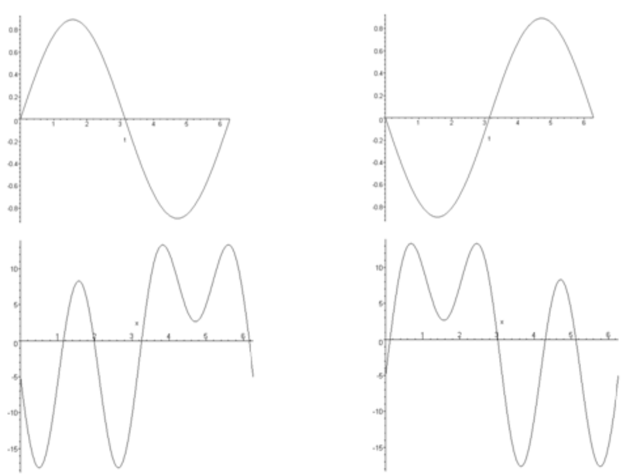

FIGURE 2. Graphs of the function (3.14).

In theorem 3.1 we proved that the nonlinear Ritz approximation of the functional $V(u, \lambda)$ is the function (3.15). All critical points of this function are degenerate when λ_1 is a solution of (3.16), so for every $\lambda_1 \in R\backslash\Sigma$ we have only three or five nondegenerate critical points. Corresponding to each nondegenerate critical point we have nonlinear approximation solution of (3.1) in the form of function (3.14). These solutions have the four geometric representations shown in Figure 2.

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