

## FRACTIONAL-ORDER BOUNDARY VALUE PROBLEM WITH STURM-LIOUVILLE BOUNDARY CONDITIONS

DOUGLAS R. ANDERSON, RICHARD I. AVERY

ABSTRACT. Using the new conformable fractional derivative, which differs from the Riemann-Liouville and Caputo fractional derivatives, we reformulate the second-order conjugate boundary value problem in this new setting. Utilizing the corresponding positive fractional Green's function, we apply a functional compression-expansion fixed point theorem to prove the existence of a positive solution. We then compare our results favorably to those based on the Riemann-Liouville fractional derivative.

### 1. INTRODUCTION

The search for the existence of positive solutions and multiple positive solutions to nonlinear fractional boundary value problems has expanded greatly over the past decade; for some recent examples please see [3-9,11,14,15,17-20]. In all of these works and the references cited therein, however, the definition of the fractional derivative used is either the Caputo or the Riemann-Liouville fractional derivative, involving an integral expression and the gamma function. Recently [10, 11, 14] a new definition has been formulated and dubbed the conformable fractional derivative. In this paper, we use this fractional derivative of order  $\alpha$ , given by

$$D^\alpha f(t) := \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}, \quad D^\alpha f(0) = \lim_{t \rightarrow 0^+} D^\alpha f(t); \quad (1.1)$$

note that if  $f$  is differentiable, then

$$D^\alpha f(t) = t^{1-\alpha} f'(t), \quad (1.2)$$

where  $f'(t) = \lim_{\varepsilon \rightarrow 0} [f(t + \varepsilon) - f(t)]/\varepsilon$ . Using this new definition of the fractional derivative, we investigate a conformable fractional boundary value problem with Sturm-Liouville boundary conditions. With the fractional differential equation and fractional boundary conditions established, we find the corresponding Green's function and prove its positivity under appropriate assumptions. This work thus sets the stage for employing a functional compression-expansion fixed point theorem to prove the existence of a positive solution to the special case of conjugate boundary conditions. We then compare our existence result to that of Bai and Lü [3].

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## 2. TWO ITERATED FRACTIONAL DERIVATIVES

We begin by considering two iterated fractional derivatives in the differential operator, together with two-point boundary conditions, as illustrated in the nonlinear boundary value problem

$$-D^\beta D^\alpha x(t) = f(t, x(t)), \quad 0 \leq t \leq 1, \quad (2.1)$$

$$\gamma x(0) - \delta D^\alpha x(0) = 0 = \eta x(1) + \zeta D^\alpha x(1), \quad (2.2)$$

where  $\alpha, \beta \in (0, 1]$  and the derivatives are conformable fractional derivatives (1.1), with  $\gamma, \delta, \eta, \zeta \geq 0$  and  $d := \eta\delta + \gamma\zeta + \gamma\eta/\alpha > 0$ . Note that if  $x$  is  $\alpha$ -differentiable and  $t^{1-\alpha}x'$  is  $\beta$ -differentiable, then using (1.2) we could rewrite (2.1) as

$$-t^{1-\beta} (t^{1-\alpha}x'(t))' = f(t, x(t)), \quad 0 \leq t \leq 1,$$

where the prime indicates the derivative  $d/dt$ . Before we find Green's function for (2.1), (2.2), and prove that it is positive, we first define the  $\beta$ -fractional integral.

**Definition 2.1.** Let  $\beta \in (0, 1]$  and  $0 \leq a < b$ . A function  $f : [a, b] \rightarrow \mathbb{R}$  is  $\beta$ -fractional integrable on  $[a, b]$  if the integral

$$\int_a^b f(s) d_\beta s := \int_a^b f(s) s^{\beta-1} ds$$

exists and is finite. Thus the integral can be interpreted either as a Riemann-Stieltjes integral or an improper Riemann integral.

**Theorem 2.2.** Let  $\alpha, \beta \in (0, 1]$ . The corresponding Green's function for the homogeneous problem

$$-D^\beta D^\alpha x(t) = 0$$

satisfying the boundary conditions (2.2) is given by

$$G(t, s) = \begin{cases} \frac{1}{d}[\delta + \frac{\gamma}{\alpha}s^\alpha][\zeta + \frac{\eta}{\alpha}(1 - t^\alpha)], & s \leq t, \\ \frac{1}{d}[\delta + \frac{\gamma}{\alpha}t^\alpha][\zeta + \frac{\eta}{\alpha}(1 - s^\alpha)], & t \leq s, \end{cases} \quad (2.3)$$

where we assume the parameters satisfy  $\gamma, \delta, \eta, \zeta \geq 0$  and  $d = \eta\delta + \gamma\zeta + \gamma\eta/\alpha > 0$ .

*Proof.* We will show that

$$x(t) = \int_0^1 G(t, s)h(s)d_\beta s,$$

for  $G$  given by (2.3), is a solution to the linear boundary value problem

$$-D^\beta D^\alpha x(t) = h(t)$$

with boundary conditions (2.2).

For any  $t \in [0, 1]$ , using the branches of (2.3) we have

$$\begin{aligned} x(t) &= \frac{1}{d}[\zeta + \frac{\eta}{\alpha}(1 - t^\alpha)] \int_0^t [\delta + \frac{\gamma}{\alpha}s^\alpha]h(s)d_\beta s \\ &\quad + \frac{1}{d}[\delta + \frac{\gamma}{\alpha}t^\alpha] \int_t^1 [\zeta + \frac{\eta}{\alpha}(1 - s^\alpha)]h(s)d_\beta s. \end{aligned}$$

Taking the  $\alpha$ -fractional derivative yields

$$D^\alpha x(t) = -\frac{\eta}{d} \int_0^t [\delta + \frac{\gamma}{\alpha}s^\alpha]h(s)d_\beta s + \frac{\gamma}{d} \int_t^1 [\zeta + \frac{\eta}{\alpha}(1 - s^\alpha)]h(s)d_\beta s.$$

Checking the first boundary condition, we see that

$$\gamma x(0) - \delta D^\alpha x(0) = 0.$$

Moreover, in checking the second boundary condition we get

$$\eta x(1) + \zeta D^\alpha x(1) = 0.$$

Taking the  $\beta$ -fractional derivative of the  $\alpha$ -fractional derivative yields

$$\begin{aligned} D^\beta D^\alpha x(t) &= -\frac{\eta}{d} \left[ \delta + \frac{\gamma}{\alpha} t^\alpha \right] h(t) t^{\beta-1} t^{1-\beta} - \frac{\gamma}{d} \left[ \zeta + \frac{\eta}{\alpha} (1-t^\alpha) \right] h(t) t^{\beta-1} t^{1-\beta} \\ &= -\frac{1}{d} h(t) \left[ \eta \delta + \gamma \zeta + \frac{\gamma \eta}{\alpha} \right] = -h(t), \end{aligned}$$

which is what we set out to prove.  $\square$

**Corollary 2.3** (Fractional conjugate and right-focal problems). *Let  $\alpha, \beta \in (0, 1]$ . The corresponding Green's function for the homogeneous problem*

$$-D^\beta D^\alpha x(t) = 0$$

*satisfying the conjugate boundary conditions  $x(0) = x(1) = 0$  is given by*

$$G(t, s) = \begin{cases} \frac{1}{\alpha} s^\alpha (1-t^\alpha), & s \leq t, \\ \frac{1}{\alpha} t^\alpha (1-s^\alpha), & t \leq s, \end{cases} \quad (2.4)$$

*and the corresponding Green's function for the homogeneous problem*

$$-D^\beta D^\alpha x(t) = 0$$

*satisfying the right-focal-type boundary conditions  $x(0) = D^\alpha x(1) = 0$  is given by*

$$G(t, s) = \begin{cases} \frac{1}{\alpha} s^\alpha, & s \leq t, \\ \frac{1}{\alpha} t^\alpha, & t \leq s. \end{cases} \quad (2.5)$$

**Remark 2.4.** Note that the conformable fractional Green's function based on (1.1) and given above for the conjugate boundary conditions in (2.4) differs from that found for example in Bai and Lü [3], where the Riemann-Liouville fractional derivative is used.

**Theorem 2.5** (Bounds on Green's function). *For  $G(t, s)$  given in (2.3), we have the following bounds. First,*

$$g_1(t)G(s, s) < G(t, s) \leq G(s, s) \quad (2.6)$$

*for  $t, s \in [0, 1]$ , where*

$$g_1(t) := \min \left\{ \frac{\alpha \delta + \gamma t^\alpha}{\alpha \delta + \gamma}, \frac{\alpha \zeta + \eta(1-t^\alpha)}{\alpha \zeta + \eta} \right\}. \quad (2.7)$$

*Next, for integers  $n \geq 3$ ,*

$$\min_{\frac{1}{n} \leq t \leq 1 - \frac{1}{n}} G(t, s) \geq g_2(s)G(s, s) \quad (2.8)$$

*for  $g_2$  given by*

$$g_2(s) := \begin{cases} \frac{\alpha \zeta + \eta(1-(1-1/n)^\alpha)}{\alpha \zeta + \eta(1-s^\alpha)}, & s \in [0, r] \\ \frac{\alpha \delta + \gamma(1/n)^\alpha}{\alpha \delta + \gamma s^\alpha}, & s \in [r, 1] \end{cases} \quad (2.9)$$

for the constant

$$r = \left( \frac{\alpha\gamma\zeta + \gamma\eta + \alpha\delta\eta(n-1)^\alpha}{(\alpha\delta\eta + \alpha\gamma\zeta + \gamma\eta)n^\alpha - \gamma\eta(n-1)^\alpha + \gamma\eta} \right)^{1/\alpha} \in \left( \frac{1}{n}, 1 - \frac{1}{n} \right], \quad (2.10)$$

where  $r = 1 - 1/n$  if  $\gamma = 0$ . Finally, we also have

$$\min_{\frac{1}{n} \leq t \leq 1 - \frac{1}{n}} G(t, s) \geq g_3 G(s, s) \quad (2.11)$$

for the constant  $g_3$  given by

$$g_3 \equiv \min \left\{ \frac{\alpha\zeta + \eta(1 - (1 - 1/n)^\alpha)}{\alpha\zeta + \eta}, \frac{\alpha\delta + \gamma(1/n)^\alpha}{\alpha\delta + \gamma} \right\} \quad (2.12)$$

for all  $\alpha \in (0, 1]$ .

*Proof.* It is straightforward to see that

$$\frac{G(t, s)}{G(s, s)} = \begin{cases} \frac{\zeta + \frac{\eta}{\alpha}(1-t^\alpha)}{\zeta + \frac{\eta}{\alpha}(1-s^\alpha)}, & s \leq t, \\ \frac{\delta + \frac{\gamma}{\alpha}t^\alpha}{\delta + \frac{\gamma}{\alpha}s^\alpha}, & t \leq s; \end{cases} \quad (2.13)$$

this expression yields both inequalities in (2.6) for  $g_1$  as in (2.7).

Next, let

$$u(t, s) = \frac{1}{d} \left[ \delta + \frac{\gamma}{\alpha} t^\alpha \right] \left[ \zeta + \frac{\eta}{\alpha} (1 - s^\alpha) \right],$$

so that

$$G(t, s) = \begin{cases} u(s, t), & s \leq t, \\ u(t, s), & t \leq s. \end{cases}$$

Let  $r$  be given by (2.10). Then we have

$$\begin{aligned} \min_{\frac{1}{n} \leq t \leq 1 - \frac{1}{n}} G(t, s) &= \begin{cases} u(s, 1 - 1/n), & s \in [0, 1/n], \\ \min\{u(s, 1 - 1/n), u(1/n, s)\}, & s \in [1/n, 1 - 1/n], \\ u(1/n, s), & s \in [1 - 1/n, 1], \end{cases} \\ &= \begin{cases} u(s, 1 - 1/n), & s \in [0, r], \\ u(1/n, s), & s \in [r, 1], \end{cases} \\ &= \begin{cases} \frac{1}{d} \left[ \delta + \frac{\gamma}{\alpha} s^\alpha \right] \left[ \zeta + \frac{\eta}{\alpha} (1 - (1 - 1/n)^\alpha) \right], & s \in [0, r], \\ \frac{1}{d} \left[ \delta + \frac{\gamma}{\alpha} (1/n)^\alpha \right] \left[ \zeta + \frac{\eta}{\alpha} (1 - s^\alpha) \right], & s \in [r, 1], \end{cases} \end{aligned}$$

where  $r$  is given in (2.10). By the monotonicity of  $G(t, s)$ , we have

$$\max_{0 \leq t \leq 1} G(t, s) = G(s, s) = \frac{1}{d} \left[ \delta + \frac{\gamma}{\alpha} s^\alpha \right] \left[ \zeta + \frac{\eta}{\alpha} (1 - s^\alpha) \right], \quad s \in [0, 1].$$

Therefore if we take  $g_2$  as in (2.9), then  $G(t, s)$  satisfies (2.8). Now since

$$\frac{\alpha\zeta + \eta(1 - (1 - 1/n)^\alpha)}{\alpha\zeta + \eta(1 - s^\alpha)} \geq \frac{\alpha\zeta + \eta(1 - (1 - 1/n)^\alpha)}{\alpha\zeta + \eta}, \quad s \in [0, r]$$

and

$$\frac{\alpha\delta + \gamma(1/n)^\alpha}{\alpha\delta + \gamma s^\alpha} \geq \frac{\alpha\delta + \gamma(1/n)^\alpha}{\alpha\delta + \gamma}, \quad s \in [r, 1],$$

we could in (2.8) use the constant  $g_3$  given in (2.12) instead of (2.9). This constant is well defined and strictly positive, since  $d > 0$  in (2.3) implies neither  $\delta$  and  $\gamma$  nor  $\eta$  and  $\zeta$  can simultaneously be zero.  $\square$

**Remark 2.6.** For conjugate boundary conditions,  $\gamma = \eta = 1$  and  $\delta = \zeta = 0$ ; if  $\alpha = 1$  and  $n = 4$ , then  $g_3 \equiv 1/4$ , the constant used in [13, (3.4)]. For right-focal boundary conditions,  $\gamma = \zeta = 1$  and  $\delta = \eta = 0$ , so that clearly  $g_3 \equiv (1/n)^\alpha$  for all  $\alpha \in (0, 1]$  and all integers  $n \geq 3$ . In the conjugate case specifically, the constant bound (2.12) is new for fractional derivatives, as the standard Riemann-Liouville fractional derivative does not allow one to calculate a single constant bound; see [3, Remark 2.2].

The following corollary is needed in Section 4 for the main existence theorem and example found there.

**Corollary 2.7.** *Let  $\alpha, \beta \in (0, 1]$ . For every  $s \in [0, 1]$  we have*

$$\max_{t \in [0, 1]} G(t, s) \leq \left( \frac{1}{1 - (3/4)^\alpha} \right) \min_{t \in [1/4, 3/4]} G(t, s),$$

where  $G(t, s)$  is the Green's function (2.4) for the homogeneous problem

$$-D^\beta D^\alpha x(t) = 0$$

satisfying the conjugate boundary conditions  $x(0) = x(1) = 0$ .

*Proof.* By (2.4),  $\max_{t \in [0, 1]} G(t, s) = G(s, s)$ . Then

$$\begin{aligned} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{G(t, s)}{G(s, s)} &= \begin{cases} \frac{1-t^\alpha}{1-s^\alpha}, & 0 \leq s \leq t \leq 3/4 \\ \frac{t^\alpha}{s^\alpha}, & 1/4 \leq t \leq s \leq 1 \end{cases} \\ &\geq \begin{cases} \frac{1-(3/4)^\alpha}{1-(0)^\alpha}, & 0 \leq s \leq 3/4 \\ \frac{(1/4)^\alpha}{(1)^\alpha}, & 1/4 \leq s \leq 1 \end{cases} \\ &= 1 - (3/4)^\alpha, \end{aligned}$$

since  $(1/4)^\alpha \geq 1 - (3/4)^\alpha$  for all  $\alpha \in (0, 1]$ . One could also use (2.11) and (2.12).  $\square$

### 3. FIXED POINT PRELIMINARIES

In this section we will state the fixed point theorem and the definitions that are used in the fixed point theorem which will be used to verify the existence of a positive solution to the fractional-order boundary value problem with conjugate boundary conditions.

**Definition 3.1.** Let  $E$  be a real Banach space. A nonempty closed convex set  $P \subset E$  is called a *cone* if it satisfies the following two conditions:

- (i)  $x \in P, \lambda \geq 0$  implies  $\lambda x \in P$ ;
- (ii)  $x \in P, -x \in P$  implies  $x = 0$ .

Every cone  $P \subset E$  induces an ordering in  $E$  given by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

**Definition 3.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 3.3.** A map  $\xi$  is said to be a nonnegative continuous concave functional on a cone  $P$  of a real Banach space  $E$  if  $\xi : P \rightarrow [0, \infty)$  is continuous and

$$\xi(tx + (1-t)y) \geq t\xi(x) + (1-t)\xi(y)$$

for all  $x, y \in P$  and  $t \in [0, 1]$ . Similarly we say the map  $\phi$  is a nonnegative continuous convex functional on a cone  $P$  of a real Banach space  $E$  if  $\phi : P \rightarrow [0, \infty)$  is continuous and

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

for all  $x, y \in P$  and  $t \in [0, 1]$ . We say the map  $\psi$  is a sub-linear functional if

$$\psi(tx) \leq t\psi(x) \quad \text{for all } x \in P, t \in [0, 1].$$

**Definition 3.4.** Let  $P$  be a cone in a real Banach space  $E$  and  $\Omega$  be a bounded open subset of  $E$  with  $0 \in \Omega$ . Then a continuous functional  $\phi : P \rightarrow [0, \infty)$  is said to satisfy property (A1) if one of the following conditions hold:

- (i)  $\phi$  is convex,  $\phi(0) = 0$ ,  $\phi(x) \neq 0$  if  $x \neq 0$ , and  $\inf_{x \in P \cap \partial\Omega} \phi(x) > 0$ ,
- (ii)  $\phi$  is sublinear,  $\phi(0) = 0$ ,  $\phi(x) \neq 0$  if  $x \neq 0$ , and  $\inf_{x \in P \cap \partial\Omega} \phi(x) > 0$ ,
- (iii)  $\phi$  is concave and unbounded.

**Definition 3.5.** Let  $P$  be a cone in a real Banach space  $E$  and  $\Omega$  be a bounded open subset of  $E$  with  $0 \in \Omega$ . Then a continuous functional  $\phi : P \rightarrow [0, \infty)$  is said to satisfy property (A2) if one of the following conditions hold:

- (i)  $\phi$  is convex,  $\phi(0) = 0$  and  $\phi(x) \neq 0$  if  $x \neq 0$ ,
- (ii)  $\phi$  is sublinear,  $\phi(0) = 0$  and  $\phi(x) \neq 0$  if  $x \neq 0$ ,
- (iii)  $\phi(x + y) \geq \phi(x) + \phi(y)$  for all  $x, y \in P$ ,  $\phi(0) = 0$ ,  $\phi(x) \neq 0$  if  $x \neq 0$ .

The following theorem is Avery, Henderson, and O'Regan's functional compression-expansion fixed point theorem [2], which generalized the functional compression fixed point theorems of Anderson-Avery [1] and Sun-Zhang [17].

**Theorem 3.6.** Let  $\Omega_1$  and  $\Omega_2$  be two bounded open sets in a Banach Space  $E$  such that  $0 \in \Omega_1$  and  $\bar{\Omega}_1 \subseteq \Omega_2$  and  $P$  is a cone in  $E$ . Suppose  $A : P \cap (\bar{\Omega}_2 - \Omega_1) \rightarrow P$  is completely continuous,  $\xi$  and  $\psi$  are nonnegative continuous functionals on  $P$ , and one of the two conditions:

- (K1)  $\xi$  satisfies property (A1) with  $\xi(Ax) \geq \xi(x)$ , for all  $x \in P \cap \partial\Omega_1$ , and  $\psi$  satisfies property (A2) with  $\psi(Ax) \leq \psi(x)$ , for all  $x \in P \cap \partial\Omega_2$ , or
- (K2)  $\xi$  satisfies property (A2) with  $\xi(Ax) \leq \xi(x)$ , for all  $x \in P \cap \partial\Omega_1$ , and  $\psi$  satisfies property (A1) with  $\psi(Ax) \geq \psi(x)$ , for all  $x \in P \cap \partial\Omega_2$ ,

is satisfied. Then  $A$  has at least one fixed point in  $P \cap (\bar{\Omega}_2 - \Omega_1)$ .

#### 4. EXISTENCE OF A POSITIVE SOLUTION

Let the Banach space  $E = C[0, 1]$  be endowed with the maximum norm,

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|,$$

and define the cone  $P \subset E$  by

$$P = \left\{ x \in E : x \text{ is nonnegative on } [0, 1], \text{ and} \right.$$

$$\left. \|x\| \leq \left( \frac{1}{1 - (3/4)^\alpha} \right) \min_{t \in [1/4, 3/4]} x(t) \right\}.$$

Let the nonnegative continuous functionals  $\phi$  and  $\psi$  be defined on the cone  $P$  by

$$\psi(x) = \min_{t \in [1/4, 3/4]} x(t) \quad \text{and} \quad (4.1)$$

$$\phi(x) = \max_{t \in [0, 1]} x(t) = \|x\|. \quad (4.2)$$

The following theorem is our main result.

**Theorem 4.1.** *Let  $\alpha, \beta \in (0, 1]$  and suppose there exists positive numbers  $r$  and  $R$  such that  $0 < (\frac{1}{1-(3/4)^\alpha})r < R$ , and suppose  $f$  satisfies the following conditions:*

- (i)  $f(s, x) \leq R(\alpha + \beta)(2\alpha + \beta)$  for all  $s \in [0, 1]$  and all  $x \in [0, R]$ ,
- (ii)  $f(s, x) \geq rN$  for all  $s \in [1/4, 3/4]$  and for all  $x \in [r, \frac{r}{1-(3/4)^\alpha}]$ ,

where

$$N = \left( \left(1 - \left(\frac{3}{4}\right)^\alpha\right) \int_{1/4}^{3/4} G(s, s) d_\beta s \right)^{-1}$$

and

$$\begin{aligned} \left(1 - \left(\frac{3}{4}\right)^\alpha\right) \int_{1/4}^{3/4} G(s, s) d_\beta s &= \left(1 - \left(\frac{3}{4}\right)^\alpha\right) \left\{ \left(\frac{3}{4}\right)^{\alpha+\beta} \left[ \frac{1}{\alpha + \beta} - \frac{(3/4)^\alpha}{2\alpha + \beta} \right] \right. \\ &\quad \left. - \left(\frac{1}{4}\right)^{\alpha+\beta} \left[ \frac{1}{\alpha + \beta} - \frac{(1/4)^\alpha}{2\alpha + \beta} \right] \right\} \end{aligned}$$

Then, the second order conjugate boundary value problem has at least one positive solution  $x^*$  such that

$$r \leq \min_{t \in [1/4, 3/4]} x^*(t) \quad \text{and} \quad \max_{t \in [0, 1]} x^*(t) \leq R.$$

*Proof.* Define the completely continuous operator  $A$  by

$$Ax(t) = \int_0^1 G(t, s) f(s, x(s)) d_\beta s$$

then if we can show that  $A$  has a fixed point in  $P$  then we have verified the existence of a positive solution. Let  $x \in P$ , then from properties of  $G(t, s)$  we have that  $Ax(t) \geq 0$  and

$$\begin{aligned} \phi(Ax) &= \max_{t \in [0, 1]} \int_0^1 G(t, s) f(s, x(s)) d_\beta s \\ &\leq \left( \frac{1}{1 - (3/4)^\alpha} \right) \min_{t \in [1/4, 3/4]} \int_0^1 G(t, s) f(s, x(s)) d_\beta s \\ &= \left( \frac{1}{1 - (3/4)^\alpha} \right) \psi(Ax); \end{aligned}$$

thus,  $Ax \in P$  and we have verified that  $A : P \rightarrow P$ .

For all  $x \in P$  we have  $\psi(x) \leq \phi(x)$ , thus if we let

$$\Omega_1 = \{x : \psi(x) < r\} \quad \text{and} \quad \Omega_2 = \{x : \phi(x) < R\}$$

we have that  $0 \in \Omega_1$  and  $\bar{\Omega}_1 \subseteq \Omega_2$ , since if  $x \in \bar{\Omega}_1$  then

$$\min_{t \in [1/4, 3/4]} x(t) \leq r$$

hence since  $x \in P$  we have

$$\max_{t \in [0, 1]} x(t) \leq \left( \frac{1}{1 - (3/4)^\alpha} \right) \min_{t \in [1/4, 3/4]} x(t) \leq \left( \frac{1}{1 - (3/4)^\alpha} \right) r < R.$$

Clearly  $\Omega_1$  and  $\Omega_2$  being bounded open subsets of  $P$ .

**Claim 1:** If  $x \in P \cap \partial\Omega_2$ , then  $\phi(Ax) \leq \phi(x)$ . Let  $x \in \partial\Omega_2$ , thus  $\phi(x) = R$  hence by condition (i) we have

$$\begin{aligned}\phi(Ax) &= \max_{t \in [0,1]} \int_0^1 G(t,s) f(s, x(s)) d_\beta s \\ &\leq R(\alpha + \beta)(2\alpha + \beta) \int_0^1 G(s,s) d_\beta s \\ &= R = \phi(x).\end{aligned}$$

**Claim 2:** If  $x \in P \cap \partial\Omega_1$ , then  $\psi(Ax) \geq \psi(x)$ . Let  $x \in \partial\Omega_1$ , thus  $\psi(x) = r$  and  $\|x\| \leq \frac{r}{1 - (\frac{3}{4})^\alpha}$ , hence by condition (ii) we have

$$\begin{aligned}\psi(Ax) &= \min_{t \in [1/4, 3/4]} \int_0^1 G(t,s) f(s, x(s)) d_\beta s \\ &\geq rN \left(1 - \left(\frac{3}{4}\right)^\alpha\right) \int_{1/4}^{3/4} G(s,s) d_\beta s \\ &= r = \psi(x).\end{aligned}$$

Clearly  $\phi$  satisfies property (A1)(i) and  $\psi$  satisfies property (A2)(iii) thus the hypothesis (K1) of Theorem 3.6 is satisfied, and therefore  $A$  has a fixed point in  $\bar{\Omega}_2 - \Omega_1$ .  $\square$

**Example 4.2.** To compare our results with those in [3, Example 3.1], where the authors use the Riemann-Liouville fractional derivative of order  $3/2 \in (1, 2]$ , we take  $\alpha = 1$ ,  $\beta = 1/2$ ,  $r = 11/1000$ ,  $R = 9/25$ , and  $f(s, x) = 1 + (1/4) \sin s + x^2$  in Theorem 4.1 to get the following. One can check that  $0 < 4r < R$ , and  $f$  satisfies the following conditions:

(i)

$$f(s, x) \leq \frac{15}{4}R = \frac{27}{20}$$

for all  $s \in [0, 1]$  and all  $x \in [0, 9/25]$ ,

(ii)

$$f(s, x) \geq \frac{960r}{33\sqrt{3} - 17} = \frac{264}{25(33\sqrt{3} - 17)}$$

for all  $s \in [1/4, 3/4]$  and for all  $x \in [r, 4r]$ .

Thus by Theorem 4.1 the  $\frac{3}{2}$ -order conjugate boundary value problem

$$-D^{0.5}x'(t) = 1 + \frac{1}{4} \sin t + x(t)^2, \quad x(0) = x(1) = 0$$

has at least one positive solution  $x^*$  such that

$$\frac{11}{1000} \leq \min_{t \in [1/4, 3/4]} x^*(t) \quad \text{and} \quad \max_{t \in [0, 1]} x^*(t) \leq \frac{9}{25}.$$

In [3, Example 3.1], the result is the existence of a positive solution  $x^\dagger$  such that

$$\frac{1}{14} \leq \max_{t \in [0, 1]} x^\dagger(t) \leq 1,$$

with no information on the minimum value of the function.

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## ADDENDUM POSTED ON FEBRUARY 4, 2016

In response to a question from a reader, we clarify that “conformable fractional derivative” means “at  $\alpha = 1$  only”; i.e. as  $\alpha$  approaches 1, we recover the full derivative. It obvious that as  $\alpha$  approaches 0, we do not recover the identity operator. End of addendum

DOUGLAS R. ANDERSON  
DEPARTMENT OF MATHEMATICS, CONCORDIA COLLEGE, MOORHEAD, MN 56562, USA  
*E-mail address:* `andersod@cord.edu`

RICHARD I. AVERY  
COLLEGE OF ARTS AND SCIENCES, DAKOTA STATE UNIVERSITY, MADISON, SD 57042, USA  
*E-mail address:* `rich.avery@dsu.edu`