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MULTIPLE POSITIVE SOLUTIONS FOR A KIRCHHOFF TYPE PROBLEM

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Abstract. In this article, we study the existence and multiplicity of positive solutions of a Kirchhoff type equation on a smooth bounded domain $\Omega \subset \mathbb{R}^3$, and we show that the number of positive solutions of the equation depends on the topological properties of the domain. The technique is based on Ljusternik-Schnirelmann category and Morse theory.

1. Introduction

This article concerns the multiplicity of positive solutions to the elliptic problem

$$\begin{gathered} -\Big(\varepsilon^2 a + \varepsilon b \int_\Omega |\nabla u|^2\Big)\Delta u + u = |u|^{p-1}u, \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^3$, $\varepsilon > 0$, $a, b > 0$ are constants, $3 < p < 5$.

In recent years, some mathematicians considered the problem

$$\begin{gathered} -\Big(a + b \int_\Omega |\nabla u|^2\Big)\Delta u = f(x,u), \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain. See for example[2, 9, 10, 14, 17, 18, 19, 20].

When $a = 1$, $b = 0$, $\mathbb{R}^3$ is replaced by $\mathbb{R}^N$, and $|u|^{p-1}u$ is replaced by $f(u)$, Equation (1.1) reduces to

$$\begin{gathered} -\varepsilon^2 \triangle u + u = f(u), \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{gathered} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^N$. Benci and Cerami [4] used Morse theory to estimate the number of positive solutions of the problem (1.3). They proved that for ε sufficiently small the number of positive solutions depends on the topology of Ω, actually on the Poincaré polynomial of Ω, $P_t(\Omega)$, defined below. Candela and Lazzo [8] considered the same equation with mixed Dirichlet-Neumann boundary conditions and $f(t) = |t|^{p-2}t$. It was proved that the number of positive solutions is

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influenced by the topology of the part Γ_1 of the boundary $\partial\Omega$ where ε is sufficiently small. Recently, Benci, Bonanno, Ghimenti and Micheletti [3, 13] proved that the number of solutions of (1.2) with $f(t) = |t|^{p-2}t$ on a smooth bounded domain $\Omega \subset \mathbb{R}^3$ depends on the topological properties of the domain. More recently, Ghimenti and Micheletti [12] extended the result in [3, 13] to the Klein-Gordon-Maxwell and Schröedinger-Maxwell system and showed that the geometry of the 3-dimensional Riemannian manifold has effects on the number of solutions of both systems.

Moreover, as far as we known, the existence and multiplicity of nontrivial solutions to the Kirchhoff equation have not ever been studied by using Morse theory. Motivated by the works described above and the fact, we will try to get the multiplicity of positive solutions to (1.1) by using Ljusternik-Schnirelmann category and Morse theory. So in this paper we shall fill this gap.

Our main results read as follows.

Theorem 1.1. *Let $3 < p < 5$. For $\varepsilon > 0$ sufficiently small, the problem* (1.1) *has at least* $\operatorname{cat}(\Omega)$ *positive solutions.*

Theorem 1.2. *Let $3 < p < 5$. Assume that for $\varepsilon > 0$ sufficiently small all the solutions of the problem* (1.1) *are nondegenerate. Then there are at least* $2P_1(\Omega)-1$ *positive solutions,*

2. Notation and preliminary results

Throughout this article, we use the following norms for $u \in H_0^1(\Omega)$:

$$\|u\|_\varepsilon = \Big(\frac{1}{\varepsilon^3}\int_\Omega \varepsilon^2|\nabla u|^2dx\Big)^{1/2}, \quad |u|_{\varepsilon,p} = \Big(\frac{1}{\varepsilon^3}\int |u|^p dx\Big)^{1/p},$$
$$\|u\| = \Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^{1/2}, \quad |u|_p = \Big(\int_{\mathbb{R}^3}|u|^p dx\Big)^{1/p}$$

and we denote by H_ε the Hilbert space $H_0^1(\Omega)$ endowed with $\|\cdot\|_\varepsilon$ norm.

Following the work by He and Zou [15], we let $U(x)$ be the positive ground state solution of

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^3}|\nabla u|^2\Big)\Delta u + u = |u|^{p-1}u, \quad x \in \mathbb{R}^3 \\ u \in H^1(\mathbb{R}^3), u(x) > 0, \quad x \in \mathbb{R}^3 \end{gathered} \tag{2.1}$$

and $I_\infty(U) = m_\infty = \inf_{u\in M_\infty} I_\infty(u)$, where

$$I_\infty(u) = \frac{a}{2}\|u\|^2 + \frac{1}{2}|u|_2^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p+1}|u^+|_{p+1}^{p+1}.$$
$$M_\infty = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G_\infty(u) = \langle I'_\infty(u), u\rangle = 0\}$$

For $\varepsilon > 0$ we set $U_\varepsilon(x) = U(x/\varepsilon)$. Obviously $U_\varepsilon(x)$ is the solution of the problem

$$\begin{gathered} -\Big(\varepsilon^2 a + \varepsilon b\int_\Omega |\nabla u|^2\Big)\Delta u + u = |u|^{p-1}u, \quad x \in \Omega \\ u \in H_0^1(\Omega), \quad u > 0, \quad x \in \Omega \end{gathered}$$

Now we shall recall some topological tools.

Definition 2.1 ([16])**.** Let X a topological space and consider a closed subset $A \subset X$. We say that A has category k relative to $X(\operatorname{cat}_X(A) = k)$ if A is covered by k closed sets $A_i, i = 1, 2, \dots, k$, which are contractible in X, and k is the minimum integer with this property. We simply denote $\operatorname{cat}(X) = \operatorname{cat}_X(X)$.

Remark 2.2 ([5])**.** Let X_1 and X_2 be topological space. If $g_1 : X_1 \to X_2$ and $g_2 : X_2 \to X_1$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on X_1, then $\mathrm{cat}(X_1) \leq \mathrm{cat}(X_2)$.

Definition 2.3. Let X is a topological space and let $H_k(X)$ denotes its k-th homology group with coefficients in Q. The Poincaré polynomial $P_t(X)$ of X is defined as the following power series in t,

$$P_t(X) = \sum_{k\geq 0} (\dim H_k(X)) t^k .$$

If X is a compact space, we have that $\dim H_k(X) < \infty$ and this series is finite. In the case $P_t(X)$ is a polynomial and not a formal series.

Remark 2.4 ([4])**.** Let X and Y be topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continuous operators such that $g \circ f$ is homotopic to the identity on X, then $P_t(Y) = P_t(X) + Z(t)$ where $Z(t)$ is a polynomial with nonnegative coefficients.

3. Proof of main results

To prove our main results, we consider the functional $I_\varepsilon \in C^2(H_\varepsilon, R)$, defined by

$$I_\varepsilon(u) = \frac{a}{2}\|u\|_\varepsilon^2 + \frac{1}{2}|u|_{\varepsilon,2}^2 + \frac{b}{4}\|u\|_\varepsilon^4 - \frac{1}{p+1}|u^+|_{\varepsilon,p+1}^{p+1}$$

Obviously, there exists a one to one correspondence between the nontrivial solutions of problem (1.1) and the nonzero critical points of I_ε on H_ε.

As the functional I_ε is not bounded below on H_ε, we introduce the manifold

$$M_\varepsilon = \{u \in H_\varepsilon \backslash \{0\} : G_\varepsilon(u) = \langle I'_\varepsilon(u), u\rangle = 0\}$$

Next, we present some properties of I_ε and M_ε.

Lemma 3.1. *(1) For any $u \in H_\varepsilon \backslash \{0\}$, there is a unique $t_\varepsilon > 0$ such that $u_{t_\varepsilon}(x) = t_\varepsilon u(x) \in M_\varepsilon$.*

(2) For any $\varepsilon > 0$, M_ε is a C^1 submanifold of H_ε, and there exists $\sigma_\varepsilon > 0$ and $K_\varepsilon > 0$ such that for any $u \in M_\varepsilon$

$$\|u\|_\varepsilon \geq \sigma_\varepsilon, \quad I_\varepsilon(u) \geq K_\varepsilon.$$

(3) It holds (PS) condition for the functional I_ε on M_ε.

Proof. (1) For any $u \in H_\varepsilon \backslash \{0\}$ and $t > 0$, set $u_t(x) = tu(x)$. Consider

$$\Upsilon_\varepsilon(t) = I_\varepsilon(u_t) = \frac{a}{2}t^2\|u\|_\varepsilon^2 + \frac{1}{2}t^2|u|_{\varepsilon,2}^2 + \frac{b}{4}t^4\|u\|_\varepsilon^4 - \frac{1}{p+1}t^{p+1}|u^+|_{\varepsilon,p+1}^{p+1}.$$

By computing, we known that Υ_ε has a unique critical point $t_\varepsilon > 0$ corresponding to its maximum. Then $\Upsilon_\varepsilon(t_\varepsilon) = \max_{t>0} \Upsilon_\varepsilon(t)$ and $\Upsilon'_\varepsilon(t_\varepsilon) = 0$. So $G_\varepsilon(u_{t_\varepsilon}) = 0$ and $u_{t_\varepsilon} \in M_\varepsilon$.

(2) By lemma 3.1 (1), $M_\varepsilon \neq \emptyset$. If $u \in M_\varepsilon$, using that $G_\varepsilon(u) = 0$ and $3 < p < 5$, we have

$$\langle G'_\varepsilon(u), u\rangle = a(2 - (p+1))\|u\|_\varepsilon^2 + (2 - (p+1))|u|_{\varepsilon,2}^2 + b(4 - (p+1))\|u\|_\varepsilon^4 < 0.$$

So M_ε is a C^1 manifold. Using that $G_\varepsilon(u) = 0$ and the Sobolev embedding, we have

$$a\|u\|_\varepsilon^2 + |u|_{\varepsilon,2}^2 + b\|u\|_\varepsilon^4 = |u^+|_{\varepsilon,p+1}^{p+1} \leq C\|u\|_\varepsilon^{p+1},$$

$$a \le C\|u\|_\varepsilon^{p-1}.$$

So the conclusion $\|u\|_\varepsilon \ge \sigma_\varepsilon$ follows. For any $u \in M_\varepsilon$,

$$I_\varepsilon(u) = a(\frac{1}{2} - \frac{1}{p+1})\|u\|_\varepsilon^2 + (\frac{1}{2} - \frac{1}{p+1})|u|_{\varepsilon,2}^2 + b(\frac{1}{4} - \frac{1}{p+1})\|u\|_\varepsilon^4.$$

Using $3 < p < 5$ and $\|u\|_\varepsilon \ge \sigma_\varepsilon$, the conclusion $I_\varepsilon(u) \ge K_\varepsilon$ follows.

(3) Let $\{u_n\}$ is (PS) sequence for I_ε on M_ε, that is

$$I_\varepsilon(u_n) \to c,\ I'_\varepsilon|_{M_\varepsilon}(u_n) \to 0.$$

Then it is easy to prove that $\|u_n\|_\varepsilon$ is bounded. Going if necessary to a subsequence, we can assume that $\|u_n\|_\varepsilon^2 \to A(>0)$, $u_n \rightharpoonup u$ in H_ε, $u_n \to u$ in $L^s(\Omega)$ $(1 \le s < 6)$. Obviously, we have

$$\rho_\varepsilon(u) = a\|u\|_\varepsilon^2 + |u|_{\varepsilon,2}^2 + bA\|u\|_\varepsilon^2 - |u^+|_{\varepsilon,p+1}^{p+1} = 0.$$

Set $\omega_n = u_n - u$. By Brézis-Lieb Lemma, we have $\|\omega_n\|_\varepsilon^2 = \|u_n\|_\varepsilon^2 - \|u\|_\varepsilon^2 + o_n(1)$. Since $\langle I'_\varepsilon(u_n), u_n\rangle = o_n(1)$, we obtain

$$(a + bA)\|\omega_n\|_\varepsilon^2 + \rho_\varepsilon(u) = o_n(1).$$

This concludes the proof. □

By Lemma 3.1 (2), we obtain $I_\varepsilon|_{M_\varepsilon}$ is bounded from below. By using Lagrange multiplier method, we known that M_ε contains every nonzero solution of problem (1.1), and define the minimax m_ε as

$$m_\varepsilon = \inf_{u \in M_\varepsilon} I_\varepsilon(u)$$

Proof of Theorem 1.1. Since the functional $I_\varepsilon \in C^2$ is bounded below and satisfies the (PS) condition on the complete manifold M_ε, we have, by the classical Ljusternik-Schnirelmann category result [7], that I_ε has at least $catI_\varepsilon^d$ critical points in the sublevel

$$I_\varepsilon^d = \{u \in H_\varepsilon : I_\varepsilon(u) \le d\}$$

In the following, we will prove that, for ε and δ sufficiently small, it holds

$$\mathrm{cat}(\Omega) \le \mathrm{cat}(M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta})$$

To prove this, we build two continuous functions

$$\Phi_\varepsilon : \Omega^- \to M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}, \tag{3.1}$$

$$\beta : M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta} \to \Omega^+, \tag{3.2}$$

where

$$\Omega^- = \{x \in \Omega : d(x, \partial\Omega) < r\}, \quad \Omega^+ = \{x \in \mathbb{R}^3 : d(x, \partial\Omega) < r\}$$

with $r > 0$ small enough so that $\mathrm{cat}(\Omega^-) = \mathrm{cat}(\Omega^+) = \mathrm{cat}(\Omega)$. Following the idea in [6], we can find two functions Φ_ε and β such that $\beta \circ \Phi_\varepsilon : \Omega^- \to \Omega^+$ is homotopic to the immersion $i : \Omega^- \to \Omega^+$. By Remark 2.2 we obtain the inequality which completes the proof. □

Proof of Theorem 1.2. By Remark 2.4, (3.1) and (3.2), we have

$$P_t(M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}) = P_t(\Omega) + Z(t)$$

where $Z(t)$ is a polynomial with nonnegative coefficients. Since $\inf_\varepsilon m_\varepsilon = c > 0$, we have

$$P_t(I_\varepsilon^{m_\infty+\delta}, I_\varepsilon^{\frac{c}{2}}) = tP_t(M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}), \tag{3.3}$$

$$P_t(H_\varepsilon, I_\varepsilon^{m_\infty+\delta}) = t(P_t(I_\varepsilon^{m_\infty+\delta}, I_\varepsilon^{\frac{c}{2}}) - t)\,. \tag{3.4}$$

By Morse theory we have

$$\sum_{u\in\mathcal{K}_\varepsilon} t^{\mu(u)} = P_t(H_\varepsilon, I_\varepsilon^{m_\infty+\delta}) + P_t(I_\varepsilon^{m_\infty+\delta}, I_\varepsilon^{\frac{c}{2}}) + (1+t)Q_\varepsilon(t) \tag{3.5}$$

where $\mathcal{K}_\varepsilon$ be the set of critical points of I_ε, $\mu(u)$ is the Morse index of u, $Q_\varepsilon(t)$ is a polynomial with nonnegative coefficients. Using this relation with (3.3)–(3.5), we obtain

$$\sum_{u\in\mathcal{K}_\varepsilon} t^{\mu(u)} = tP_t(\Omega) + t^2(P_t(\Omega) - 1) + t(1+t)Q_\varepsilon(t) \tag{3.6}$$

Theorem 1.2 easily follows by evaluating the power series (3.6) for $t = 1$. $\square$

4. The function Φ_ε

For $\xi \in \Omega^-$ we define the function

$$\omega_{\xi,\varepsilon}(x) = U_\varepsilon(x-\xi)\chi_r(|x-\xi|) \tag{4.1}$$

where χ_r is a smooth cut off function $\chi_r \equiv 1$ for $t \in [0, \frac{r}{2})$, $\chi_r \equiv 0$ for $t > r$ and $|\chi_r'(t)| \le 2/r$.

We define $\Phi_\varepsilon : \Omega^- \to M_\varepsilon$ by

$$\Phi_\varepsilon(\xi) = t_\varepsilon(\omega_{\xi,\varepsilon})\omega_{\xi,\varepsilon}(x)$$

Remark 4.1. We have that the following limits hold uniformly with respect to $\xi \in \Omega^-$,

$$\|\omega_{\xi,\varepsilon}\|_\varepsilon \to \|U\|, \quad |\omega_{\xi,\varepsilon}|_{\varepsilon,p} \to |U|_p\,.$$

Proposition 4.2. *For any $\varepsilon > 0$ the map Φ_ε is continuous. Moreover for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then $I_\varepsilon(\Phi_\varepsilon(\xi)) < m_\infty + \delta$.*

Proof. Obviously, Φ_ε is continuous. We claim that $t_\varepsilon(\omega_{\xi,\varepsilon}) \to 1$ uniformly with respect to $\xi \in \Omega^-$. In fact, by Lemma 3.1 $t_\varepsilon(\omega_{\xi,\varepsilon})$ is the unique solution of

$$at\|\omega_{\xi,\varepsilon}\|_\varepsilon^2 + t|\omega_{\xi,\varepsilon}|_{\varepsilon,2}^2 + bt^3\|\omega_{\xi,\varepsilon}\|_\varepsilon^4 = t^p|\omega_{\xi,\varepsilon}^+|_{\varepsilon,p+1}^{p+1}$$

By Remark 4.1 we have the claim.

Now, by Remark 4.1 and the above claim we have

$$\begin{aligned}
&I_\varepsilon(\Phi_\varepsilon(\xi)) \\
&= (\frac{1}{2} - \frac{1}{p+1})at_\varepsilon^2\|\omega_{\xi,\varepsilon}\|_\varepsilon^2 + (\frac{1}{2} - \frac{1}{p+1})t_\varepsilon^2|\omega_{\xi,\varepsilon}|_{\varepsilon,2}^2 + (\frac{1}{4} - \frac{1}{p+1})bt_\varepsilon^4\|\omega_{\xi,\varepsilon}\|_\varepsilon^4 \\
&\to (\frac{1}{2} - \frac{1}{p+1)})a\|U\|^2 + (\frac{1}{2} - \frac{1}{p+1})|U|^2 + (\frac{1}{4} - \frac{1}{p+1})b\|U\|^4 = m_\infty
\end{aligned}$$

this completes the proof. $\square$

Remark 4.3. Note that $\limsup_{\varepsilon\to 0} m_\varepsilon \le m_\infty$.

5. The map β

For $u \in M_\varepsilon$ we can define a point $\beta(u) \in \mathbb{R}^3$ by

$$\beta(u) = \frac{\int_\Omega x|u^+|^{p+1}dx}{\int_\Omega |u^+|^{p+1}dx}$$

The function β is well defined in M_ε since if $u \in M_\varepsilon$ then $u^+ \neq 0$.

We shall prove that if $u \in M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ then $\beta(u) \in \Omega^+$. First of all we consider partitions of the compact manifold Ω. Given $\varepsilon > 0$, a finite partition $P_\varepsilon = \{P_j^\varepsilon\}_{j\in\Lambda_\varepsilon}$ is called a "good" partition if: for any $j \in \Lambda_\varepsilon$ the set P_j^ε is closed; $P_i^\varepsilon \cap P_j^\varepsilon \subseteq \partial P_i^\varepsilon \cap \partial P_j^\varepsilon$ for $i \neq j$; there exist $r_1(\varepsilon), r_2(\varepsilon) > 0$ such that, for any j, there exists a point $q_j^\varepsilon \in P_j^\varepsilon$ such that

$$B(q_j^\varepsilon, \varepsilon) \subset P_j^\varepsilon \subset B(q_j^\varepsilon, r_2(\varepsilon)) \subset B(q_j^\varepsilon, r_1(\varepsilon)),$$

with $r_1(\varepsilon) \geq r_2(\varepsilon) \geq C\varepsilon$ for some positive constant C; lastly, there exists a finite number $\iota \in N$ such that every $x \in \Omega$ is contained in at most ι balls $B(q_j^\varepsilon, r_1(\varepsilon))$, where ι does not depends on ε.

Lemma 5.1. *There exists $\gamma > 0$ such that, for any $\delta > 0$ and any $\varepsilon \in (0, \varepsilon_0(\delta))$ where $\varepsilon_0(\delta)$ is as in Proposition 4.2, given any "good" partition P_ε of the domain Ω and for any $u \in M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ there exists a set P_j^ε such that*

$$\frac{1}{\varepsilon^3}\int_{P_j^\varepsilon} |u^+|^{p+1}dx \geq \gamma$$

Proof. Taking into account that $G_\varepsilon(u) = 0$ we have

$$\begin{aligned} a\|u\|_\varepsilon^2 \leq |u^+|_{\varepsilon,p+1}^{p+1} &= \sum_j \frac{1}{\varepsilon^3}\int_{P_j^\varepsilon} |u^+|^{p+1}dx \\ &\leq \sum_j |u_j^+|_{\varepsilon,p+1}^{p+1} = \sum_j |u_j^+|_{\varepsilon,p+1}^{p-1}|u_j^+|_{\varepsilon,p+1}^2 \\ &\leq \max_j\{|u_j^+|_{\varepsilon,p+1}^{p-1}\}\sum_j |u_j^+|_{\varepsilon,p+1}^2 \end{aligned}$$

where u_j^+ is the restriction of the function u^+ on the set P_j^ε.

Arguing as in [3], we prove that there exists a constant $C > 0$ such that

$$\sum_j |u_j^+|_{\varepsilon,p+1}^2 \leq C\iota\|u^+\|_\varepsilon^2,$$

thus

$$\max_j\{|u_j^+|_{\varepsilon,p+1}^{p-1}\} \geq \frac{a}{C\iota}$$

that concludes the proof. □

Proposition 5.2. *For any $\eta \in (0,1)$ there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and any $\varepsilon \in (0, \varepsilon_0(\delta))$ as in Proposition 4.2, for any $u \in M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ there exists a point $q = q(u) \in \Omega$ such that*

$$\frac{1}{\varepsilon^3}\int_{B(q,\frac{r}{2})} |u^+|^{p+1}dx > (1-\eta)\frac{2(p+1)}{p-1}\Big(4m_\infty + \frac{2a^2 - 2a\sqrt{a^2+3bm_\infty}}{b}\Big)$$

Proof. We prove only the proposition for any $u \in M_\varepsilon \cap I_\varepsilon^{m_\varepsilon+2\delta}$. Indeed, by this result and by Remark 4.3 we get

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_\infty$$

Hence it holds $I_\varepsilon^{m_\infty+\delta} \subset I_\varepsilon^{m_\varepsilon+2\delta}$ for δ, ε small enough. So the thesis holds.

We argue by contradiction. Suppose that there exists $\eta \in (0,1)$ such that we can find vanishing sequences $\{\delta_k\}$, $\{\varepsilon_k\}$ and a sequence $\{u_k\} \subset M_{\varepsilon_k} \cap I_\varepsilon^{m_{\varepsilon_k}+2\delta_k}$ such that

$$\begin{aligned} m_{\varepsilon_k} &\le I_{\varepsilon_k}(u_k) \\ &= (\frac{1}{2} - \frac{1}{p+1})a\|u_k\|_{\varepsilon_k}^2 + (\frac{1}{2} - \frac{1}{p+1})|u_k|_{\varepsilon_k,2}^2 + (\frac{1}{4} - \frac{1}{p+1})b\|u_k\|_{\varepsilon_k}^4 \\ &\le m_{\varepsilon_k} + 2\delta_k \le m_\infty + 3\delta_k. \end{aligned} \tag{5.1}$$

for k large enough, and for any $q \in \Omega$,

$$\frac{1}{\varepsilon_k^3}\int_{B(q,\frac{r}{2})} |u_k^+|^{p+1}dx \le (1-\eta)\frac{2(p+1)}{p-1}(4m_\infty + \frac{2a^2 - 2a\sqrt{a^2+3bm_\infty}}{b}). \tag{5.2}$$

By Ekeland variational principle and by definition of M_{ε_k} we can assume that

$$I'_{\varepsilon_k}(u_k) \to 0. \tag{5.3}$$

By Lemma 5.1, there exists a set $P_k^{\varepsilon_k} \in P_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon^3}\int_{P_k^{\varepsilon_k}} |u_k^+|^{p+1}dx \ge \gamma. \tag{5.4}$$

So we can choose a point $q_k \in \overset{\circ}{P}{}_k^{\varepsilon_k}$, and define, for $z \in \Omega_{\varepsilon_k} = \frac{1}{\varepsilon_k}(\Omega - q_k)$,

$$\omega_k(z) = u_k(\varepsilon_k z + q_k) = u_k(x),$$

where $x \in \Omega$. We obtain that $\omega_k \in H_0^1(\Omega_{\varepsilon_k})$. By (5.1), we have

$$\|\omega_k\|_{H_0^1(\Omega_{\varepsilon_k})}^2 \le C.$$

So we obtain $\omega_k \rightharpoonup \omega$ in $H_{loc}^1(\mathbb{R}^3)$, $\omega_k \to \omega$ in $L_{loc}^s(\mathbb{R}^3)(2 \le s < 6)$ and $\|\omega_k\|^2 \to A_1$. Thus we prove that $\omega \not\equiv 0$ and $A_1 > 0$ by (5.4).

Next we claim

$$\lim_{k\to\infty} \frac{dist(q_k, \partial\Omega)}{\varepsilon_k} = \infty. \tag{5.5}$$

We argue by contradiction. Suppose that

$$\lim_{k\to\infty} \frac{dist(q_k, \partial\Omega)}{\varepsilon_k} = d < \infty.$$

It is easy to verify that ω is a solution of

$$\begin{gathered} -(a + bA_1)\Delta u + u = |u|^{p-1}u, \quad x \in \mathbb{R}_+^3 \\ u(x) = 0, \quad x \in \partial\mathbb{R}_+^3 \end{gathered} \tag{5.6}$$

where $\mathbb{R}_+^3$ is a half space. We know that (5.6) has no nontrivial solution from the work by [1, 11]. So $\omega \equiv 0$, this contradicts with $\omega \not\equiv 0$. This concludes the claim.

By (5.5), Ω_{ε_k} converges to the whole space $\mathbb{R}^3$ as $k \to \infty$. Using (5.1) (5.3) and computing, we have

$$I_\infty(\omega_k) \to m_\infty, \quad I'_\infty(\omega_k) \to 0. \tag{5.7}$$

This implies $\{\omega_k\} \subset M_\infty$ is a minimizing sequences for m_∞. Arguing as in [15], we have $\omega_k \to \omega$, $\omega \in M_\infty$, $I_\infty(\omega) = m_\infty$ and $I'_\infty(\omega) = 0$.

By using (5.7) and Pohozaev identity, we have

$$\begin{gathered}
\frac{a}{2}\|\omega_k\|^2 + \frac{1}{2}|\omega_k|_2^2 + \frac{b}{4}\|\omega_k\|^4 - \frac{1}{p+1}|\omega_k|_{p+1}^{p+1} = m_\infty + o_k(1), \\
a\|\omega_k\|^2 + |\omega_k|_2^2 + b\|\omega_k\|^4 - |\omega_k|_{p+1}^{p+1} = o_k(1) \\
\frac{a}{2}\|\omega_k\|^2 + \frac{3}{2}|\omega_k|_2^2 + \frac{b}{2}\|\omega_k\|^4 - \frac{3}{p+1}|\omega_k|_{p+1}^{p+1} = o_k(1)
\end{gathered}$$

Thus, we obtain that

$$|\omega_k|_{p+1}^{p+1} \to \frac{2(p+1)}{p-1}\Big(4m_\infty + \frac{2a^2 - 2a\sqrt{a^2+3bm_\infty}}{b}\Big)$$

So by $\omega_k \to \omega$, for T and k large enough, we have

$$\int_{B(0,T)} |\omega_k^+|^{p+1} dz > (1-\eta)\frac{2(p+1)}{p-1}\Big(4m_\infty + \frac{2a^2 - 2a\sqrt{a^2+3bm_\infty}}{b}\Big)$$

On the other hand by (5.2) and the definition of ω_k, for any $T > 0$ we have, for k large enough,

$$\begin{aligned}
\int_{B(0,T)} |\omega_k^+|^{p+1} dz &\le \frac{1}{\varepsilon_k^3}\int_{B(q_k,\varepsilon_k T)} |u_k|^{p+1} dx \\
&\le \frac{1}{\varepsilon_k^3}\int_{B(q_k,\frac{r}{2})} |u_k|^{p+1} dx \\
&\le (1-\eta)\frac{2(p+1)}{p-1}(4m_\infty + \frac{2a^2 - 2a\sqrt{a^2+3bm_\infty}}{b}).
\end{aligned}$$

This leads to a contradiction. $\square$

Proposition 5.3. *There exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$ and any $\varepsilon \in (0,\varepsilon(\delta_0))$ as in Proposition 5.2, for any $u \in M_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ it holds $\beta(u) \in \Omega^+$. Moreover the composition*

$$\beta \circ \Phi_\varepsilon : \Omega^- \to \Omega^+$$

is homotopic to the immersion $i : \Omega^- \to \Omega^+$.

Proof. Arguing by contradiction, we suppose that there exist sequences $\{\delta_k\}, \{\varepsilon_k\} \subset R$ and $\{u_k\} \subset M_{\varepsilon_k} \cap I_\varepsilon^{m_\infty+\delta_k}$ such that $\delta_k, \varepsilon_k \to 0^+$, as $k \to \infty$, and $\beta(u_k) \notin \Omega^+$ for all k.

By Ekeland variational principle and by definition of M_{ε_k} we can assume that $I'_{\varepsilon_k}(u_k) \to 0$. So by Proposition 5.2 we can find $q_k \in \Omega$ such that

$$\frac{\frac{1}{\varepsilon_k^3}\int_{B(q_k,\frac{r}{2})} |u_k|^{p+1} dx}{\frac{1}{\varepsilon_k^3}\int_\Omega |u_k|^{p+1} dx} > \frac{(1-\eta)\frac{2(p+1)}{p-1}\big(4m_\infty + \frac{2a^2-2a\sqrt{a^2+3bm_\infty}}{b}\big)}{\frac{2(p+1)}{p-1}\big(4(m_\infty+\delta_k) + \frac{2a^2-2a\sqrt{a^2+3b(m_\infty+\delta_k)}}{b}\big)}$$

Finally,

$$\begin{aligned}
&|\beta(u_k) - q_k| \\
&\le \frac{|\frac{1}{\varepsilon_k^3}\int_\Omega (x-q_k)|u_k|^{p+1} dx|}{\frac{1}{\varepsilon_k^3}\int_\Omega |u_k|^{p+1} dx}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{|\frac{1}{\varepsilon_k^3}\int_{B(q_k,\frac{r}{2})}(x-q_k)|u_k|^{p+1}dx|}{\frac{1}{\varepsilon_k^3}\int_\Omega |u_k|^{p+1}dx} + \frac{|\frac{1}{\varepsilon_k^3}\int_{\Omega\setminus B(q_k,\frac{r}{2})}(x-q_k)|u_k|^{p+1}dx|}{\frac{1}{\varepsilon_k^3}\int_\Omega |u_k|^{p+1}dx} \\
&\le \frac{r}{2} + 2\operatorname{diam}(\Omega)\Big(1 - \frac{(1-\eta)\frac{2(p+1)}{p-1}(4m_\infty + \frac{2a^2-2a\sqrt{a^2+3bm_\infty}}{b})}{\frac{2(p+1)}{p-1}(4(m_\infty+\delta_k) + \frac{2a^2-2a\sqrt{a^2+3b(m_\infty+\delta_k)}}{b})}\Big)
\end{aligned}
$$

The above expression implies that $\beta(u_k) \in \Omega^+$, which contradicts $\beta(u_k) \notin \Omega^+$. $\square$

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