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SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS VIA COUPLED FIXED POINT

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Abstract. In this article, we investigate the existence and uniqueness of a solution for the fractional differential equation by introducing some new coupled fixed point theorems for the class of mixed monotone operators with perturbations in the context of partially ordered complete metric space.

1. Introduction and preliminaries

In the previous decade, one of the most attractive research subject is to investigate the existence and uniqueness of a fixed point of certain operator in the setting of complete metric space endowed with a partial order (see e.g. [1]-[24] and related reference therein). Recently, CB. Zhai [20] proved some results on a class of mixed monotone operators with perturbations. The aim of this article is to propose a method for the existence and uniqueness of a solution of certain fractional differential equations by following the paper by Zhai [20]. For this purpose, we shall consider some coupled fixed point theorems for a class of mixed monotone operators with perturbations on ordered Banach spaces with the different conditions that was introduced by Zhai [20]. On the other hand, our result are finer than the results of Zhai [20] since we obtain the existence and uniqueness of coupled fixed points without assuming continuity of compactness of the operator.

For the sake of completeness of the paper, we present here some basic definitions, notations and known results.

Suppose $(E, \|\cdot\|)$ is a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ or $x > y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P,\ \lambda \geq 0 \Longrightarrow \lambda x \in P$; (ii) $x \in P,\ -x \in P \Longrightarrow x = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. Also we define the order interval $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ for all $x_1, x_2 \in E$. We say that and operator $A : E \to E$ is increasing whenever $x \leq y$ implies $Ax \leq Ay$. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$, such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalent relation. Given

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$e > \theta$, we denote by P_e the set $P_e = \{x \in E | x \sim e\}$. It is easy to see that $P_e \subset P$ is convex and $\lambda P_e = P_e$ for all $\lambda > 0$. If $P \neq \phi$ and $e \in P$, it is clear that $P_e = P$.

Definition 1.1 ([8, 9]). $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \le u_2$, $v_1 \ge v_2$ imply $A(u_1, v_1) \le A(u_2, v_2)$. The element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

The following conditions were was assumed in [21]:

- (A1) there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$,
- (A2) for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that
$$A(tu, t^{-1}v) \ge \varphi(t)A(u, v). \tag{1.1}$$

Lemma 1.2 ([21]). *Assume that* (A1), (A2) *hold. Then* $A : P_h \times P_h \to P_h$*; and there exist* $u_0, v_0 \in P_h$ *and* $r \in (0, 1)$ *such that*
$$rv_0 \le u_0 < v_0, \quad u_0 \le A(u_0, v_0) \le A(v_0, u_0) \le v_0.$$

Definition 1.3 ([20]). An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies
$$A(tx) \ge tA(x), \quad \forall t \in (0, 1),\ x \in P.$$

The following result can be found in Zhai and Zhang [21].

Theorem 1.4 ([21]). *Let P be a normal cone in E. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies:*

- (A3) *there exists $h \in P$ with $h \neq \theta$ such that $T(h, h) \in P_h$;*
- (A4) *for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that*
$$T(tu, t^{-1}v) \ge \varphi(t)T(u, v). \tag{1.2}$$

Then

1. $T : P_h \times P_h \to P_h$*;*
2. *there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \le u_0 < v_0$, $u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0$;*
3. *T has a unique fixed point x^* in P_h;*
4. *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*
$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2 \dots,$$
we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

2. Main result

In this section, we state and prove our main results. First, we consider the mixed monotone operator $A : P \times P \to P$. The following conditions will be assumed:

- (A5) there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$,
- (A6) for any $u, v \in P$ and $s, t \in (0, 1)$ such that $s \le t$, there exists $\varphi(t) \in (t^2, 1]$ and φ is decreasing such that
$$A(tu, t^{-1}v) + A(tu, s^{-1}v) \ge 2\frac{\varphi(t)}{t}A(u, v). \tag{2.1}$$

Lemma 2.1. *Assume* (A5), (A6) *hold. Then* $A : P_h \times P_h \to P_h$*; and there exist* $u_0, v_0 \in P_h$ *and* $r \in (0, 1)$ *such that*
$$rv_0 \le u_0 < v_0, u_0 \le A(u_0, v_0) \le A(v_0, u_0) \le v_0.$$

Proof. For $s \le t$ from condition (A6) we obtain

$$A(t^{-1}x, ty) \le \frac{1}{2\frac{\varphi(t)}{t}}(A(x,y) + A(x, \frac{t}{s}y)), \quad \forall s,t \in (0,1),\ x,y \in P. \tag{2.2}$$

For any $u, v \in P_h$, there exist $\mu_1, \mu_2 \in (0,1)$, such that

$$\mu_1 h \le u \le \frac{1}{\mu_1}h,\ \mu_2 h \le v \le \frac{1}{\mu_2}h.$$

Let $\mu = \min\{\mu_2, \mu_1\}$. Then $\mu \in (0,1)$. From (2.2) and the mixed monotone properties of operator A and regarding $0 < \mu$, there exists $0 < \mu' < \mu$ such that

$$\begin{aligned}
A(u,v) &\le A(\frac{1}{\mu_1}h, \mu_2 h) \le A(\frac{1}{\mu}h, \mu h)\\
&\le \frac{1}{2(\frac{\varphi(\mu)}{\mu})}(A(h,h) + A(h, \frac{\mu}{\mu'}h))\\
&\le \frac{1}{2(\frac{\varphi(\mu)}{\mu})}(A(h,h) + A(h,h))\\
&= \frac{1}{(\frac{\varphi(\mu)}{\mu})}(A(h,h)) \le \frac{1}{\varphi(\mu)}A(h,h).
\end{aligned}$$

Regarding the inequality

$$A(u,v) \ge A(\mu_1 h, \frac{1}{\mu_2}h) \ge A(\mu h, \frac{1}{\mu}h),$$

we derive that

$$\begin{aligned}
2A(u,v) &\ge A(\mu h, \frac{1}{\mu}h) + A(\mu h, \frac{1}{\mu}h)\\
&\ge 2(\frac{\varphi(\mu)}{\mu})(A(h,h) + A(h,h))\\
&= 2(\frac{\varphi(\mu)}{\mu})(A(h,h)) \ge 2\varphi(\mu)A(h,h).
\end{aligned}$$

Tehrefore, we obtain

$$A(u,v) \ge \varphi(\mu)A(h,h).$$

It follows from $A(h,h) \in P_h$ that $A(u,v) \in P_h$. Hence we have $A : P_h \times P_h \to P_h$. Since $A(h,h) \in P_h$, we can choose a sufficiently small number $t_0 \in (0,1)$ such that

$$t_0 h \le A(h,h) \le \frac{1}{t_0}h. \tag{2.3}$$

For $k > 2$ we have

$$t_0^k h \le A(h,h) \le \frac{1}{t_0^k}h\,. \tag{2.4}$$

Put $u_0 = {t_0}^k h$ and $v_0 = \frac{1}{t_0^k}h$. Evidently, $u_0, v_0 \in P_h$ and $u_0 = {t_0}^{2k}v_0 < v_0$. Take any $r \in (0, {t_0}^{2k}]$, then $r \in (0,1)$ and $u_0 \ge r v_0$. By the mixed monotone properties of A, we have $A(u_0, v_0) \le A(v_0, u_0)$. Because $t_0 \in (0,1)$, then there exists $s_0 \in (0,1)$ such that $0 < s_0 \le t_0$. Further, combining condition (A2) with (2.3), and since $s_0 \le t_0$ we have

$$A(u_0, v_0) = A(t_0^k h, \frac{1}{t_0^k}h)$$

$$
\begin{aligned}
&\geq 2(\frac{\varphi(t_0^k)}{t_0^k})A(h,h) - A(h,h) \\
&\geq (2(\frac{\varphi(t_0)}{t_0^2}) - 1)A(h,h) > A(h,h) \\
&\geq t_0^k h = u_0,
\end{aligned}
$$

and

$$
\begin{aligned}
A(v_0, u_0) &= A(\frac{1}{t_0^k}h, t_0^k h) \\
&\leq \frac{1}{2(\frac{\varphi(t_0^k)}{t_0^k})}(A(h,h) + A(h, \frac{t_0^k}{s_0^k}h)) \\
&\leq \frac{1}{2(\frac{\varphi(t_0^k)}{t_0^k})}2A(h,h) \\
&\leq A(h,h) \leq \frac{1}{t_0^k}h = v_0.
\end{aligned}
$$

Consequently, we have $u_0 \leq A(u_0, v_0) \leq A(v_0, u_0) \leq v_0$. □

Corollary 2.2. *If in* (2.1) *put* $s = t$ *then we obtain* (1.2). *Consequently the Lemma 2.1 yields the Lemma 1.2.*

Theorem 2.3. *Suppose that* P *is a normal cone of* E*, and* (A5), (A6) *hold. Then operator* A *has a unique fixed point* x^* *in* P_h*. Moreover, for any initial* $x_0, y_0 \in P_h$*, constructing successively the sequences*

$$
x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad n = 1, 2, \dots,
$$

we have $\|x_n - x^*\| \to 0, \|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof. From Lemma 2.1, there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that

$$
rv_0 \leq u_0 < v_0, u_0 \leq A(u_0, v_0) \leq A(v_0, u_0) \leq v_0.
$$

Construct recursively the sequences

$$
u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \dots.
$$

Evidently $u_1 \leq v_1$. By the mixed monotone properties of A, we obtain

$$
u_n \leq v_n \ , \ n = 1, 2, \dots.
$$

It also follows from Lemma 2.1 and the mixed monotone properties of A that

$$
u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \tag{2.5}
$$

Note that $u_0 \geq rv_0$. We can get $u_n \geq u_0 \geq rv_0 \geq rv_n, \ n = 1, 2, \dots$. Let

$$
t_n = \sup\{t > 0 | u_n \geq tv_n\}, \quad s_n = \sup\{s > 0 | u_n \geq sv_n\}, \quad s_n \leq t_n \ n = 1, 2, \dots.
$$

Thus we have $u_n \geq t_n v_n, u_n \geq s_n v_n, n = 1, 2, \dots$, then $u_n \geq t_n v_n \geq s_n v_n$, also $u_{n+1} \geq u_n \geq t_n v_n \geq t_n v_{n+1} \geq s_n v_{n+1}, n = 1, 2, \dots$. Therefore, $t_{n+1} \geq t_n$, *i.e.*, t_n is increasing with $t_n \subset (0,1]$. Suppose $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$. Then from condition (A2) and $t_n \leq t^*$, we have $A(u_n, v_n) \geq A(t_n v_n, \frac{1}{t_n} u_n)$ and $A(u_n, v_n) \geq A(t_n v_n, \frac{1}{s_n} u_n)$, so

$$
u_{n+1} = A(u_n, v_n)
$$

$$\begin{aligned}
&\geq \frac{1}{2}(A(t_n v_n, \frac{1}{t_n}u_n) + A(t_n v_n, \frac{1}{s_n}u_n)) \\
&\geq \frac{\varphi(t_n)}{t_n}A(v_n, u_n) \geq \frac{\varphi(t^*)}{t_n}A(v_n, u_n) \\
&= \frac{\varphi(t^*)}{t_n}v_{n+1}.
\end{aligned}$$

By the definition of t_n, $t_{n+1} \geq \frac{\varphi(t^*)}{t_n}$. Let $n \to \infty$, we obtain $t^{*2} \geq \varphi(t^*) > t^{*2}$, which is a contradiction. Thus, $\lim_{n\to\infty} t_n = 1$. For any natural number p we have

$$\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - t_n v_n = (1 - t_n)v_n \leq (1 - t_n)v_0,$$
$$\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (1 - t_n)v_0.$$

Since the cone P is normal, we have

$$\|u_{n+p} - u_n\| \leq M(1 - t_n)\|v_0\| \to 0,$$
$$\|v_n - v_{n+p}\| \leq M(1 - t_n)\|v_0\| \to 0,$$

as $n \to \infty$, where M is the normality constant of P. So we can claim that u_n and v_n are Cauchy sequences. Since E is complete, there exist u^*, v^* such that $u_n \to u^*, v_n \to v^*$, as $n \to \infty$. By (2.5), we know that $u_n \leq u^* \leq v^* \leq v_n$ with $u^*, v^* \in P_h$ and

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n)v_0.$$

Further

$$\|v^* - u^*\| \leq M(1 - t_n)\|v_0\| \to 0 \quad (n \to \infty),$$

and thus $u^* = v^*$. Let $x^* := u^* = v^*$ and then we obtain

$$u_{n+1} = A(u_n, v_n) \leq A(x^*, x^*) \leq A(v_n, u_n) = v_{n+1}.$$

Let $n \to \infty$, then we obtain $x^* = A(x^*, x^*)$. That is, x^* is a fixed point of A in P_h. Next we shall prove that x^* is the unique fixed point of A in P_h. In fact, suppose $\bar{x}$ is a fixed point of A in P_h. Since $x^*, \bar{x} \in P_h$, there exists positive numbers $\bar{\mu}_1, \bar{\mu}_2, \bar{\lambda}_1, \bar{\lambda}_2 > 0$ such that

$$\bar{\mu}_1 h \leq x^* \leq \bar{\lambda}_1, \quad \bar{\mu}_2 h \leq \bar{x} \leq \bar{\lambda}_2 h.$$

Then we obtain

$$\bar{x} \leq \bar{\lambda}_2 h = \frac{\bar{\lambda}_2}{\bar{\mu}_1}\bar{\mu}_1 h \leq \frac{\bar{\lambda}_2}{\bar{\mu}_1}x^*, \quad \bar{x} \geq \bar{\mu}_2 h = \frac{\bar{\mu}_2}{\bar{\lambda}_1}\bar{\lambda}_1 h \geq \frac{\bar{\mu}_2}{\bar{\lambda}_1}x^*.$$

Let $e_1 = \sup\{t > 0 | tx^* \leq \bar{x} \leq t^{-1}x^*\}$. Evidently, $0 < e_1 \leq 1, e_1 x^* \leq \bar{x} \leq \frac{1}{e_1}x^*$. Next we prove $e_1 = 1$. If $0 < e_1 < 1$, then $\bar{x} = A(\bar{x}, \bar{x}) \geq A(e_1 x^*, \frac{1}{e_1}x^*)$, then

$$\begin{aligned}
2A(\bar{x}, \bar{x}) &\geq 2A(e_1 x^*, \frac{1}{e_1}x^*) = A(e_1 x^*, \frac{1}{e_1}x^*) + A(e_1 x^*, \frac{1}{e_1}x^*) \\
&\geq 2(\frac{\varphi(e_1)}{e_1})A(x^*, x^*).
\end{aligned}$$

So we have

$$A(\bar{x}, \bar{x}) \geq (\frac{\varphi(e_1)}{e_1})A(x^*, x^*) \geq \frac{\varphi(e_1)}{e_1}A(x^*, x^*) \geq \varphi(e_1)A(x^*, x^*).$$

Since $\varphi(e_1) > e_1$, this contradicts the definition of e_1. Hence $e_1 = 1$, and we obtain $\bar{x} = x^*$. Therefore, A has a unique fixed point x^* in P_h. Note that $[u_0, v_0] \subset P_h$, then we know that x^* is the unique fixed point of A in $[u_0, v_0]$.

Now we construct the sequences recursively as follows:

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$

for any initial points $x_0, y_0 \in P_h$. Since $x_0, y_0 \in P_h$ we can choose small numbers $e_2, e_3 \in (0,1)$ such that

$$e_2 h \le x_0 \le \frac{1}{e_2} h, \quad e_3 h \le y_0 \le \frac{1}{e_3} h.$$

Let $e^* = \min\{e_2, e_3\}$. Then $e^* \in (0,1)$ and

$$e^* h \le x_0, \quad y_0 \le \frac{1}{e^*} h.$$

We can choose a sufficiently large positive integer m such that

$$\big[\frac{\varphi(e^*)}{e^*}\big]^m \ge \frac{1}{e^*}.$$

Put $\bar{u}_0 = {e^*}^m h, \bar{v}_0 = \frac{1}{{e^*}^m} h$, it easy to see that $\bar{u}_0, \bar{v}_0 \in P_h$ and $\bar{u}_0 < x_0, y_0 < \bar{v}_0$. Let

$$\bar{u}_n = A(\bar{u}_{n-1}, \bar{v}_{n-1}), \quad \bar{v}_n = A(\bar{v}_{n-1}, \bar{u}_{n-1}), \quad n = 1, 2, \dots.$$

Analogously, it follows that there exists $y^* \in P_h$ such that $A(y^*, y^*) = y^*$ and $\lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \bar{v}_n = y^*$. By the uniqueness of fixed point of operator A in P_h. We get $x^* = y^*$ and by induction $\bar{u}_n \le x_n, y_n \le \bar{v}_n$, $n = 1, 2, \dots$. Since cone P is normal we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$. $\square$

Theorem 2.4. *Let $\alpha \in (0,1)$, $A : P \times P \to P$ be a mixed monotone operator satisfying*

$$A(tx, t^{-1}y) + A(tx, s^{-1}y) \ge 2t^{2\alpha-1} A(x, y), \quad s, t \in (0,1),\ s \le t\ x, y \in P. \tag{2.6}$$

Suppose that $B : P \to P$ is an in increasing sub-homogeneous operator. Assume also that

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $A(x, y) \ge \delta_0 Bx$ for all $x, y \in P$.*

Then

(1) *$A : P_h \times P_h \to P_h, B : P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*
$$rv_0 \le u_0 < v_0, \quad u_0 \le A(u_0, v_0) + Bu_0 \le A(v_0, u_0) + Bv_0 \le v_0;$$
(3) *the operator $A(x, x) + Bx = x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*
$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \quad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \dots,$$
we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Proof. Notice that from (2.6) and Definition 1.3, we have

$$A(\frac{1}{t}x, ty) \le \frac{1}{2t^{2\alpha-1}} (A(x, y) + A(x, \frac{t}{s} y)), \tag{2.7}$$

and $B(\frac{1}{t}x) \le \frac{1}{t} Bx$ for $s, t \in (0,1), x, y \in P$ and $s \le t$.

Since $A(h_0, h_0), Bh_0 \in P_h$, there exist constants $\lambda_1, \lambda_2, \nu_1, \nu_2 > 0$ such that

$$\lambda_1 h \le A(h_0, h_0) \le \lambda_2 h, \quad \nu_1 h \le Bh_0 \le \nu_2 h\,.$$

Also from $h_0 \in P_h$, there exists a constant $t_0 \in (0,1)$ such that $t_0 h \le h_0 \le \frac{1}{t_0} h$, and let $s_0 \in (0,1)$ such that $s_0 \le t_0$, then we have

$$s_0 h \le t_0 h \le h_0 \le \frac{1}{t_0} h \le \frac{1}{s_0} h.$$

From $s_0 \le t_0$, (2.6), (2.7) and the mixed monotone properties of operator A, we have

$$A(h,h) \ge A(t_0 h_0, \frac{1}{t_0} h_0), \quad A(h,h) \ge A(t_0 h_0, \frac{1}{s_0} h_0).$$

So we have

$$2A(h,h) \ge 2t_0^{2\alpha-1} A(h_0,h_0)\,.$$

By combining the inequalities above, we have

$$A(h,h) \ge t_0^{2\alpha-1} A(h_0,h_0) \ge t_0^{2\alpha} A(h_0,h_0) \ge \lambda_1 t_0^{2\alpha} h,$$

and

$$\begin{aligned} A(h,h) &\le A(\frac{1}{t_0} h_0, t_0 h_0) \le \frac{1}{2t_0^{2\alpha-1}} (A(h_0,h_0) + A(h_0, \frac{t_0}{s_0} h_0)) \\ &\le \frac{1}{t_0^{2\alpha}} A(h_0,h_0) \le \frac{\lambda_2}{t_0^{\alpha}} h. \end{aligned}$$

Noting that $\frac{\lambda_2}{t_0^{2\alpha}}, \lambda_1 t_0^2 \alpha > 0$, we can get $A(h,h) \in P_h$. By Definition 1.3 and the monotone property of operator B, we have

$$Bh \le B(\frac{1}{t_0} h_0) \le \frac{1}{t_0} B h_0 \le \frac{\nu_2}{t_0} h, \quad Bh \ge B(t_0 h_0) \ge t_0 B h_0 \ge \nu_1 t_0 h.$$

Next we show $B : P_h \to P_h$. For any $x \in P_h$, we can choose a sufficiently small number $\mu \in (0,1)$ such that

$$\mu h \le x \le \frac{1}{\mu} h.$$

Consequently,

$$Bx \le B(\frac{1}{\mu} h) \le \frac{1}{\mu} \frac{\nu_2}{t_0} h, \quad Bx \ge B(\mu h) \ge \mu t_0 \nu_1 h.$$

Evidently, we have $\frac{\nu_2}{\mu t_0}, \mu t_0 \nu_1 > 0$. Thus $Bx \in P_h$; that is, $B : P_h \to P_h$. So the conclusion (1) holds. Now we define an operator $T = A + B$ by $T(x,y) = A(x,y) + Bx$. Then $T : P \times P \to P$ is a mixed monotone operator and $T(h,h) \in P_h$. In the following we show that there exists $\varphi(t) \in (t,1]$ with respect to $s,t \in (0,1), s \le t$ such that

$$T(tx, t^{-1} y) + T(tx, s^{-1} y) \ge 2(\frac{\varphi(t)}{t}) A(x,y), \quad \forall, x, y \in P.$$

Consider the function

$$f(t) = \frac{t^{2\beta-1} - t}{t^{2\alpha-1} - t^{2\beta-1}},$$

for $t \in (0,1)$, where $\beta \in (\alpha,1)$. It is easy to prove that f is increasing in $(0,1)$ and

$$\lim_{t\to 0^+} f(t) = 0, \quad \lim_{t\to 1^-} f(t) = \frac{1-\beta}{\beta-\alpha}.$$

Further, fixing $t \in (0,1)$, we have

$$\lim_{\beta\to 1^-} f(t) = \lim_{\beta\to 1^-} \frac{t^{2\beta-1} - t}{t^{2\alpha-1} - t^{2\beta-1}} = 0.$$

So there exists $\beta_0(t) \in (0,1)$ with respect to t such that

$$\frac{t^{2\beta_0(t)-1} - t}{t^{2\alpha-1} - t^{2\beta_0(t)-1}} \leq \delta_0, \quad t \in (0,1).$$

Hence we have

$$A(x,y) \geq \delta_0 Bx \geq \frac{t^{2\beta_0(t)-1} - t}{t^{2\alpha-1} - t^{2\beta_0(t)-1}} Bx, \quad \forall t \in (0,1),\ x,y \in P.$$

Then we obtain

$$t^{2\alpha-1}A(x,y) + tBx \geq t^{2\beta_0(t)-1}[A(x,y) + Bx], \quad \forall t \in (0,1),\ x,y \in P.$$

Consequently, for any $t \in (0,1)$ and $x,y \in P$,

$$\begin{aligned} T(tx,t^{-1}y) + T(tx,s^{-1}y) &= A(tx,t^{-1}y) + B(tx) + A(tx,s^{-1}y) + B(tx) \\ &\geq 2t^{2\alpha-1}A(x,y) + 2tBx \\ &\geq 2t^{2\beta_0(t)-1}(A(x,y) + Bx) \\ &= 2t^{2\beta_0(t)-1}T(x,y). \end{aligned}$$

Let $\varphi(t) = t^{2\beta_0(t)},\ t \in (0,1)$. Then $\varphi(t) \in (t^2, 1]$ and

$$T(tx,t^{-1}y) + T(tx,s^{-1}y) \geq 2(\frac{\varphi(t)}{t})A(x,y),$$

for any $s,t \in (0,1)$ and $x,y \in P$. Hence the condition (A2) in Lemma 2.1 is satisfied. By Lemma 2.1 we conclude that: (a) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0, u_0 \leq T(u_0,v_0) \leq T(v_0,u_0) \leq v_0$; (b) T has a unique fixed point x^* in P_h; (c) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1,2,\dots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$. That is, conclusions (2)–(4) hold. □

Corollary 2.5. *Let $\alpha \in (0,1)$, $A : P \times P \to P$ is a mixed monotone operator. Assume* (2.6) *holds and there is $h_0 > \theta$ such that $A(h_0,h_0) \in P_h$. Then*

(1) *$A : P_h \times P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*
$$rv_0 \leq u_0 < v_0, \quad u_0 \leq A(u_0,v_0) \leq A(v_0,u_0) \leq v_0;$$
(3) *the operator $A(x,x) = x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*
$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1,2,\dots,$$
we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3. Solution to fractional differential equations

In this section, we shall propose a method for showing the existence and uniqueness of a solution for the fractional differential equation

$$\begin{gathered} \frac{D^\alpha}{Dt}u(s,t) + f(s,t,u(s,t),v(s,t)) = 0, \\ 0 < \epsilon < T, \quad T \geq 1, \quad t \in [\epsilon,T], \quad 0 < \alpha < 1, \quad s \in [a,b] \end{gathered} \tag{3.1}$$

subject to the condition

$$u(s,\zeta) = u(s,T), \quad (s,\zeta) \in [a,b] \times (\epsilon,t), \tag{3.2}$$

where D^α is the Riemann-Liouville fractional derivative of order α. We will suppose that $a, b \in (0, \infty)$, $a < b$. Let

$$E = C([a, b] \times [\epsilon, T]).$$

Consider the Banach space of continuous functions on $[a, b] \times [\epsilon, T]$ with sup norm and set

$$P = \{y \in C([a, b] \times [\epsilon, T]) : \min_{(s,t) \in [a,b] \times [\epsilon, T]} y(s, t) \geq 0\}.$$

Then P is a normal cone.

Lemma 3.1. *Let $(s, t) \in [a, b] \times [\epsilon, T], (s, \zeta) \in [a, b] \times (\epsilon, t)$ and $0 < \alpha < 1$. Then the problem*

$$\frac{D^\alpha}{Dt} u(s, t) + f(s, t, u(s, t), v(s, t)) = 0$$

with the boundary value condition $u(s, \zeta) = u(s, T)$ has a solution u_0 if and only if u_0 is a solution of the fractional integral equation

$$u(s, t) = \int_\epsilon^T G(t, \xi) f(s, \xi, u(s, \xi), v(s, t)) d\xi,$$

where

$$G(t, \xi) = \begin{cases} \frac{t^{\alpha-1}(\zeta-\xi)^{\alpha-1} - t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1} - T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & \epsilon \leq \xi \leq \zeta \leq t \leq T, \\ \frac{-t^{\alpha-1} - (T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1} - T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & \epsilon \leq \zeta \leq \xi \leq t \leq T, \\ \frac{-t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1} - T^{\alpha-1})\Gamma(\alpha)}, & \epsilon \leq \zeta \leq t \leq \xi \leq T. \end{cases}$$

Proof. From $\frac{D^\alpha}{Dt} u(s, t) + f(s, t, u(s, t), v(s, t)) = 0$ and the boundary condition, it is easy to see that $u(s, t) - c_1 t^{\alpha-1} = -I_\epsilon^\alpha f(s, t, u(s, t), v(s, t))$. By the definition of a fractional integral, we obtain

$$\begin{aligned} u(s, t) &= c_1 t^{\alpha-1} - \int_\epsilon^\zeta \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi, \\ u(s, \zeta) &= c_1 T^{\alpha-1} - \int_\epsilon^\zeta \frac{(\zeta-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi, \\ u(s, T) &= c_1 T^{\alpha-1} - \int_\epsilon^T \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi. \end{aligned}$$

Since $u(s, \zeta) = u(s, T)$, we obtain

$$\begin{aligned} c_1 &= \frac{1}{\zeta^{\alpha-1} - T^{\alpha-1}} \int_\epsilon^\zeta \frac{(\zeta-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi \\ &\quad - \frac{1}{\zeta^{\alpha-1} - T^{\alpha-1}} \int_\epsilon^T \frac{(T-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi. \end{aligned}$$

Hence

$$\begin{aligned} u(s, t) &= \frac{t^{\alpha-1}}{\zeta^{\alpha-1} - T^{\alpha-1}} \int_\epsilon^\zeta \frac{(\zeta-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi \\ &\quad - \frac{t^{\alpha-1}}{\zeta^{\alpha-1} - T^{\alpha-1}} \int_\epsilon^T \frac{(T-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi \\ &\quad - \int_\epsilon^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s, \xi, u(s, \xi), v(s, \xi)) d\xi \end{aligned}$$

$$= \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,u(s,\xi),v(s,\xi)) d\xi.$$

This completes the proof. □

Theorem 3.2. *Let $0 < \epsilon < T$ and let $f(s,t,u(s,t),v(s,t))$ be function in the space $C([a,b],[\epsilon,T],[0,\infty],[0,\infty])$, that is increasing in u, decreasing in v, with positive values. Also assume that for any $u,v \in P$ and $c,c' \in (0,1)$ with $c' \le c$, there exists $\varphi(c) \in (c^2,1]$ and φ is decreasing such that*

$$\int_{\epsilon}^{T} G(t,\xi) f(s,\xi,cu(s,\xi),c^{-1}v(s,\xi)) d\xi + \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,cu(s,\xi),{c'}^{-1}v(s,\xi)) d\xi$$
$$\ge 2\frac{\varphi(c)}{c} \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,u(s,\xi),v(s,\xi)) d\xi,$$

and $f(s,t,u(s,t),v(s,t)) = 0$, whenever $G(s,t) < 0$. Also assume that there exist $M_1 > 0$, $M_2 > 0$ and $h \ne \theta \in P$ such that

$$M_1 h(t) \le \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,u(s,\xi),v(s,\xi)) d\xi \le M_2 h(t),$$

for all $t \in [\epsilon,T]$, where $G(t,s)$ is the green function defined in Lemma 3.1. Then problem (3.1) with the boundary value condition (3.2) has unique solution u^*.

Proof. By using Lemma (3.1), the problem is equivalent to the integral equation

$$u(s,t) = \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,u(s,\xi),v(s,\xi)) d\xi,$$

where

$$G(t,\xi) = \begin{cases} \frac{t^{\alpha-1}(\eta-\xi)^{\alpha-1} - t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} & \epsilon \le \xi \le \eta \le t \le T \\ \frac{-t^{\alpha-1}-(T-\xi)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} & \epsilon \le \eta \le \xi \le t \le T \\ \frac{-t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} & \epsilon \le \eta \le t \le \xi \le T \end{cases}$$

Define the operator $A : P \times P \to E$ by

$$A(u(s,t),v(s,t)) = \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,u(s,\xi),v(s,\xi)) d\xi.$$

Then u is solution for the problem if and only if $u = A(u,u)$. It is easy to see to check that the operator A is increasing in u and decreasing in v on P. By assumptions of theorem we have;

(A7) there exists $h \in P$ with $h \ne \theta$ such that

$$M_1 h(t) \le \int_{\epsilon}^{T} G(t,\xi) f(s,\xi,u(s,\xi),v(s,\xi)) d\xi \le M_2 h(t),$$

thus $A(h,h) \in P_h$,

(A8) for any $u,v \in P$ and $c,c' \in (0,1)$ such that $c' \le c$, there exists $\varphi(c) \in (c^2,1]$ and φ is decreasing such that

$$A(cu,c^{-1}v) + A(cu,{c'}^{-1}v) \ge 2\frac{\varphi(c)}{c} A(u,v).$$

Now by using theorem (2.3), the operator A has a unique fixed point u^* in P_h. Therefore the boundary value problem (3.1) has unique solution u^*. □

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