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NONLINEAR INITIAL BOUNDARY-VALUE PROBLEMS WITH RIESZ FRACTIONAL DERIVATIVE

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ABSTRACT. We consider an initial boundary-value problem for a nonlinear partial differential equation with fractional derivative of Riesz type on a half-line. We study local and global existence of solutions in time, as well as the asymptotic behavior of solutions for large time.

1. INTRODUCTION

We study the existence of local and global solutions, and the asymptotic behaviour, for the initial boundary-value problem

$$\begin{gathered} u_t + \mathcal{N}(u) + |\partial_x|^\alpha u = 0, \quad t > 0,\ x > 0; \\ u(x,0) = u_0(x), \quad x > 0, \\ u(0,t) = h(t), \quad t > 0. \end{gathered} \tag{1.1}$$

where $\mathcal{N}(u) = |u|^\sigma u$, $\alpha < \sigma < \alpha + 1$, $2/5 < \alpha < 1$, and $|\partial_x|^\alpha$ is a fractional derivative of Riesz type defined by

$$|\partial_x|^\alpha u = \mathcal{R}^{1-\alpha+[\alpha]}\partial_x^{[\alpha]+1}u. \tag{1.2}$$

Here, $[\alpha]$ denotes the integer part of the number $\alpha > 0$, $\alpha \notin \mathbb{Z}$, and $\mathcal{R}^\alpha$ is the modified Riesz Potential (see [15]),

$$\mathcal{R}^\alpha u = \frac{1}{2\Gamma(\alpha)\sin(\frac{\pi}{2}\alpha} \int_0^{+\infty} \frac{\operatorname{sgn}(x-y)}{|x-y|^{1-\alpha}} u(y)dy.$$

The Cauchy problem for nonlinear nonlocal dissipative equations has been extensively studied. In particular, the large time asymptotic behavior for the Cauchy problem for different nonlinear equations is investigated in [10] and the references therein.

Boundary value problems arise in many applications and play an important role in the contemporary mathematical physics. For the study of the effect of the boundary data on the qualitative properties of the solution the reader is referred to [2, 3, 4, 5, 6, 7, 16] and references therein.

The general theory of nonlinear nonlocal equations on a half-line was developed in [9], where the pseudodifferential operator $\mathbb{K}$ ($|\partial_x|^\alpha$ in our case) defined on a

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half-line was introduced by virtue of the inverse Laplace transformation; it is given by

$$\mathbb{K}u = \frac{1}{2\pi i}\sum_j C_j \int_{-i\infty+\sigma}^{i\infty+\sigma} e^{px}p^{\alpha_j}\Big(\widehat{u}(p) - \sum_{l=1}^{[\alpha_j]} \partial_x^{l-1}u(0)p^{-l}\Big)dp.$$

Note that the symbol $K(p) = \sum_j C_j p^{\alpha_j}$ is analytic in the complex right half-plane. We emphasize that the operator $|\partial_x|^\alpha$ in equation (1.2) has a nonanalytic nonhomogeneous symbol $K(p) = |p|^\alpha$ and the general theory in [9] can not be applied to problem (1.1) directly.

As far as we know there are few results on the initial-boundary value problems with pseudodifferential equations having a nonanalytic symbol. The case of rational symbols $K(p)$ with poles in the complex right half-plane was studied in [11, 12], where it was proposed a new method for constructing the Green operator based on the introduction of necessary conditions at the singular points of the symbol $K(p)$. In [13] the initial-boundary value problem for a pseudodifferential equation with a nonanalytic homogeneous symbol $K(p) = |p|^{1/2}$ was studied, a theory of sectionally analytic functions was implemented for proving that the initial-boundary value problem is well-posed. Since the symbol $K(p) = |p|^{1/2}$ does not grow fast at infinity, no boundary data is needed. Finally, the case of $K(p) = |p|^\alpha$, $\alpha \in (1/2, 1)$, was studied in [1]. In the present work we consider the nonlinear version of the problem [14], considering $\alpha \in (2/5, 1)$. In order to construct a Green operator we follow the methods used in [1, 14].

To state precisely the results of the present paper we give some notations. We denote $\langle t\rangle = 1+t$, $\{t\} = \frac{t}{\langle t\rangle}$. Here and below p^α is the main branch of the complex analytic function in the complex half-plane $\mathrm{Re}(p) \geq 0$, so that $1^\alpha = 1$ (we make a cut along the negative real axis $(-\infty, 0)$). Note that due to the analyticity of p^α for all $\mathrm{Re}(p) > 0$ the inverse Laplace transform gives us 0 for all $x < 0$. The direct Laplace transformation $\mathcal{L}_{x\to p}$ is given by

$$\widehat{u}(p) \equiv \mathcal{L}_{x\to p}\{u\} = \int_0^{+\infty} e^{-px}u(x)dx$$

and the inverse Laplace transformation $\mathcal{L}^{-1}_{p\to x}$ is defined by

$$u(x) \equiv \mathcal{L}^{-1}_{p\to x}\{\widehat{u}\} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}u(p)dp.$$

The norm in weighted Lebesgue space $L^{s,\mu}(\mathbb{R}^+) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^{s,\mu}} < \infty\}$ is given by

$$\|\varphi\|_{L^{s,\mu}} = \Big(\int_0^{+\infty} x^{\mu s}|\varphi(x)|^s dx\Big)^{\frac{1}{s}}$$

for $\mu > 0$, $1 < s < \infty$ and

$$\|\varphi\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^+}|\varphi(x)|.$$

We now introduce the spaces for initial data and solutions on $\mathbb{R}^+$:

$$\mathbb{Z}^\nu = \{\phi \in L^1 \cap L^{1,\nu} \cap L^\infty : \|\phi\|_{\mathbb{Z}^\nu} < \infty\},$$

with the norm

$$\|\phi\|_{\mathbb{Z}^\nu} = \|\phi\|_{L^1} + \|\phi\|_{L^{1,\nu}} + \|\phi\|_{L^\infty},$$

$$X^\mu = \{\varphi \in C([0,\infty); L^1) \cap C((0,\infty); L^s \cap L^{s,\mu} \cap L^\infty) : \|\varphi\|_{X^\mu} < \infty\},$$

where $s \geq 1$, with the norm

$$\|\varphi\|_{X^\mu} = \sup_{t>0}(t^{\frac{1}{\alpha}(1-\frac{1}{s})}\|\varphi\|_{L^s} + t^{\frac{1}{\alpha}(1-\frac{1}{s}-\mu)}\|\varphi\|_{L^{s,\mu}} + \{t\}^\gamma \langle t\rangle^{\frac{1}{\alpha}}\|\varphi\|_{L^\infty})$$

where $0 < \gamma < 1$ and $|1 - \frac{1}{s} - \mu| < \alpha$. We also define the space

$$B^\alpha = \{h \in C^1(0,\infty) : \|h\|_{B^\alpha} < \infty\}$$

for the boundary data, where the norm

$$\|h\|_{B^\alpha} = \sup_{t>0}\langle t\rangle^{\frac{1}{\alpha}}(|h(t)| + \langle 1+t\rangle|h'(t)|).$$

Now, to state the main results we introduce $\Lambda(s), \Upsilon(s) \in L^\infty(\mathbb{R}^+)$ by the formulae

$$\Lambda(s) = C\int_{-i\infty}^{i\infty} d\xi e^\xi \int_{-i\infty}^{i\infty} e^{ps}\frac{p^{\alpha/2}}{K(p)+\xi}L(p,\xi)dp \tag{1.3}$$

and

$$\begin{aligned}\Upsilon(s) = &-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^\xi \int_{-i\infty}^{i\infty} e^{ps}\frac{Y^+(p,\xi)}{K(p)+\xi}I^+(p,\xi)dp \\ &+ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{ps-K(p)}\frac{K(p)}{p}dp,\end{aligned} \tag{1.4}$$

where

$$L(p,\xi) = (\Gamma_\xi^+ - \frac{1}{K(p)+\xi})^2 + \Gamma_{\xi\xi}^+ + \frac{1}{(K(p)+\xi)^2},$$
$$I(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{K(q)}{q}dq.$$

Here, $Y^+(p,\xi) = (-i)^\alpha e^{\Gamma^+(p,\xi)}$,

$$\Gamma^+(p,\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\ln^-(q-p)(\frac{K'(q)}{K(q)+\xi} - \frac{K_1'(q)}{K_1(q)+\xi})dq,$$

$K(p) = |p|^\alpha$, $K_1(p) = p^\alpha$ and $\ln^-(z) = \ln|z| + i\arg^-(z)$ for $-\frac{3}{2}\pi < \arg^-(z) \leq \frac{\pi}{2}$.

Theorem 1.1. *Let the initial data $u_0 \in \mathbb{Z}^\mu$ and the boundary value $h \in B^\alpha$ be such that the correspondent norms are sufficiently small. Also, suppose that $h(t) = bt^\beta + \mathcal{O}(t^{\beta-\delta})$, where $\beta < -\frac{1}{\alpha} - \frac{1}{2} - \gamma$ and $\delta, \gamma > 0$. Then, the initial-boundary value problem* (1.1) *has unique global solution $u \in X^\mu$ and the following asymptotic is valid*

$$u(x,t) = t^{-\frac{1}{\alpha}-\frac{1}{2}}(a\Lambda(xt^{-1/\alpha}) + b\Upsilon(xt^{-1/\alpha})) + \mathcal{O}(t^{-\frac{1}{\alpha}-\frac{1}{2}-\gamma}),$$

for $t \to \infty$ in $L^\infty(\mathbb{R}^+)$, where

$$a = \int_0^{+\infty} y^{\alpha/2}u_0(y)dy - \int_0^{+\infty} d\tau \int_0^{+\infty} y^{\alpha/2}\mathcal{N}(u(\tau))dy.$$

2. Preliminaries

Definition 2.1. A function $\phi(q)$ is said to satisfy the Hölder condition on the curve L, if there exist positive constants C and λ such that

$$|\phi(q_1) - \phi(q_2)| \le C|q_1 - q_2|^{\lambda}, \tag{2.1}$$

for any $q_1, q_2 \in L$.

The proof of the next theorem can be found in [8].

Theorem 2.2. *Let $\phi(q)$ be a complex function, which obeys the Hölder condition (2.1) for all finite q and tends to a finite limit $\phi(\infty)$ as $q \to \infty$, such that for large q the following inequality holds*

$$|\phi(q) - \phi(\infty)| \le C|q|^{\mu}, \quad \mu > 0.$$

Then, the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q - z} dq$$

defines an analytic function in the left and right semi-planes. Here and below these functions will be denoted by $F^+(z)$ and $F^-(z)$, respectively. These functions have the limiting values $F^+(p)$ and $F^-(p)$ at all points of the imaginary axis, $\mathrm{Re}(p) = 0$, *on approaching the contour from the left and from the right, respectively. These limiting values are expressed by the Sokhotski-Plemelj formulae,*

$$F^{\pm}(p) = \frac{1}{2\pi i} \fint_{-i\infty}^{i\infty} \frac{\phi(q)}{q - p} dq \pm \frac{1}{2}\phi(p). \tag{2.2}$$

Subtracting the formulae (2.2) we obtain the following equivalent formula

$$F^+(p) - F^-(p) = \phi(p), \tag{2.3}$$

which will be frequently employed hereafter. Now, we consider the following linear initial-boundary value problem on half-line

$$\begin{gathered} u_t + |\partial_x|^{\alpha} u = 0, \quad x, t > 0; \\ u(x, 0) = u_0(x), \quad x > 0, \\ u(0, t) = h(t), \quad t > 0. \end{gathered} \tag{2.4}$$

Setting $K(q) = |q|^{\alpha}$, $K_1(q) = q^{\alpha}$, we define

$$\mathcal{G}(t)\phi = \int_0^{+\infty} G(x, y, t)\phi(y) dy,$$

$$\mathcal{H}(x)h = \int_0^t H(x, t - \tau)h(\tau) d\tau, \quad H(x, t) = \partial_t \partial_y^{-1} G(x, y, t)|_{y=0},$$

where the function $G(x, y, t)$ is given by

$$G(x, y, t) = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} Z^-(p, \xi, y) dp, \tag{2.5}$$

where

$$Z^-(p, \xi, y) = \lim_{\substack{z \to p \\ \mathrm{Re} z > 0}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q - z} \frac{1}{Y^+(q, \xi)} e^{-qy} dq.$$

for $x > 0$, $y > 0$, $t > 0$. Here, $Y^+(p,\xi) = (-i)^\alpha e^{\Gamma^+(p,\xi)}$, where

$$\Gamma^+(p,\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln^-(q-p)\Big(\frac{K'(q)}{K(q)+\xi} - \frac{K_1'(q)}{K_1(q)+\xi}\Big)dq$$

and $\ln^-(z) = \ln|z| + i\arg^-(z)$, $-\frac{3}{2}\pi < \arg^-(z) \le \frac{\pi}{2}$, where the integrals are understood in the sense of the principal values.

Proposition 2.3. *If $u_0(x) \in \mathbb{Z}^\nu$, there exist a unique solution $u(x,t)$ for the initial-boundary value problem* (2.4)*, which has an integral representation*

$$u = \mathcal{G}(t)u_0 + \mathcal{H}(x)h, \quad x > 0,\ t > 0. \tag{2.6}$$

Proof. To obtain an integral representation for solutions of problem (2.4) we suppose that there exist a solution $u(x,t)$, which is continued by zero outside of $x > 0$:

$$u(x,t) = 0, \quad \text{for all } x < 0.$$

Let $\phi(q)$ be a function of the complex variable q, which obeys the Hölder condition (2.1) for all q, such that $\mathrm{Re}(q) = 0$. We define the operator $\mathbb{P}$ by

$$\mathbb{P}_{q\to z}\{\phi(q)\} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\phi(q)dq, \quad \mathrm{Re}(z) \neq 0.$$

Applying the Laplace transform with respect to x to $|\partial_x|^\alpha u$, for $\mathrm{Re}(q) > 0$ we obtain

$$\mathcal{L}_{x\to q}\{|\partial_x|^\alpha u(x,t)\} = \mathbb{P}_{p\to q}\{K(p)(\widehat{u}(p,t) - \frac{u(0,t)}{p})\}. \tag{2.7}$$

Since $\widehat{u}(q,t)$ is analytic, for all $\mathrm{Re}(q) > 0$, we have

$$\widehat{u}(q,t) = \mathbb{P}_{p\to q}\{\widehat{u}(p,t)\}. \tag{2.8}$$

Applying the Laplace transform with respect to x to problem (2.4) and using (2.7) and (2.8), for $\mathrm{Re}(p) = 0$, we obtain

$$\begin{gathered}\mathbb{P}_{p\to q}\big\{\widehat{u}_t(p,t) + K(p)\big(\widehat{u}(p,t) - \frac{h(t)}{p}\big)\big\} = 0,\\ \widehat{u}(p,0) = \widehat{u}_0(p),\end{gathered} \tag{2.9}$$

We rewrite (2.9) in the form

$$\begin{gathered}\widehat{u}_t(p,t) + K(p)\widehat{u}(p,t) - \frac{K(p)}{p}h(t) = \Phi(p,t);\\ \widehat{u}(p,0) = \widehat{u}_0(p),\end{gathered} \tag{2.10}$$

with some function $\Phi(p,t)$ such that for $\mathrm{Re}(p) = 0$

$$\mathbb{P}_{p\to q}\{\Phi(p,t)\} = 0 \quad \text{and} \quad |\Phi(p,t)| \le C|p|^{\alpha-1-\gamma}, \quad |p| > 1,\ \gamma > 0. \tag{2.11}$$

Applying the Laplace transform with respect to time variable to (2.10) we find

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\Big(\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{h}(\xi) + \widehat{\Phi}(p,\xi)\Big), \tag{2.12}$$

where $\mathrm{Re}(p) = 0$ and $\mathrm{Re}(\xi) > 0$. Here, the functions $\widehat{\widehat{u}}(p,\xi)$, $\widehat{h}(\xi)$, and $\widehat{\Phi}(p,\xi)$ are the Laplace transforms for $\widehat{u}(p,t)$, $h(t)$, and $\Phi(p,t)$ with respect to time, respectively. In order to obtain an integral formula for solutions to the problem (2.4) it is necessary to know the function $\Phi(p,t)$. We find the function $\widehat{\Phi}(p,\xi)$ using the analytic properties of the function $\widehat{\widehat{u}}$ in the right-half complex planes $\mathrm{Re}(p) > 0$ and

$\mathrm{Re}(\xi) > 0$. The equation (2.8) and the Sokhotski-Plemelj formulae (2.2) imply, for $\mathrm{Re}(p) = 0$,

$$\widehat{\widehat{u}}(p,\xi) = -\frac{1}{\pi i} \fint_{-i\infty}^{i\infty} \frac{1}{q-p} \widehat{\widehat{u}}(q,\xi) dq. \tag{2.13}$$

In view of Sokhotski-Plemelj formulae (2.2), via (2.12) the condition (2.13) can be written as

$$\Theta^+(p,\xi) = -\Lambda^+(p,\xi), \tag{2.14}$$

where the sectionally analytic functions $\Theta(p,\xi)$ and $\Lambda(p,\xi)$ are given by Cauchy type integrals

$$\Theta(z,\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{K(q)+\xi} \widehat{\Phi}(q,\xi) dq, \tag{2.15}$$

$$\Lambda(z,\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{K(q)+\xi} (\widehat{u}_0(q) + \frac{K(q)}{q} \widehat{h}(\xi)) dq. \tag{2.16}$$

To perform the condition (2.14) in the form of a nonhomogeneous Riemann-Hilbert problem we introduce the sectionally analytic function

$$\Omega(z,\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{K(q)}{K(q)+\xi} \widehat{\Phi}(q,\xi) dq. \tag{2.17}$$

Taking into account the assumed condition (2.11), we obtain

$$\Theta^-(p,\xi) = -\frac{1}{\xi} \Omega^-(p,\xi). \tag{2.18}$$

Also observe that from (2.15) and (2.17) by Sokhotski-Plemelj formulae,

$$K(p)(\Theta^+ - \Theta^-) = \Omega^+ - \Omega^- = \frac{K(p)}{K(p)+\xi} \widehat{\Phi}. \tag{2.19}$$

Substituting (2.14) and (2.18) into this equation we obtain for $\mathrm{Re}\ p = 0$

$$\Omega^+(p,\xi) = W(p,\xi)\Omega^-(p,\xi) + g(p,\xi), \tag{2.20}$$

where $W = \frac{K(p)+\xi}{\xi}$ and $g = -K(p)\Lambda^+$. Equation (2.20) is the boundary condition for a nonhomogeneous Riemann-Hilbert problem. It is required to find two functions for some fixed point ξ, $\mathrm{Re}\ \xi > 0$: $\Omega^+(z,\xi)$, analytic in the left-half complex plane $\mathrm{Re}\ z < 0$ and $\Omega^-(z,\xi)$, analytic in the right-half complex plane $\mathrm{Re}\ z > 0$, which satisfy on the contour $\mathrm{Re}\ p = 0$ the relation (2.20).

Note that bearing in mind formula (2.19) we can find the unknown function $\widehat{\Phi}(p,\xi)$, which involved in the formula (2.12), by the relation

$$\widehat{\Phi}(p,\xi) = \frac{K(p)+\xi}{K(p)} (\Omega^+(p,\xi) - \Omega^-(p,\xi)). \tag{2.21}$$

The method for solving the Riemann problem $F^+(p) = \phi(p)F^-(p) + \varphi(p)$ is based on the following results. The proofs may be found in [8].

Lemma 2.4. *An arbitrary function $\varphi(p)$ given on the contour $\mathrm{Re}\, p = 0$, satisfying the Hölder condition, can be uniquely represented in the form $\varphi(p) = U^+(p) - U^-(p)$, where $U^\pm(p)$ are the boundary values of the analytic functions $U^\pm(z)$ and the condition $U^\pm(\infty) = 0$ holds. These functions are determined by formula*

$$U(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \varphi(q) dq$$

Lemma 2.5. *An arbitrary function $\phi(p)$ given on the contour $\operatorname{Re} p = 0$, satisfying the Hölder condition, and having zero index,*

$$\operatorname{ind}\phi(p) := \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\ln\phi(p) =,$$

is uniquely representable as the ratio of the functions $X^+(p)$ and $X^-(p)$, constituting the boundary values of functions, $X^+(z)$ and $X^-(z)$, analytic in the left and right complex semi-plane and having in these domains no zero. These functions are determined to within an arbitrary constant factor and given by formula

$$X^\pm(z) = e^{\Gamma^\pm(z)}, \quad \Gamma(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\ln\phi(q)dq.$$

In the formulations of Lemmas 2.4 and 2.5 the coefficient $\phi(p)$ and the free term $\varphi(p)$ of the Riemann problem are required to satisfy the Hölder condition on the contour $\operatorname{Re} p = 0$. This restriction is essential. On the other hand, it is easy to observe that both functions $W(p,\xi)$ and $g(p,\xi)$ do not have limiting value as $p \to \pm i\infty$. So we can not find the solution using $\ln W(p,\xi)$. The principal task now is to get an expression equivalent to the boundary value problem (2.20), such that the conditions of Lemmas are satisfied. First, we introduce the function

$$\phi(p,\xi) = \Big(\frac{K(p)+\xi}{K_1(p)+\xi}\Big)\frac{w^-(p)}{w^+(p)}, \quad w^\pm(p) = \frac{1}{(p\mp z_0)^{\alpha/2}}, \tag{2.22}$$

where $K(p) = |p|^\alpha$, $K_1(p) = p^\alpha$ and $z_0 > 0$. We make a cut in the plane z: $(-\infty,-z_0]\cup[z_0,\infty)$. Owing to the manner of performing the cut the functions $w^-(z)$, $K_1(z)$ are analytic for $\operatorname{Re} z > 0$ and the function $w^+(z)$ is analytic for Re $z < 0$.

We observe that the function $\phi(p,\xi)$ given on the contour Re $p = 0$, satisfies the Hölder condition and $K_1(p)+\xi$ does not vanish for any Re $\xi > 0$. Also we have $\operatorname{ind}\phi(p,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\ln\phi(p,\xi) = 0$. Therefore in accordance with Lemma 2.5, the function $\phi(p,\xi)$ can be represented as

$$\phi(p,\xi) = \frac{X^+(p,\xi)}{X^-(p,\xi)}, \quad X^\pm(p,\xi) = \lim_{\substack{z\to p\\ \pm\operatorname{Re} z<0}} e^{\Gamma_0(z,\xi)}, \tag{2.23}$$

where $\Gamma_0(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\phi(q,\xi)dq$. From equations (2.22) and (2.23) we obtain

$$\frac{Y^+}{Y^-} = \frac{K(p)+\xi}{K_1(p)+\xi}, \tag{2.24}$$

where $Y^\pm = e^{\Gamma^\pm}w^\pm$. Now, we show that $Y^\pm$ do not depend of z_0. Integrating by parts we obtain for Γ_0:

$$\Gamma_0 = \lim_{R\to\infty}\frac{1}{2\pi i}\ln(q-z)\ln\phi(q,\xi)\Big|_{-iR}^{iR} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\ln(q-z)\partial_q\ln\phi(q,\xi)dq. \tag{2.25}$$

Using that $\ln\phi(\pm iR,\xi)\to 0$, as $R\to\infty$, we obtain

$$\Gamma_0(z,\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\ln(q-z)\partial_q\ln\phi(q,\xi)dq. \tag{2.26}$$

We define $\ln^{\pm}(z) = \ln|z| + i\arg^{\pm}(z)$, where $-\frac{3}{2}\pi < \arg^{-}(z) \le \frac{\pi}{2}$ and $-\frac{3}{2}\pi \le \arg^{+}(z) < \frac{\pi}{2}$. Then, for $\operatorname{Re} p = 0$,

$$\Gamma_0^{\pm}(p,\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln^{\mp}(q-p)\partial_q \ln\phi(q,\xi)dq. \tag{2.27}$$

Now, since

$$\partial_q \ln\phi(q,\xi) = F(q,\xi) + \frac{\alpha z_0}{(q-z_0)(q+z_0)},$$

where $F(q,\xi) = \frac{K'(q)}{K(q)+\xi} - \frac{K_1'(q)}{K_1(q)+\xi}$, and via Cauchy's Residue Theorem,

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln^{\pm}(q-p)\frac{\alpha z_0}{(q-z_0)(q+z_0)}dq = \pm i\pi\frac{\alpha}{2} + \ln^{\pm}(p\pm z_0)^{\alpha/2},$$

from (2.27) follows that

$$\Gamma^{\pm}(p,\xi) = \mp i\pi\frac{\alpha}{2} + \ln^{\mp}(p\mp z_0)^{\alpha/2} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln^{\mp}(q-p)F(q,\xi)dq.$$

Therefore,

$$Y^{\pm}(p,\xi) = (\mp i)^{\alpha} e^{\Gamma^{\pm}(p,\xi)}, \tag{2.28}$$

where

$$\Gamma^{\pm}(p,\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln^{\mp}(q-p)F(q,\xi)dq.$$

We note that equation (2.24) is equivalent to

$$\frac{K(p)+\xi}{\xi} = \frac{Y^+}{Y^-}(\frac{K_1(p)+\xi}{\xi}).$$

Now, we return to the nonhomogeneous Riemann-Hilbert problem defined by the boundary condition (2.20). We substitute the above equation in (2.20) and add $-\frac{\xi}{Y^+}\Lambda^+$ in both sides to get

$$\frac{\Omega^+ - \xi\Lambda^+}{Y^+} = \Big(\frac{K_1(p)+\xi}{\xi}\Big)\frac{\Omega^-}{Y^-} - \Big(\frac{K(p)+\xi}{Y^+}\Big)\Lambda^+. \tag{2.29}$$

On the other hand, by Sokhotski-Plemelj formulae and (2.16),

$$\Lambda^+ - \Lambda^- = \frac{1}{K(p)+\xi}\Big(\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{h}(\xi)\Big).$$

Now, we substitute Λ^+ from this equation in formula (2.29), then by (2.24) we arrive to

$$\frac{\Omega^+ - \xi\Lambda^+}{Y^+} = \frac{K_1(p)+\xi}{\xi}\Big(\frac{\Omega^- - \xi\Lambda^-}{Y^-}\Big) - \frac{1}{Y^+}\Big(\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{h}(\xi)\Big). \tag{2.30}$$

In subsequent consideration we shall have to use the following property of the limiting values of a Cauchy type integral, the statement of which we now quote. The proofs may be found in [8].

Lemma 2.6. *If L is a smooth closed contour and $\phi(q)$ a function that satisfies the Hölder condition on L, then the limiting values of the Cauchy type integral*

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{1}{q-z}\phi(q)dq$$

also satisfy this condition.

Since $\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{h}(\xi)$ satisfies on $\mathrm{Re}(p) = 0$ the Hölder condition, on basis of Lemma 2.6 the function $\frac{1}{Y^+}(\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{h}(\xi))$ also satisfies this condition. Therefore, in accordance with Lemma 2.4 it can be uniquely represented in the form of the difference of the functions $U^+(p,\xi)$ and $U^-(p,\xi)$, constituting the boundary values of the analytic function $U(z,\xi)$, given by formula

$$U(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\Big(\widehat{u}_0(q) + \frac{K(q)}{q}\widehat{h}(\xi)\Big)dq. \tag{2.31}$$

Therefore, the equation (2.30) takes the form

$$\frac{\Omega^+ - \xi\Lambda^+}{Y^+} + U^+ = \frac{K_1(p)+\xi}{\xi}\Big(\frac{\Omega^- - \xi\Lambda^-}{Y^-}\Big) + U^-.$$

The last relation indicates that the function $\frac{\Omega^+ - \xi\Lambda^+}{Y^+} + U^+$, analytic in $\mathrm{Re}(z) < 0$, and the function $\frac{K_1(p)+\xi}{\xi}(\frac{\Omega^- - \xi\Lambda^-}{Y^-}) + U^-$, analytic in $\mathrm{Re}(z) > 0$, constitute the analytic continuation of each other through the contour $\mathrm{Re}(z) = 0$. Consequently, they are branches of a unique analytic function in the entire plane. Moreover, this function has a zero in infinite. According to Liouville theorem this function is identically zero. Thus, we obtain the solution of the Riemann-Hilbert problem defined by the boundary condition (2.20),

$$\begin{gathered}\Omega^+(p,\xi) = -Y^+(p,\xi)U^+(p,\xi) + \xi\Lambda^+(p,\xi)\\ \Omega^-(p,\xi) = -\frac{\xi}{K_1(p)+\xi}Y^-(p,\xi)U^-(p,\xi) + \xi\Lambda^-(p,\xi).\end{gathered} \tag{2.32}$$

Now, we proceed to find the unknown function $\widehat{\Phi}$ of (2.12) for the solution $\widehat{\widehat{u}}$ of problem (2.4). First, we represent Ω^- as the limiting value of analytic functions on the left-hand side complex semi-plane. From equation (2.24) and Sokhotski-Plemelj formulae we obtain $\Omega^- = -\frac{\xi}{K(p)+\xi}Y^+U^+ + \xi\Lambda^+$. Now, making use of (2.32) and the last equation, we obtain

$$\Omega^+ - \Omega^- = -\frac{K(p)}{K(p)+\xi}Y^+U^+.$$

Thus, by formula (2.21), $\widehat{\Phi} = -Y^+U^+$. We observe that $\widehat{\Phi}$ is boundary value of a function analytic in the left-hand side complex semi-plane and therefore satisfies our basic assumption (2.11). Having determined the function $\widehat{\Phi}$ and bearing in mind formula (2.12) we determine the required function: $\widehat{\widehat{u}} = \frac{1}{K(p)+\xi}(\widehat{u}_0(p) - Y^+U^+)$. Now we prove that, in accordance with last relation, the function $\widehat{\widehat{u}}$ constitutes the limiting value of an analytic function in $\mathrm{Re}(z) > 0$. In fact, making use of Sokhotski-Plemelj formulae and using (2.24), we obtain $\widehat{\widehat{u}} = -\frac{1}{K_1(p)+\xi}Y^-U^-$. Thus, the function $\widehat{\widehat{u}}$ is the limiting value of an analytic function in $\mathrm{Re}(z) > 0$. We note the fundamental importance of the proven fact, the solution $\widehat{\widehat{u}}$ constitutes an analytic function in $\mathrm{Re}(z) > 0$ and, as a consequence, its inverse Laplace transform vanish for all $x < 0$. We now return to solution u of the problem (2.4). Taking inverse Laplace transform with respect to time and space variables, we obtain (2.6). Proposition 2.3 has been proved. □

In the following lemma we collect some preliminary estimates for the Green operator $\mathcal{G}(t)$

Lemma 2.7. *If $\Lambda(s)$ as defined in (1.3), then exists $C>0$ such that*

$$\|\mathcal{G}(t)\phi - \vartheta t^{-\frac{1}{\alpha}-\frac{1}{2}}\Lambda((\cdot)t^{-1/\alpha})\|_{L^\infty} \le Ct^{-\frac{1}{\alpha}-\frac{1}{2}-\gamma}\|\phi\|_{L^{1,\frac{\alpha}{2}}}, \quad \gamma>0, \tag{2.33}$$

$$\|\mathcal{G}(t)\phi\|_{L^\infty} \le C\{t\}^{-\gamma}\langle t\rangle^{-1/\alpha}(\|\phi\|_{L^1}+\|\phi\|_{L^\infty}), \quad 0<\gamma<1, \tag{2.34}$$

$$\|\mathcal{G}(t)\phi\|_{L^{s,\mu}} \le Ct^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{s}-\mu)}\|\phi\|_{L^r} + Ct^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{s})}\|\phi\|_{L^{r,\mu}}, \tag{2.35}$$

where $\vartheta = \int_0^{+\infty} y^{\alpha/2}\phi(y)dy$, $|\frac{1}{r}-\frac{1}{s}-\mu|<\alpha$ and $1\le r\le s\le\infty$.

Proof. First, we estimate the function $Y^+ = (-i)^\alpha e^{\Gamma^+(p,\xi)}$, where

$$\Gamma^+(p,\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln^-(q-p)F(q,\xi)dq$$

and $F(q,\xi) = \frac{K'(q)}{K(q)+\xi} - \frac{K_1'(q)}{K_1(q)+\xi}$. Using the relation

$$\ln^-(q-p) = \pi i + \ln^+(p) + \ln^+\big(1-\frac{q}{p}\big) + 2\pi i\theta(\operatorname{Im} p)\theta(\operatorname{Im} q - \operatorname{Im} p),$$

where θ is the Heaviside step function, we obtain

$$\Gamma^+(p,\xi) = \frac{\alpha}{2}(\pi i + \ln^+(p)) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\ln^+\big(1-\frac{q}{p}\big)F(q,\xi)dq + \mathcal{O}(p^{-\alpha\gamma}\xi^\gamma),$$

where $0<\gamma<1$. Then, separating the integral, $|q|<|p|$ and $|q|<|p|$, and using $\ln(1-z)=\mathcal{O}(z^\gamma)$, for $|z|<1$ and $0<\gamma<1$, and the equation

$$\ln^+\big(1-\frac{q}{p}\big) = -\pi i + \ln^+(q) - \ln^+(p) + \ln^+(1-\frac{p}{q}) + 2\pi i\varsigma(\operatorname{Im} p, \operatorname{Im} q),$$

where $\varsigma(x,y) = \theta(x-y)(\theta(x)\theta(y)-\theta(-x)\theta(-y)) + \theta(-x)$, we obtain

$$\Gamma^+(p,\xi) = \frac{\alpha}{2}(\pi i + \ln^+(p)) + \mathcal{O}\big(p^{\alpha(1-\gamma)}\xi^{\gamma-1}\big), \quad 0<\gamma<1. \tag{2.36}$$

Therefore, from (2.28) and (2.36) follows

$$Y^+(p,\xi) = p^{\alpha/2} + \mathcal{O}\big(p^{\alpha(\frac{3}{2}-\gamma)}\xi^{\gamma-1}\big), \quad 0<\gamma<1, \tag{2.37}$$

$$\frac{1}{Y^+(p,\xi)} = \mathcal{O}(p^{-\alpha/2}). \tag{2.38}$$

We note that in the above formulas $p^\beta = e^{\beta\ln^+(p)}$. Now, we show that

$$\partial_\xi^j\Gamma^+(p,\xi) = \mathcal{O}(p^{-j\alpha\gamma}\xi^{j(\gamma-1)}), \quad 0<\gamma<1,\ j=1,2. \tag{2.39}$$

In fact, from Cauchy's Theorem we obtain

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\ln^-(q-p)\frac{K_1'(q)}{(K_1(q)+\xi)^2}dq = 0,$$

then integrating by parts

$$\begin{aligned}\partial_\xi^j\Gamma^+(p,\xi) &= \frac{(-1)^{j+1}j}{2\pi i}\int_{-i\infty}^{i\infty}\ln^-(q-p)\frac{K'(q)}{(K(q)+\xi)^{j+1}}dq\\ &= \frac{(-1)^j}{2\pi i}\Big(\frac{\ln^-(q-p)}{(K(q)+\xi)^j}\Big|_{-i\infty}^{i\infty} - \int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{dq}{(K(q)+\xi)^j}\Big)\\ &= \mathcal{O}(p^{-j\alpha\gamma}\xi^{j(\gamma-1)}).\end{aligned}$$

As a direct consequence of (2.38) and (2.39) we obtain

$$\partial_\xi(\frac{1}{Y^+}) = -\frac{\Gamma_\xi^+}{Y^+} = \mathcal{O}(p^{-\alpha\gamma-\frac{\alpha}{2}}\xi^{\gamma-1}), \tag{2.40}$$

$$\partial_\xi^2(\frac{1}{Y^+}) = \frac{1}{Y^+}\Big((\Gamma_\xi^+)^2 - \Gamma_{\xi\xi}^+\Big) = \mathcal{O}\big(p^{-2\alpha\gamma-\frac{\alpha}{2}}\xi^{2(\gamma-1)}\big). \tag{2.41}$$

Then, using $e^{-qy} - 1 = \mathcal{O}(q^\mu y^\mu)$, $0 < \mu < 1$, (2.40) and (2.41), we obtain for $Z_0^+(p,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-qy}-1}{q-p}\frac{dq}{Y^+(q,\xi)}$,

$$\partial_\xi^j Z_0^+(p,\xi,y) = \mathcal{O}(y^\mu p^{\mu-\alpha(j\gamma+\frac{1}{2})}\xi^{j(\gamma-1)}), \quad 0 < \mu < 1, \tag{2.42}$$

where $j = 1, 2$. Moreover, using $\frac{1}{q-p} = \frac{1}{q} + \frac{p}{q(q-p)}$, we express Z_0^+ in the form

$$Z_0^+(p,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-qy}-1}{q}\frac{dq}{Y^+(q,\xi)} + \frac{p}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-qy}-1}{q(q-p)}\frac{dq}{Y^+(q,\xi)}.$$

Thus, from (2.38) we arrive to

$$Z_0^+(p,\xi,y) = C_0 y^{\alpha/2} + \mathcal{O}(y^\mu p^{\mu-\frac{\alpha}{2}}), \quad \mu > \frac{\alpha}{2}, \tag{2.43}$$

where $C_0 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-q}-1}{q^{1+\frac{\alpha}{2}}}dq$. Now, we consider the function $\frac{Y^+}{K(p)+\xi}Z_0^+$. From (2.37), (2.39) and (2.42), we obtain

$$\begin{aligned}
&\partial_\xi^2\Big(\frac{Y^+}{K(p)+\xi}Z_0^+\Big) \\
&= \frac{Y^+}{K(p)+\xi}\Big(\Big(\Gamma_\xi^+ - \frac{1}{K(p)+\xi}\Big)^2 Z_0^+ + 2\Big(\Gamma_\xi^+ - \frac{1}{K(p)+\xi}\Big)\partial_\xi Z_0^+ \\
&\quad + \Big(\Gamma_{\xi\xi}^+ + \frac{1}{(K(p)+\xi)^2}\Big)Z_0^+ + \partial_\xi^2 Z_0^+\Big) \\
&= \frac{1}{K(p)+\xi}\big(Y^+ Z_0^+ L + \mathcal{O}(y^\mu p^{\mu-2\alpha\gamma}\xi^{2(\gamma-1)})\big),
\end{aligned}$$

where

$$L(p,\xi) = \Big(\Gamma_\xi^+ - \frac{1}{K(p)+\xi}\Big)^2 + \Gamma_{\xi\xi}^+ + \frac{1}{(K(p)+\xi)^2} = \mathcal{O}\big(p^{-2\alpha\gamma}\xi^{2(\gamma-1)}\big).$$

Then, using (2.37) and (2.43) we obtain the estimate

$$\partial_\xi^2\Big(\frac{Y^+}{K(p)+\xi}Z_0^+\Big) = \frac{C_0 y^{\alpha/2}}{K(p)+\xi}\Big(p^{\alpha/2}L + \mathcal{O}(p^{\mu-2\alpha\gamma}\xi^{2(\gamma-1)})\Big), \quad \mu > \frac{\alpha}{2}. \tag{2.44}$$

On the other hand, by Sokhotski-Plemelj formulae and Cauchy's Residue Theorem, we represent the Green function G, (2.5), as

$$\begin{aligned}
G(x,y,t) = &-\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}Z^+(p,\xi,y)dp \\
&+ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-K(p)t}dp.
\end{aligned}$$

Now, by Cauchy's Theorem we obtain

$$Z = \frac{1}{Y^+} + Z_0, \quad \text{where} \quad Z_0(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-qy}-1}{q-z}\frac{dq}{Y^+(q,\xi)}.$$

Therefore, we can represent G in the form

$$\begin{aligned} G = &-\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}Z_0^+(p,\xi,y)dp \\ &+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}(e^{-py}-1)dp = G_1+G_2. \end{aligned} \tag{2.45}$$

Using $(e^{-py}-1)=\mathcal{O}(p^\mu y^\mu)$, $0<\mu<1$, we obtain for the second integral in the above equation

$$G_2(x,y,t)=\mathcal{O}(t^{-\frac{1}{\alpha}(1+\mu)}y^\mu). \tag{2.46}$$

To estimate G_1, we use the analyticity of the integrand in the left-half semiplane and integrate by parts two times with respect to ξ to obtain

$$G_1(x,y,t)=Ct^{-2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\partial_\xi^2\Big(\frac{Y^+(p,\xi)}{K(p)+\xi}Z_0^+(p,\xi,y)\Big)dp.$$

Then, by (2.44) and (2.46) follows

$$G(x,y,t)=y^{\alpha/2}t^{-\frac{1}{\alpha}(1+\frac{\alpha}{2})}\Lambda(xt^{-1/\alpha})+\mathcal{O}(y^{\alpha/2}t^{-\frac{1}{\alpha}(1+\mu)}),\quad \mu>\frac{\alpha}{2}, \tag{2.47}$$

where $2/5<\alpha<1$ and

$$\Lambda(s)=C\int_{-i\infty}^{i\infty} d\xi e^{\xi}\int_{-i\infty}^{i\infty} e^{ps}\frac{p^{\alpha/2}}{K(p)+\xi}L(p,\xi)dp.$$

Multiplying by a function ϕ and integrating with respect to y in (2.47) we obtain

$$\Big|\int_0^{+\infty}(G(x,y,t)-y^{\alpha/2}t^{-\frac{1}{\alpha}(1+\frac{\alpha}{2})}\Lambda(xt^{-1/\alpha}))\phi(y)dy\Big|\le Ct^{-\frac{1}{\alpha}(1+\mu)}\|\phi\|_{L^{1,\frac{\alpha}{2}}}.$$

Therefore,

$$\Big\|\mathcal{G}(t)\phi-t^{-\frac{1}{\alpha}(1+\frac{\alpha}{2})}\Lambda((\cdot)t^{-1/\alpha})\int_0^{+\infty}y^{\alpha/2}\phi(y)dy\Big\|_{L^\infty}\le Ct^{-\frac{1}{\alpha}(1+\mu)}\|\phi\|_{L^{1,\frac{\alpha}{2}}},$$

where $\mu>\frac{\alpha}{2}$. Thus, the first estimate in Lemma 2.7 has been proved. Now, we are going to prove the second estimate in Lemma 2.7. First, for large t. From (2.45), integrating by parts the first summand on the right hand side two times with respect to ξ we obtain

$$\begin{aligned} G(x,y,t) = &Ct^{-2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\partial_\xi^2\Big(\frac{Y^+(p,\xi)}{K(p)+\xi}Z^+(p,\xi,y)\Big)dp \\ &+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-K(p)t}dp = J_1+J_2. \end{aligned} \tag{2.48}$$

We have

$$\partial_\xi^2(\frac{Y^+}{K(p)+\xi}Z^+)=\frac{1}{K(p)+\xi}\mathcal{O}(p^{-2\alpha\gamma}\xi^{2(\gamma-1)}),$$

$0<\gamma<1$, then $J_1=\mathcal{O}(t^{-1/\alpha})$, provided $\frac{1}{3}<\alpha<1$, also $J_2=\mathcal{O}(t^{-1/\alpha})$. Therefore, $G=\mathcal{O}(t^{-1/\alpha})$ and

$$\|\mathcal{G}(t)\phi\|_{L^\infty}=\sup_{x\in\mathbb{R}^+}\Big|\int_0^{+\infty}G(x,y,t)\phi(y)dy\Big|\le t^{-1/\alpha}\|\phi\|_{L^1}. \tag{2.49}$$

Now, for small t, using the inequality $|e^z| \le |z|^{-\gamma}$, for $\mathrm{Re}(z) < 0$ and $\gamma > 0$, we obtain

$$\partial_\xi^2\Big(\frac{Y^+}{K(p)+\xi}Z^+\Big) = \frac{1}{K(p)+\xi}\mathcal{O}(y^{-\gamma}p^{-\gamma-2\alpha\gamma_1}\xi^{2(\gamma_1-1)}),$$

$\gamma > 0$, $0 < \gamma_1 < 1$. Then, from (2.48) we obtain $J_1 = \mathcal{O}(y^{-\gamma}t^{-\frac{1}{\alpha}(1-\gamma)})$, $\gamma > 0$. On the other hand, we write the second summand as

$$J_2(r,t) = \frac{1}{2\pi i}\int_{\mathcal{C}_\pm} e^{pr-K(p)t}dp, \quad \pm r > 0,$$

where

$$\mathcal{C}_\pm = \{p \in (\infty e^{-i(\frac{\pi}{2}\pm\varepsilon)}, 0) \cup (0, \infty e^{i(\frac{\pi}{2}\pm\varepsilon)})\}, \quad \varepsilon > 0.$$

Making the change of variable $p = zt^{-1/\alpha}$ and using the inequality $|e^z| \le |z|^{-\gamma}$, for $\mathrm{Re}(z) < 0$ and $\gamma > 0$, we obtain $J_2 = \mathcal{O}(t^{-\frac{1}{\alpha}(1-\gamma)}|x-y|^{-\gamma})$, where $x, y > 0$. Therefore, $G = \mathcal{O}\big(t^{-\frac{1}{\alpha}(1-\gamma)}(|x-y|^{-\gamma} + y^{-\gamma})\big)$. Thus,

$$\begin{aligned}
\|\mathcal{G}(t)\phi\|_{L^\infty} &\le \sup_{x\in\mathbb{R}^+}\int_0^{+\infty}|G(x,y,t)||\phi(y)|dy \\
&\le Ct^{-\frac{1}{\alpha}(1-\gamma)}\int_0^{+\infty}(|x-y|^{-\gamma} + y^{-\gamma})|\phi(y)|dy \\
&\le Ct^{-\frac{1}{\alpha}(1-\gamma)}(\|\phi\|_{L^1} + \|\phi\|_{L^\infty}).
\end{aligned}$$

The second estimate in Lemma 2.7 has been proved. Let us introduce the operators

$$\mathcal{J}_1(t)\phi = \theta(x)\int_0^{+\infty} J_1(x,y,t)\phi(y)dy, \tag{2.50}$$

$$\mathcal{J}_2(t)\phi = \theta(x)\int_0^{+\infty} J_2(x-y,t)\phi(y)dy, \tag{2.51}$$

Then, the operator $\mathcal{G}(t)$ can be written in the form

$$\mathcal{G}(t) = \mathcal{J}_1(t) + \mathcal{J}_2(t). \tag{2.52}$$

Now, we are going to prove the third estimate in Lemma 2.7. First, we estimate the operator $\mathcal{J}_2$. Making the change of variable $p = qt^{-1/\alpha}$, we obtain

$$|J_2(r,t)| \le Ct^{-1/\alpha}. \tag{2.53}$$

On the other hand, making $z = t^{-1/\alpha}r$, we obtain

$$J_2 = \frac{t^{-1/\alpha}}{2\pi i}\int_{-i\infty}^{i\infty} e^{qz-K(q)}dq.$$

Then, integrating by parts the last equation we obtain

$$J_2(r,t) = \frac{t^{-1/\alpha}}{2\pi i}\Big(\frac{1}{z}\Big)\int_{-i\infty}^{i\infty} e^{-K(q)}de^{qz} = \frac{t^{-1/\alpha}}{2\pi i}\Big(\frac{\alpha}{z}\Big)\int_{-i\infty}^{i\infty}\frac{K(q)}{q}e^{qz-K(q)}dq$$

Thus,

$$|J_2(r,t)| \le Ct^{-1/\alpha}\frac{1}{|z|^{1+\gamma}}\int_{\mathcal{C}_\pm}|q|^{\alpha-1-\gamma}e^{-C|q|^\alpha}|dq|,$$

for $\pm r > 0$, where $\mathcal{C}_\pm$ are defined as above. Therefore,

$$|J_2(r,t)| \le Ct^{-1/\alpha}\frac{1}{|z|^{1+\gamma}}, \quad \gamma < \alpha. \tag{2.54}$$

Finally, from the inequalities (2.53) and (2.54) we have

$$|J_2(r,t)| \le C \frac{t^{-1/\alpha}}{1+(t^{-1/\alpha}|r|)^{1+\gamma}}, \quad \gamma < \alpha. \tag{2.55}$$

Lets write some well known inequalities:

- Young's inequality. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} \ge 1$. Then, the convolution $h(x) \equiv \int_{\mathbb{R}} f(x-y)g(y)dy$ belongs to $L^r(\mathbb{R})$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and the Young's inequality

$$\|h\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \tag{2.56}$$

holds.

- Minkowski's Inequality. Let $f, g \in L^p$ and $1 \le p \le \infty$, then

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}. \tag{2.57}$$

Now, we estimate the operator $\mathcal{J}_2$, defined in (2.50). Using the inequality $x^\mu \le |x-y|^\mu + y^\mu$, where $0 \le \mu \le 1$, and Minkowski's inequality (2.57), we obtain

$$\begin{aligned} \|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} &\le \Big(\int_0^{+\infty} \Big(\int_0^{+\infty} |x-y|^\mu |J_2(x-y,t)||\phi(y)|dy \Big)^s dx \Big)^{1/s} \\ &\quad + \Big(\int_0^{+\infty} \Big(\int_0^{+\infty} y^\mu |J_2(x-y,t)||\phi(y)|dy \Big)^s dx \Big)^{1/s}. \end{aligned}$$

Then, Young's inequality (2.56) implies

$$\|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} \le \|J_2(\cdot,t)\|_{L^{p,\mu}} \|\phi\|_{L^r} + \|J_2(\cdot,t)\|_{L^p} \|\phi\|_{L^{r,\mu}}, \tag{2.58}$$

where $\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - 1$, $1 \le p, r \le \infty$, $1 \le \frac{1}{p} + \frac{1}{r} \le 2$ and $0 \le \mu \le 1$. Then, by the inequality (2.55) and the change of variables $x = t^{-1/\alpha}|r|$, we obtain

$$\|J_2(\cdot,t)\|_{L^{p,\mu}} \le Ct^{-\frac{1}{\alpha}(1-\frac{1}{p}-\mu)} \Big(\int_{-\infty}^{+\infty} (\frac{|x|^\mu}{(1+|x|)^{1+\gamma}})^p dx \Big)^{1/p}.$$

Thus, $\|J_2(\cdot,t)\|_{L^{p,\mu}} \le Ct^{-\frac{1}{\alpha}(1-\frac{1}{p}-\mu)}$, provided $1+\gamma-\mu > \frac{1}{p}$. Using $\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - 1$, it follows

$$\|J_2(\cdot,t)\|_{L^{p,\mu}} \le Ct^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{s}-\mu)}, \tag{2.59}$$

where $\frac{1}{r} - \frac{1}{s} - \mu + \gamma > 0$. We note that $-\frac{1}{\alpha}(\frac{1}{r} - \frac{1}{s} - \mu) < 1$, since $\gamma < \alpha$. Substituting (2.59) in (2.58), we obtain

$$\|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} \le Ct^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{s}-\mu)} \|\phi\|_{L^r} + Ct^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{s})} \|\phi\|_{L^{r,\mu}}, \tag{2.60}$$

where $-\frac{1}{\alpha}(\frac{1}{r} - \frac{1}{s} - \mu) < 1$, $1 \le s, r \le \infty$ and $0 \le \mu \le 1$. Now, we estimate the operator $\mathcal{J}_1$, defined in (2.51). First, we note that using the Cauchy theorem we obtain

$$\begin{aligned} &\int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p,\xi)}{K(p)+\xi} Z^+(p,\xi,y) dp \\ &= \int_0^{+\infty} e^{-px} \Big(\frac{1}{(-ip)^\alpha + \xi} - \frac{1}{(ip)^\alpha + \xi} \Big) Y^+(-p,\xi) Z^+(-p,\xi,y) dp. \end{aligned}$$

Then, by inequality

$$\Big| \frac{1}{(-ip)^\alpha + \xi} - \frac{1}{(ip)^\alpha + \xi} \Big| \le C \frac{|p|^{\alpha(1-\gamma)}}{||p|^{2\alpha} + \xi^2|^{1-\gamma} |\xi|^\gamma}, \tag{2.61}$$

$0<\gamma<1$, we obtain for J_1

$$|J_1| \le C\int_{-i\infty}^{i\infty}|d\xi||e^{i\xi t}|\int_0^{+\infty}dp\frac{e^{-px}p^{\alpha(1-\gamma)+\frac{\alpha}{2}}}{|p^{2\alpha}+\xi^2|^{1-\gamma}|\xi|^{\gamma}}\int_{\mathcal{C}_3}\frac{e^{-C|q|y}}{|q-p|}\frac{|dq|}{|q|^{\alpha/2}}.$$

Thus, by $|\mathcal{J}_2(t)\phi| \le \int_0^{+\infty}|J_2(x,y,t)||\phi(y)|dy$ and

$$\int_0^{+\infty}e^{-C|q|y}|\phi(y)|dy \le \|e^{-C|q|(\cdot)}\|_{L^l}\|\phi\|_{L^r} = C|q|^{-\frac{1}{l}}\|\phi\|_{L^r},$$

where $\frac{1}{l}+\frac{1}{r}=1$, we obtain

$$|\mathcal{J}_2(t)\phi| \le C\|\phi\|_{L^r}\int_{-i\infty}^{i\infty}|d\xi||e^{i\xi t}|\int_0^{+\infty}\frac{e^{-px}p^{\alpha(1-\gamma)}}{|p^{2\alpha}+\xi^2|^{1-\gamma}|\xi|^{\gamma}p^{\frac{1}{l}}}dp.$$

Then, using $\|e^{-p(\cdot)}\|_{L^{s,\mu}} = Cp^{-\frac{1}{s}-\mu}$, $p>0$, and $\frac{1}{l}+\frac{1}{r}=1$, we obtain

$$\|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} \le C\|\phi\|_{L^r}\int_{-i\infty}^{i\infty}|d\xi||e^{i\xi t}|\int_0^{+\infty}\frac{p^{\alpha(1-\gamma)}}{|p^{2\alpha}+\xi^2|^{1-\gamma}|\xi|^{\gamma}p^{1-\frac{1}{r}+\frac{1}{s}+\mu}}dp.$$

Therefore,

$$\|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} \le Ct^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{s}-\mu)}\|\phi\|_{L^r}, \tag{2.62}$$

where $|\frac{1}{r}-\frac{1}{s}-\mu|<\alpha$, $1\le r\le s\le\infty$ and $0\le\mu<1$. Finally, from estimates (2.60) and (2.62) we obtain the third estimate in Lemma 2.7. Then, we have proved Lemma 2.7. □

As a direct consequence of the above lemma, taking $r=1$, we obtain the estimate

$$\|\mathcal{G}(t)\phi\|_{X^{\mu}} \le C\|\phi\|_{\mathbb{Z}^{\mu}}.$$

Now, we collect some estimates for operator $\mathcal{H}$ defined by

$$\mathcal{H}(x)h = \int_0^t H(x,t-\tau)h(\tau)d\tau, \quad H(x,t) = \partial_t\partial_y^{-1}G(x,y,t)|_{y=0}. \tag{2.63}$$

Lemma 2.8. *The following estimates are valid*

$$\|\mathcal{H}(\cdot)h\|_{X^{\mu}} \le C\|h\|_{B^{\alpha}}, \quad \mu+\frac{1}{s}<\alpha,$$
$$\mathcal{H}(x)h = bt^{\beta+\gamma}\Lambda_1(xt^{-1/\alpha}) + \mathcal{O}(t^{\beta}),$$

for $h(t)=bt^{\beta}+\mathcal{O}(t^{\beta-\gamma})$, where $\beta<0$, $\gamma>0$ and

$$\Upsilon(r) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi}\int_{-i\infty}^{i\infty}e^{pr}\frac{Y^{+}(p,\xi)}{K(p)+\xi}I^{+}(p,\xi)dp + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{pr-K(p)}\frac{K(p)}{p}dp.$$

Proof. First, we note that

$$H(x,t) = H_1(x,t) + H_2(x,t), \tag{2.64}$$

where

$$H_1(x,t) = -\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}e^{px}\frac{Y^{+}(p,\xi)}{K(p)+\xi}I^{+}(p,\xi)dp,$$
$$H_2(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{px-K(p)t}\frac{K(p)}{p}dp.$$

Here,

$$I(z,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{K(q)}{q}dq.$$

For H_2, making the change of variable $q=pt^{-1/\alpha}$, we obtain $|H_2|\le Ct^{-1}$. On the other hand, for H_1, we consider two domains: $|p|<1$ and $|p|>1$. For $|p|<1$, we use $Y^+I^+=\mathcal{O}(p^{-1+\alpha})$; and for $|p|>1$, we integrate by parts two times with respect to ξ and use

$$\partial_\xi^2(\frac{Y^+}{K(p)+\xi}I^+)=\frac{p^{-1+\alpha}}{K(p)+\xi}\mathcal{O}(p^{-2\alpha\gamma}\xi^{2(\gamma-1)})$$

to obtain $|H_1|\le Ct^{-1}$. Therefore,

$$|H(x,t)|\le Ct^{-1}. \tag{2.65}$$

By the Cauchy's Residue Theorem, for $\operatorname{Re}(z)<0$, we have

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K(q)+\xi}{Y^+(q,\xi)}\frac{dq}{q}-\xi\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{dq}{q}\\&=-\frac{1}{2}\frac{1}{(-z)}\frac{\xi}{Y^+(0,\xi)}-\xi\Big(\frac{1}{z}\frac{1}{Y^+(z,\xi)}+\frac{1}{2}\frac{1}{(-z)}\frac{1}{Y^+(0,\xi)}\Big)\\&=\frac{\xi}{zY^+(0,\xi)}-\frac{\xi}{zY^+(z,\xi)}.\end{aligned}\tag{2.66}$$

Also, integrating H with respect to t and substituting (2.66) we obtain

$$\partial_t^{-1}H(x,t)=-\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}d\xi\frac{e^{\xi t}}{Y^+(0,\xi)}\int_{-i\infty}^{i\infty}e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}\frac{dp}{p}. \tag{2.67}$$

Then,

$$|\partial_t^{-1}H(x,t)|\le C \tag{2.68}$$

Now, we divide the integration domain and integrate by parts (2.63) to obtain

$$\begin{aligned}\mathcal{H}(x)h&=\int_0^{t/2}H(x,t-\tau)h(\tau)d\tau-h(\tau)\partial_\tau^{-1}H(x,t-\tau)\big|_{t/2}^{t}\\&\quad+\int_{t/2}^{t}h'(\tau)\partial_\tau^{-1}H(x,t-\tau)d\tau.\end{aligned}$$

Therefore, from (2.65), (2.68) and last equation we have

$$\|\mathcal{H}(\cdot)h\|_{L^\infty}\le C\langle t\rangle^{-1/\alpha}\|h\|_{B^\alpha}. \tag{2.69}$$

We estimate the operator $\mathcal{H}$ in norm $L^{s,\mu}$. First, we estimate H_1 from (2.64),

$$\|H_1(\cdot,t)\|_{L^{s,\mu}}\le C\int_{\mathcal{C}_1}|d\xi|e^{-\lambda|\xi|t}\int_{\mathcal{C}_2}\frac{1}{|K(p)+\xi|}\frac{|dp|}{|p|^{1-\alpha+\frac{1}{s}+\mu}}.$$

Making the change of variables $p=p_1t^{-1/\alpha}$, $\xi=\xi_1t^{-1}$, we obtain

$$\|H_1(\cdot,t)\|_{L^{s,\mu}}\le Ct^{\frac{1}{\alpha}(\frac{1}{s}+\mu)-1},\quad \frac{1}{s}+\mu<\alpha. \tag{2.70}$$

On the other hand for H_2 we have

$$\|H_2(\cdot,t)\|_{L^{s,\mu}}\le C\int_{-i\infty}^{i\infty}e^{-K(p)t}\frac{|dp|}{|p|^{1-\alpha+\frac{1}{s}+\mu}}. \tag{2.71}$$

Thus,

$$\|H_2(\cdot,t)\|_{L^{s,\mu}} \le Ct^{\frac{1}{\alpha}(\frac{1}{s}+\mu)-1}, \quad \frac{1}{s}+\mu<\alpha. \tag{2.72}$$

Therefore, from (2.70) and (2.72) we conclude

$$\|\mathcal{H}(\cdot)h\|_{L^{s,\mu}} \le C\|h\|_{B^\alpha}\int_0^t (t-\tau)^{\frac{1}{\alpha}(\frac{1}{s}+\mu)-1}\langle t\rangle^{-1/\alpha}d\tau \le C\langle t\rangle^{-\frac{1}{\alpha}(1-\frac{1}{s}-\mu)}\|h\|_{B^\alpha},$$

where $\frac{1}{s}+\mu<\alpha$. Finally, we estimate the operator $\mathcal{H}$ in norm L^1. By Cauchy's Residue Theorem we have for $\mathrm{Re}(z)>0$,

$$\begin{aligned} I(z,\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K(q)+\xi}{Y^+(q,\xi)}\frac{dq}{q} - \xi\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{dq}{q} \\ &= -\frac{K(z)+\xi}{zY^+(z,\xi)} + \frac{\xi}{zY^+(0,\xi)}. \end{aligned}$$

Then, we obtain the formula

$$H(x,t) = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\Big(\frac{Y^+(p,\xi)\xi}{(K(p)+\xi)Y^+(0,\xi)}-1\Big)\frac{dp}{p}. \tag{2.73}$$

Now, we substitute the equation

$$\frac{Y^+(p,\xi)\xi}{(K(p)+\xi)Y^+(0,\xi)}-1 = \Big(\frac{Y^+(p,\xi)-Y^+(0,\xi)}{Y^+(0,\xi)(K(p)+\xi)}\Big)\xi - \frac{K(p)}{K(p)+\xi}$$

in (2.73), and change ξ and p by ξt^{-1} and $pt^{-1/\alpha}$, respectively, to obtain

$$H(x,t) = t^{-1}J_1(s) + t^{-1}J_2(s), \quad s = xt^{-1/\alpha}, \tag{2.74}$$

where

$$J_1(s) = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\frac{d\xi}{Y^+(0,\xi)}e^{\xi}\int_{-i\infty}^{i\infty} e^{ps}\Big(\frac{Y^+(p,\xi)-Y^+(0,\xi)}{K(p)+\xi}\Big)\xi\frac{dp}{p} \tag{2.75}$$

and

$$J_2(s) = -\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi}\int_{-i\infty}^{i\infty} e^{ps}\frac{K(p)}{K(p)+\xi}\frac{dp}{p}. \tag{2.76}$$

We estimate J_2. First, for $x<1$, we obtain

$$\int_0^1 |J_2(xt^{-1/\alpha})|dx \le Ct^{\frac{1}{\alpha}}. \tag{2.77}$$

On the other hand, for $x>1$, applying Cauchy's Theorem, we write

$$J_2(s) = -\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1} d\xi e^{\xi}\int_{-i\infty}^{i\infty} e^{ps}\frac{K(p)}{K(p)+\xi}\frac{dp}{p},$$

where

$$\mathcal{C}_1 = \{\xi\in(-i\infty,-i)\cup\mathcal{S}\cup(i,i\infty)\},$$

$\mathcal{S}=\{\xi=e^{i\theta}:\theta\in[-\frac{\pi}{2},-\frac{3\pi}{2}]\}$. Substituting $1=\frac{K(p)+\xi}{\xi}-\frac{K(p)}{\xi}$ in last equation, we obtain

$$J_2 = -\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{d\xi}{\xi}e^{\xi}\int_{-i\infty}^{i\infty} e^{ps}K(p)\frac{dp}{p} + \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{d\xi}{\xi}e^{\xi}\int_{-i\infty}^{i\infty} e^{ps}\frac{K(p)^2}{K(p)+\xi}\frac{dp}{p}.$$

By Cauchy's Theorem, using the analyticity with respect to ξ, the first summand in last equation is zero. Thus,

$$J_2(s) = J_{21}(s) + J_{22}(s), \tag{2.78}$$

where

$$J_{21}(s) = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \frac{d\xi}{\xi} e^{\xi} \int_{\mathcal{C}_2} e^{ps} \frac{K(p)^2}{K(p)+\xi} \frac{dp}{p}, \tag{2.79}$$

$$J_{22}(s) = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \frac{d\xi}{\xi} e^{\xi} \int_{-i/s^2}^{i/s^2} e^{ps} \frac{K(p)^2}{K(p)+\xi} \frac{dp}{p}. \tag{2.80}$$

For $\epsilon > 0$ small enough,

$$\mathcal{C}_2 = \big\{p \in \frac{1}{s^2}\big(-i + \infty e^{i(\frac{\pi}{2}+\epsilon)}, -i\big) \cup \frac{1}{s^2}\big(i, i + \infty e^{i(\frac{\pi}{2}+\epsilon)}\big)\big\}.$$

Taking the L^1 norm with respect to x, we obtain

$$\|J_{21}(\cdot)\|_{L^1} \le Ct^{-1/\alpha} \int_{\mathcal{C}_1} \frac{|d\xi|}{|\xi|} \int_{\mathcal{C}_2} \frac{K(p)^2}{|K(p)+\xi|} \frac{|dp|}{|p|^2} \le Ct^{-1/\alpha}. \tag{2.81}$$

Now, for J_{22} we have

$$|J_{22}(s)| \le C \int_{\mathcal{C}_1} \frac{|d\xi|}{|\xi|} \int_{-i/s^2}^{i/s^2} \frac{K(p)^2}{|K(p)+\xi|} \frac{|dp|}{|p|} \le C\frac{1}{s^2}. \tag{2.82}$$

From (2.78), (2.81) and (2.82), we obtain for $t > 0$ and $x > 1$,

$$\int_1^{\infty} |J_2(xt^{-1/\alpha})| dx \le Ct^{\frac{1}{\alpha}}. \tag{2.83}$$

Thus, from (2.77) and (2.83) we obtain

$$\|J_2(\cdot)\|_{L^1} \le Ct^{\frac{1}{\alpha}}. \tag{2.84}$$

We estimate J_1. Substituting $\frac{\xi}{p} = \frac{K(p)+\xi}{p} - \frac{K(p)}{p}$ in (2.75), we obtain

$$J_1(s) = J_{11}(s) + J_{12}(s), \tag{2.85}$$

where

$$J_{11}(s) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \frac{d\xi}{Y^+(0,\xi)} e^{\xi} \int_{-i\infty}^{i\infty} e^{ps} (Y^+(p,\xi) - Y^+(0,\xi)) \frac{dp}{p},$$

$$J_{12}(s) = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \frac{d\xi}{Y^+(0,\xi)} e^{\xi} \int_{-i\infty}^{i\infty} e^{ps} \Big(\frac{Y^+(p,\xi) - Y^+(0,\xi)}{K(p)+\xi}\Big) K(p) \frac{dp}{p}.$$

The integrand in J_{11} is an analytic function in the left half-plane, with respect to p, and the residue at $p = 0$ is zero, then by Cauchy's Residue Theorem we conclude that $J_{11}(s) = 0$. Now, we estimate J_{12}. Using

$$Y^+(p,\xi) - Y^+(0,\xi) = \int_0^p \Gamma_q^+(q,\xi) Y^+(q,\xi) dq = \mathcal{O}(p^{\alpha(\frac{3}{2}-\gamma)} \xi^{\gamma-1}),$$

we obtain $\|J_{12}(\cdot)\|_{L^1} \le Ct^{\frac{1}{\alpha}}$. Therefore,

$$\|J_1(\cdot)\|_{L^1} \le Ct^{1/\alpha}. \tag{2.86}$$

Then, from (2.84) and (2.86) we obtain

$$\|H(\cdot,t)\|_{L^1} \le Ct^{-1+\frac{1}{\alpha}}. \tag{2.87}$$

Finally, integrating with respect to time we obtain

$$\|\mathcal{H}(\cdot)h\|_{L^1} \le C\|h\|_{B^\alpha} \int_0^t (t-\tau)^{\frac{1}{\alpha}-1} \langle t\rangle^{-1/\alpha} d\tau \le C\|h\|_{B^\alpha}.$$

Then, first estimate of Lemma 2.8 has been proved. Now we are going to obtain an asymptotic representation for $\mathcal{H}$. First, we split the integral,

$$\mathcal{H}(x)h = \int_0^{t-1} H(x,t-\tau)h(\tau)d\tau + \int_{t-1}^{t} H(x,t-\tau)h(\tau)d\tau \tag{2.88}$$

For the first integral in (2.88) we have

$$\begin{aligned}&\int_0^{t-1} H(x,t-\tau)h(\tau)d\tau \\ &= -\int_0^{t-1} H(x,t-\tau)(h(t)-h(\tau))d\tau + h(t)\int_0^{t-1} H(x,t-\tau)d\tau.\end{aligned} \tag{2.89}$$

Making the change $t-\tau = s$, we obtain

$$\begin{aligned}\int_0^{t-1} H(x,t-\tau)d\tau &= \int_1^t H(x,s)ds = \int_1^t \Upsilon(xs^{-1/\alpha})\frac{ds}{s} \\ &= \int_1^t (\Upsilon(xs^{-1/\alpha}) - \Upsilon(xt^{-1/\alpha}))\frac{ds}{s} + \ln(t)\Upsilon(xt^{-1/\alpha}) \\ &= \int_1^t (H(x,s) - \frac{t}{s}H(x,t))ds + \ln(t)\Upsilon(xt^{-1/\alpha}),\end{aligned} \tag{2.90}$$

where Υ is given by (1.4). From (2.89) and (2.90) we obtain

$$\begin{aligned}&\int_0^{t-1} H(x,t-\tau)h(\tau)d\tau \\ &= h(t)\ln(t)\Upsilon(xt^{-1/\alpha}) + h(t)\int_1^t \Big(H(x,s) - \frac{t}{s}H(x,t)\Big)ds \\ &\quad - \int_0^{t-1} H(x,t-\tau)(h(t)-h(\tau))d\tau.\end{aligned} \tag{2.91}$$

Now, since $|H(x,t)| \le Ct^{-1}$ and $h(t) = bt^{\beta} + \mathcal{O}(t^{\beta-\gamma})$, we obtain

$$\Big|\int_1^t (H(x,s) - \frac{t}{s}H(x,t))ds\Big| \le \ln(t),$$

$$\int_0^{t-1} H(x,t-\tau)(h(t)-h(\tau))d\tau = bt^{\beta} + \mathcal{O}(t^{\beta-\delta}).$$

Then, from (2.91) we obtain

$$\int_0^{t-1} H(x,t-\tau)h(\tau)d\tau = bt^{\beta+\gamma}\Upsilon(xt^{-1/\alpha}) + \mathcal{O}(t^{\beta}). \tag{2.92}$$

On the other hand, for the second integral in (2.88), integrating by parts and using $|\partial_t^{-1}H(x,t)| \le C$, we have

$$\int_{t-1}^{t} H(x,t-\tau)h(\tau)d\tau = \mathcal{O}(t^{\beta}). \tag{2.93}$$

Thus, by (2.88), (2.92) and (2.93) we obtain

$$\mathcal{H}(x)h = bt^{\beta+\gamma}\Upsilon(xt^{-1/\alpha}) + \mathcal{O}(t^{\beta}). \tag{2.94}$$

Therefore, Lemma 2.8 has been proved. □

The following theorem is a local version of Theorem 1.1, its proof is similar and will be omitted.

Theorem 2.9. *Let the initial data $u_0 \in \mathbb{Z}^\mu$ and the boundary value $h \in B^\alpha$ such that the correspondent norms are sufficiently small. Then, for some $T > 0$, there exist a unique solution*

$$u \in C([0,T]; L^1) \cap C((0,T]; L^s \cap L^{s,\mu} \cap L^\infty), \quad s \geq 1,$$

to the initial boundary-value problem (1.1).

3. Proof of Theorem 1.1

By the Local Existence Theorem 2.9, it follows that the global solution (if it exist) is unique. Indeed, on the contrary, we suppose that there exist two global solutions with the same initial data. And these solutions are different at some time $t > 0$. By virtue of the continuity of solutions with respect to time, we can find a maximal time segment $[0, T]$, where the solutions are equal, but for $t > T$ they are different. Now, we apply the local existence theorem taking the initial time T and obtain that these solutions coincide on some interval $[T, T_1]$, which give us a contradiction with the fact that T is the maximal time of coincidence. So our main purpose in the proof of Theorem 1.1 is to show the global in time existence of solutions.

First, we note that Lemma 2.7 imply for the Green operator $\mathcal{G} : \mathbb{Z}^\mu \to X^\mu$ the inequality $\|\mathcal{G}(t)u_0\|_{X^\mu} \leq C\|u_0\|_{\mathbb{Z}^\mu}$. Now, we are going to show the estimate

$$\begin{aligned} &\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)))d\tau\|_{X^\mu} \\ &\leq C\|u - v\|_{X^\mu}(\|u\|_{X^\mu} + \|v\|_{X^\mu})^\sigma, \end{aligned} \tag{3.1}$$

for all $u, v \in X$, where $\mathcal{N}(u) = |u|^\sigma u$ and $\alpha < \sigma < 1 + \alpha$. In fact, using the inequality

$$||u|^\sigma u - |v|^\sigma v| \leq C|u - v|(|u|^\sigma + |v|^\sigma),$$

we obtain

$$\begin{aligned} &\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^{1,\nu}} \\ &\leq C\|u - v\|_{L^\infty}\big(\|u\|^\sigma_{L^{\sigma,\frac{\nu}{\sigma}}} + \|v\|^\sigma_{L^{\sigma,\frac{\nu}{\sigma}}}\big) \\ &\leq C\{\tau\}^{-\gamma}\langle\tau\rangle^{-1/\alpha}\tau^{-\frac{1}{\alpha}(\sigma-1-\nu)}\|u - v\|_{X^\mu}(\|u\|_{X^\mu} + \|v\|_{X^\mu})^\sigma, \end{aligned} \tag{3.2}$$

where $\mu = \nu/\sigma \geq 0$, and

$$\begin{aligned} &\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^\infty} \\ &\leq C\|u - v\|_{L^\infty}(\|u\|^\sigma_{L^\infty} + \|v\|^\sigma_{L^\infty}) \\ &\leq C\{\tau\}^{-\gamma(\sigma+1)}\langle\tau\rangle^{-\frac{1}{\alpha}(\sigma+1)}\|u - v\|_{X^\mu}(\|u\|_{X^\mu} + \|v\|_{X^\mu})^\sigma. \end{aligned} \tag{3.3}$$

Then, estimates (3.2), (3.3), and Lemma 2.7, imply

$$\begin{aligned} &\|\mathcal{G}(t-\tau)(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)))\|_{L^{s,\mu}} \\ &\leq C(t-\tau)^{-\frac{1}{\alpha}(1-\frac{1}{s})}\tau^{-\frac{1}{\alpha}(\sigma-1)}\{\tau\}^{-\gamma}\langle\tau\rangle^{-1/\alpha}\Big((t-\tau)^{\frac{\mu}{\alpha}} + \tau^{\frac{\mu}{\alpha}}\Big) \\ &\quad \times \|u - v\|_{X^\mu}(\|u\|_{X^\mu} + \|v\|_{X^\mu})^\sigma, \end{aligned} \tag{3.4}$$

where $0 \le \mu < 1$, and

$$
\begin{aligned}
&\|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau)-\mathcal{N}(v)(\tau))\|_{L^\infty}\\
&\le C\{t-\tau\}^{-\gamma}\langle t-\tau\rangle^{-1/\alpha}\{\tau\}^{-\gamma}\langle\tau\rangle^{-1/\alpha}\Big(\tau^{-\frac{1}{\alpha}(\sigma-1)}+\{\tau\}^{-\gamma\sigma}\langle\tau\rangle^{-\sigma/\alpha}\Big)\\
&\quad\times\|u-v\|_{X^\mu}(\|u\|_{X^\mu}+\|v\|_{X^\mu})^\sigma.
\end{aligned}\tag{3.5}
$$

Now, we integrate with respect to τ inequalities (3.4) and (3.5),

$$
\begin{aligned}
&\int_0^t\|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau)-\mathcal{N}(v)(\tau))\|_{L^{s,\mu}}\\
&\le C\{t\}^{1-\frac{1}{\alpha}(\sigma-\frac{1}{s}-\mu)-\gamma}\langle t\rangle^{1-\frac{1}{\alpha}(1+\sigma-\frac{1}{s}-\mu)}\|u-v\|_{X^\mu}(\|u\|_{X^\mu}+\|v\|_{X^\mu})^\sigma,
\end{aligned}\tag{3.6}
$$

provided $\sigma < 1+\alpha$, $s < \frac{1}{1-\alpha}$, and

$$
\begin{aligned}
&\int_0^t\|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau)-\mathcal{N}(v)(\tau))\|_{L^\infty}\\
&\le C\{t\}^{1-\frac{1}{\alpha}(\sigma-1)-2\gamma}\langle t\rangle^{1-\frac{1}{\alpha}(\sigma+1)}\|u-v\|_{X^\mu}(\|u\|_{X^\mu}+\|v\|_{X^\mu})^\sigma,
\end{aligned}\tag{3.7}
$$

whenever $\sigma < 1+\alpha$. Then, for $\alpha < \sigma < 1+\alpha$, the definition of the norm in the space X^μ and the estimates (3.6) and (3.7), imply (3.1). Now, we apply the Contraction Mapping Principle on a ball with ratio $\rho > 0$ in the space X^μ, $X^\mu_\rho = \{\phi \in X^\mu : \|\phi\|_{X^\mu} \le \rho\}$, where $\rho = 3C\max\{\|u_0\|_{\mathbb{Z}^\mu}, \|h\|_{B^\alpha}\}$. First, we show that

$$
\|\mathcal{M}(u)\|_{X^\mu} \le \rho, \tag{3.8}
$$

where $u \in X^\mu_\rho$. Indeed, from the integral formula

$$
\mathcal{M}(u) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau + \mathcal{H}(x)h \tag{3.9}
$$

and the estimate (3.1) (with $v \equiv 0$) we obtain

$$
\begin{aligned}
\|\mathcal{M}(u)\|_{X^\mu} &\le \|\mathcal{G}(t)u_0\|_{X^\mu} + \|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)d\tau\|_{X^\mu} + \|\mathcal{H}(\cdot)h\|_{X^\mu}\\
&\le C\Big(\|u_0\|_{\mathbb{Z}^\mu} + \|u\|^{\sigma+1}_{X^\mu} + \|h\|_{B^\alpha}\Big)\\
&\le \frac{\rho}{3} + C\rho^{\sigma+1} + \frac{\rho}{3} \le \rho,
\end{aligned}
$$

provided $\rho > 0$ is sufficient small. Therefore, the operator $\mathcal{M}$ transforms a ball of ratio $\rho > 0$ into itself, in the space X^μ. In the same way we estimate the difference of two functions $u, v \in X^\mu_\rho$,

$$
\begin{aligned}
\|\mathcal{M}(u)-\mathcal{M}(v)\|_{X^\mu} &\le \|\int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau)-\mathcal{N}(v)(\tau))d\tau\|_{X^\mu}\\
&\le C\|u-v\|_{X^\mu}(\|u\|_{X^\mu}+\|v\|_{X^\mu})^\sigma \le C(2\rho)^\sigma\|u-v\|_{X^\mu}\\
&\le \frac{1}{2}\|u-v\|_{X^\mu}
\end{aligned}
$$

whenever $\rho > 0$ is sufficient small. Thus, $\mathcal{M}$ is a contraction mapping in X^μ_ρ. Therefore, there exist a unique solution $u \in X^\mu$ to the Cauchy problem (1.1). Now we prove the asymptotic formula

$$
u(x,t) = t^{-\frac{1}{\alpha}-\frac{1}{2}}\big(a\Lambda(xt^{-1/\alpha}) + b\Upsilon(xt^{-1/\alpha})\big) + O(t^{-\frac{1}{\alpha}-\frac{1}{2}-\gamma}), \tag{3.10}
$$

where $\gamma > 0$ and

$$a = \int_0^{+\infty} y^{\alpha/2} u_0(y) dy - \int_0^{+\infty} d\tau \int_0^{+\infty} y^{\alpha/2} \mathcal{N}(u(\tau)) dy.$$

We denote $G_0(t) = t^{-\frac{1}{\alpha}-\frac{1}{2}} \Lambda(xt^{-1/\alpha})$ and $G_1(t) = bt^{\beta+\gamma}\Upsilon(xt^{-1/\alpha})$. From Lemmas 2.7 and 2.8, we have for all $t > 1$,

$$\|\mathcal{G}(t)\phi - G_0(t) \int_0^{+\infty} y^{\alpha/2} \phi(y) dy\|_{L^\infty} \le C\langle t\rangle^{-\frac{1}{\alpha}-\frac{1}{2}-\gamma} \|\phi\|_{\mathbb{Z}^{1,\frac{\alpha}{2}}}, \tag{3.11}$$

$$\|\mathcal{H}(\cdot)h - G_1(t)\|_{L^\infty} \le C\langle t\rangle^{\beta}, \tag{3.12}$$

for $h(t) = bt^{\beta} + \mathcal{O}(t^{\beta-\gamma})$, $\beta < 0$, $\gamma > 0$. Also, in view of the definition of the norm X^μ we have

$$\begin{aligned} \Big|\int_0^{+\infty} y^{\alpha/2} \mathcal{N}(u(\tau)) dy\Big| &\le \|\mathcal{N}(u(\tau))\|_{L^{1,\frac{\alpha}{2}}} \le \|u(\tau)\|_{L^\infty}^{\sigma} \|u(\tau)\|_{L^{1,\frac{\alpha}{2}}} \\ &\le C\{\tau\}^{-\gamma\sigma} \langle\tau\rangle^{-\frac{\sigma}{\alpha}} \tau^{\frac{1}{2}} \|u\|_{X^{\alpha/2}}^{\sigma+1}. \end{aligned}$$

By a direct calculation, for $t > 1$, we have

$$\begin{aligned} &\|\int_0^{\frac{t}{2}} d\tau (G_0(t-\tau) - G_0(t)) \int_0^{+\infty} y^{\alpha/2} \mathcal{N}(u(\tau)) dy\|_{L^\infty} \\ &\le C\|u\|_{X^{\alpha/2}}^{\sigma+1} \int_0^{\frac{t}{2}} \|G_0(t-\tau) + G_0(t)\|_{L^\infty} \{\tau\}^{-\gamma\sigma} \langle\tau\rangle^{-\frac{\sigma}{\alpha}} \tau^{\frac{1}{2}} d\tau \\ &\le C\langle t\rangle^{-\frac{1}{\alpha}-\frac{1}{2}} \|u\|_{X^{\alpha/2}}^{\sigma+1} \int_0^{\frac{t}{2}} \{\tau\}^{-\gamma\sigma} \langle\tau\rangle^{-\frac{\sigma}{\alpha}} \tau^{\frac{1}{2}} d\tau \le C\langle t\rangle^{-\frac{1}{\alpha}-\frac{1}{2}-(\frac{\sigma}{\alpha}-\frac{3}{2})} \|u\|_{X^{\alpha/2}}^{\sigma+1}, \end{aligned} \tag{3.13}$$

where $\sigma > 3\alpha/2$, provided $\gamma\sigma < \frac{3}{2}$, and in the same way

$$\|G_0(t) \int_{t/2}^{\infty} d\tau \int_0^{+\infty} y^{\alpha/2} \mathcal{N}(u(\tau)) dy\|_{L^\infty} \le C\langle t\rangle^{-\frac{1}{\alpha}-\frac{1}{2}-(\frac{\sigma}{\alpha}-\frac{3}{2})} \|u\|_{X^{\alpha/2}}^{\sigma+1}, \tag{3.14}$$

provided $\sigma > 3\alpha/2$. Also we have for all $t > 1$,

$$\begin{aligned} &\|\int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G_0(t-\tau) \int_0^{+\infty} y^{\alpha/2} \mathcal{N}(u(\tau)) dy) d\tau\|_{L^\infty} \\ &\quad + \|\int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau) \mathcal{N}(u(\tau)) d\tau\|_{L^\infty} \\ &\le C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{\alpha}-\frac{1}{2}-\gamma} \|\mathcal{N}(u(\tau))\|_{L^{1,\frac{\alpha}{2}}} d\tau \\ &\quad + C \int_{\frac{t}{2}}^{t} \{t-\tau\}^{-\gamma} \langle t-\tau\rangle^{-1/\alpha} (\|\mathcal{N}(u(\tau))\|_{L^1} + \|\mathcal{N}(u(\tau))\|_{L^\infty}) d\tau \\ &\le C\langle t\rangle^{-\frac{1}{\alpha}-\frac{1}{2}-(\frac{\sigma}{\alpha}-\frac{3}{2})} \|u\|_{X^{\alpha/2}}^{\sigma+1}, \end{aligned} \tag{3.15}$$

provided $\gamma\sigma < \frac{3}{2}$ and $\sigma > \frac{3}{2}\alpha$. By virtue of the integral equation (3.9) we obtain

$$\begin{aligned} &\|u(t) - aG_0(t) - G_1(t)\|_{L^\infty} \\ &\le \|\mathcal{G}(t)u_0 - G_0(t)\int_0^{+\infty} y^{\alpha/2}u_0(y)dy\|_{L^\infty} \\ &\quad + \|\int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G_0(t-\tau)\int_0^{+\infty} y^{\alpha/2}\mathcal{N}(u(\tau))dy)d\tau\|_{L^\infty} \\ &\quad + \|\int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau\|_{L^\infty} + \|G_0(t)\int_{\frac{t}{2}}^{\infty} d\tau \int_0^{+\infty} y^{\alpha/2}\mathcal{N}(u(\tau))dy\|_{L^\infty} \\ &\quad + \|\int_0^{\frac{t}{2}} d\tau (G_0(t-\tau) - G_0(t))\int_0^{+\infty} y^{\alpha/2}\mathcal{N}(u(\tau))dy\|_{L^\infty} \\ &\quad + \|\mathcal{H}(\cdot)h - G_1(t)\|_{L^\infty}. \end{aligned} \tag{3.16}$$

All summands in the right-hand side of (3.16) are estimated, via (3.13)-(3.15), by

$$C\langle t\rangle^{-\frac{1}{\alpha}-\frac{1}{2}-\gamma}(\|u_0\|_{\mathbb{Z}^{\alpha/2}} + \|u\|^{\sigma+1}_{X^{\alpha/2}}),$$

where $0 < \gamma < \frac{\sigma}{\alpha} - \frac{3}{2}$, provided $\beta < -\frac{1}{\alpha} - \frac{1}{2} - \gamma$. Thus by last estimate the asymptotic (3.10) is valid. Theorem 1.1 has been proved.

References

[1] Árciga-Alejandre, M. P.; *Asymptotics for Nonlinear Evolution Equation with Module-Fractional Derivative on a Half-Line*, Boundary Value Problems, Volume 2011, Article ID 946143, 29 pages, doi:10.1155/2011/946143.

[2] Bashir, A.; Nieto, J. J.; *Existence Results for Nonlinear Boundary Value Problems of Fractional Integrodifferential Equations with Integral Boundary Conditions*, Boundary Value Problems, vol. **2009**, Article ID 708576, 11 pages, (2009). doi:10.1155/2009/708576.

[3] Bashir, A.; Otero-Espinar, V.; *Existence of Solutions for Fractional Differential Inclusions with Antiperiodic Boundary Conditions*, Boundary Value Problems, vol. **2009**, Article ID 625347, 11 pages, (2009). doi:10.1155/2009/625347.

[4] Belmekki, M.; Nieto, J. J.; Rodrguez-López, R.; *Existence of Periodic Solution for a Nonlinear Fractional Differential Equation*, Boundary Value Problems, vol. **2009**, Article ID 324561, 18 pages, (2009). doi:10.1155/2009/324561.

[5] Benchohra, M.; Cabada, A.; Seba, D.; *An Existence Result for Nonlinear Fractional Differential Equations on Banach Spaces*, Boundary Value Problems, vol. **2009**, Article ID 628916, 11 pages, (2009). doi:10.1155/2009/628916.

[6] Benchohra, M., Hamani, S.; *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative.* Topol. Methods Nonlinear Anal. **32** (2008), no. 1, 115–130.

[7] Caballero-Mena, J.; Harjani, J.; Sadarangani, K.; *Existence and Uniqueness of Positive and Nondecreasing Solutions for a Class of Singular Fractional Boundary Value Problems*, Boundary Value Problems, vol. **2009**, Article ID 421310, 10 pages, (2009). doi:10.1155/2009/421310.

[8] Gakhov, F. D.; *Boundary value problems*, Pergamon Press, ISBN 0-486-66275-6, (1966), pp. 562.

[9] Hayashi, N.; Kaikina, E. I.; *Nonlinear theory of pseudodifferential equations on a half-line.* North-Holland Mathematics Studies **194**. Elsevier Science B.V., Amsterdam, (2004), 319 pp.

[10] Hayashi, N.; Kaikina, E. I.; Naumkin, P. I.; Shishmarev, I. A.; *Asymptotics for dissipative nonlinear equations.* Lecture Notes in Mathematics **1884**. Springer-Verlag, Berlin, (2006), 557 pp.

[11] Kaikina, E. I.; *Subcritical pseudodifferential equation on a half-line with nonanalytic symbol.* Differential Integral Equations **18**, (2005), no. 12, 1341–1370.

[12] Kaikina, E. I.; *Nonlinear pseudoparabolic type equations on a half-line with large initial data.* Nonlinear Anal. **67**, (2007), no. 10, 2839–2858.

[13] Kaikina, E. I.; *Pseudodifferential operator with a nonanalytic symbol on a half-line*, J. of Mathematical Physics **48**, (2007), no. 11, 1341–1370.
[14] Kaikina, E. I.; *Fractional derivative of Abel type on a half-line.* Trans. Amer. Math. Soc., 364 (2012), no. 10, 5149-5172.
[15] Samko, S. G.; Kilbas, A. A.; Marichev, O. I.; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993. MR1347689 (96d:26012)
[16] Spence, E. A.; Fokas, A. S.; *A new transform method I: domain-dependent fundamental solutions and integral representations.* Proceedings of the Royal Society A, (2010) 466, 2259-2281 doi:10.1098/rspa.2009.0512.

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