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EXISTENCE OF POSITIVE SOLUTIONS FOR KIRCHHOFF PROBLEMS

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Abstract. We study problems for the Kirchhoff equation

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = \nu u^3 + Q(x)u^q, \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain, $a,b\geq 0$ and $a+b>0$, $\nu>0$, $3<q\leq 5$ and $Q(x)>0$ in Ω. By the mountain pass lemma, the existence of positive solutions is obtained. Particularly, we give a condition of Q to ensure the existence of solutions for the case of $q=5$.

1. Introduction and main results

In this article, we consider the Kirchhoff type problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = \nu u^3 + Q(x)u^q, \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain, $a,b\geq 0$ and $a+b>0$, $\nu>0, 3<q\leq 5$ are four parameters. The coefficient function Q is a positive function in Ω. When $a=0, b>0$, problem (1.1) is called degenerate, and the case of $a,b>0$ is called non-degenerate.

When $a\geq 0$ and $b>0$, problem (1.1) is called the Kirchhoff type problem. Kirchhoff type problems are often referred to as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2dx)\Delta u$ which implies that the equation in (1.1) is no longer a pointwise equation. The existence and multiplicity of solutions for the problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u), \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

on a smooth bounded domain $\Omega\subset\mathbb{R}^3$ and $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ a continuous function, has been extensively studied (see [1, 3],[7]-[23], [25]-[28],[30, 31]).

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Particularly, Sun and Tang [26] considered the problem

$$
\begin{gathered}
-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u=\lambda u^3+g(u)-h(x), \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.3}
$$

where $h\in L^2(\Omega)$ and $g\in C(\mathbb{R},\mathbb{R})$ satisfies

$$
\lim_{|t|\to\infty}\frac{g(t)}{t^3}=0. \tag{1.4}
$$

Under a Landesman-Lazer type condition, by the minimax methods, they obtained the existence of solutions for problem (1.3).

When $a=1$ and $b=0$, problem (1.1) reduces to the semilinear elliptic problem

$$
\begin{gathered}
-\Delta u=\nu u^3+\lambda u^q, \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega.
\end{gathered}
\tag{1.5}
$$

Obviously when $3<q<5$, problem (1.5) has a positive solution for all $\nu,\lambda>0$. While for $q=5,\lambda=1$, Brézis and Nirenberg [6] studied problem (1.5). By the variant of the mountain pass theorem of Ambrosetti and Rabinowitz without the (PS) condition, they obtained that there exists $\nu_0>0$ such that problem (1.5) has a positive solution for each $\nu\ge\nu_0$.

For $u\in H_0^1(\Omega)$, we define

$$
I(u)=\frac{a}{2}\int_\Omega|\nabla u|^2dx+\frac{b}{4}\Big(\int_\Omega|\nabla u|^2dx\Big)^2-\frac{\nu}{4}\int_\Omega|u|^4dx-\frac{1}{q+1}\int_\Omega Q(x)|u|^{q+1}dx,
$$

where $H_0^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\|=\big(\int_\Omega|\nabla u|^2dx\big)^{1/2}$. Note that a function u is called a weak solution of (1.1) if $u\in H_0^1(\Omega)$ such that

$$
\Big(a+b\int_\Omega|\nabla u|^2dx\Big)\int_\Omega(\nabla u,\nabla\varphi)dx-\nu\int_\Omega u^3\varphi dx-\int_\Omega Q(x)u^q\varphi dx=0, \tag{1.6}
$$

for all $\varphi\in H_0^1(\Omega)$.

We denote by ν_1 is the first eigenfunction of the eigenvalue problem

$$
\begin{gathered}
-\Big(\int_\Omega|\nabla u|^2dx\Big)\Delta u=\nu u^3, \quad x\in\Omega,\\
u=0, \quad x\in\partial\Omega.
\end{gathered}
$$

From [23], we know that $\nu_1>0$. Let S be the best Sobolev constant, namely

$$
S:=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2dx}{\big(\int_\Omega|u|^6dx\big)^{1/3}}. \tag{1.7}
$$

As well known, the function

$$
U(x)=\frac{(3\varepsilon^2)^{1/4}}{(\varepsilon^2+|x|^2)^{1/2}}, \quad x\in\mathbb{R}^3, \tag{1.8}
$$

is an extremal function for the minimum problem (1.7); that is, it is a positive solution of the problem

$$
-\Delta u=u^5, \quad \forall x\in\mathbb{R}^N.
$$

Problem (1.1) with $3<q\le5$ does not satisfy condition (1.4). It is natural to ask whether (1.1) has a positive solution. Using the mountain pass theorem, we study (1.1) and give a positive answer. It is worth pointing out that the result of the case

of $q = 5$ is more meaningful. Because of $q = 5$ is critical case and the coefficient of critical term is no longer constant. We give the suitable condition (A1) in the Theorem 1.3 below to ensure the existence of solutions to problem (1.1).

Our main results are described as follows.

Theorem 1.1. *Assume $a, b > 0$, $3 < q < 5$ and $Q \in L^{\frac{6}{5-q}}(\Omega)$ is a positive function, then* (1.1) *possesses a positive solution u^* for all $\nu > 0$, and $I(u^*) > 0$.*

Remark 1.2. Obviously, Theorem 1.1 does not apply to (1.4). For all $\nu > 0$, we obtain the existence of positive solutions for problem (1.1). For the degenerate case, that is $a = 0$, $b > 0$, we can also obtain that problem (1.1) possesses a positive solution for all $0 < \nu < b\nu_1$.

Theorem 1.3. *Assume $a, b > 0$, $q = 5$, $Q \in C(\overline{\Omega})$ is a positive function and satisfies the assumption*

(A1) *There exists $x_0 \in \Omega$ such that $Q(x_0) = Q_M = \max_{x\in\overline{\Omega}} Q(x)$ and*
$$Q(x) - Q(x_0) = o(|x - x_0|), \quad \text{as } x \to x_0.$$

Then there exists $\nu^* > 0$ such that (1.1) *possesses a positive solution u^* for all $\nu > \nu^*$, and $I(u^*) > 0$.*

Remark 1.4. This case is the critical exponent problem, and Theorem 1.3 does not apply to (1.4). When $Q(x) \equiv 1$, the Kirchhoff type problems with critical exponent have been considered by several papers, such as [1, 9, 12] [20]-[22], [27, 28, 30]. Particularly, problem (1.1) with $Q(x) \equiv 1$ was been considered in [21]. However, there exists a flaw in the proof of [21, Theorem 1.3] with the case $\theta = 4$.

To our best knowledge, problem (1.1) with $Q(x)$ not constant has not been considered yet. When $Q(x)$ is not constant, the analysis of the compactness becomes complicated, which results in much difficulty. It is worth pointing out that (A1) ensures the existence of solutions. Obviously, Theorem 1.3 extends the corresponding result of [21].

This article is organized as follows. In Section 2, we consider the case of $3 < q < 5$ and prove Theorem 1.1 by the variational methods. We study the critical case of problem (1.1) with $q = 5$ and give the proof of Theorem 1.3 in Section 3.

2. The case $3 < q < 5$

In this section, suppose that $Q \in L^{\frac{6}{5-q}}(\Omega)$ is a positive function and $3 < q < 5$. We will prove Theorem 1.1 by the mountain pass theorem. Before proving Theorem 1.1, we give the following lemma.

Lemma 2.1. *Assume $a, b > 0$, $3 < q < 5$ and $Q \in L^{\frac{6}{5-q}}(\Omega)$ is a positive function, then the functional I satisfies the (PS)c condition for all $\nu > 0$.*

Proof. Suppose that $\{u_n\}$ is a (PS)c sequence of I, that is,
$$I(u_n) \to c, \quad I'(u_n) \to 0, \tag{2.1}$$
as $n \to +\infty$. We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. In fact, from (2.1) one has
$$\begin{aligned} 1 + c + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{a}{4}\|u_n\|^2 + \big(\frac{1}{4} - \frac{1}{q+1}\big) \int_\Omega Q(x)|u_n|^{q+1} dx \end{aligned}$$

$$\geq \frac{a}{4}\|u_n\|^2.$$

Hence, we conclude that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u, \quad \text{weakly in } H_0^1(\Omega), \\ u_n \to u, \quad \text{strongly in } L^s(\Omega),\ 1 \leq s < 6, \\ u_n(x) \to u(x), \quad \text{a.e. in } \Omega, \end{gathered} \tag{2.2}$$

as $n \to \infty$. Now, we only need to prove that $u_n \to u$ as $n \to \infty$ in $H_0^1(\Omega)$. As usually, letting $w_n = u_n - u$, we need prove that $\|w_n\| \to 0$ as $n \to \infty$. By the Vitali theorem (see [24, p.133]), we claim that

$$\lim_{n\to\infty} \int_\Omega Q(x)|u_n|^{q+1}dx = \int_\Omega Q(x)|u|^{q+1}dx. \tag{2.3}$$

Indeed, we only need to prove that $\{\int_\Omega Q(x)|u_n|^{p+1}dx, n \in N\}$ is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, by the Sobolev embedding theorem, then exists a constant $C > 0$ such that $|u_n|_6 \leq C < \infty$. From the Hölder inequality, for every $\varepsilon > 0$, setting $\delta > 0$, when $E \subset \Omega$ with $\operatorname{meas} E < \delta$, we have

$$\int_E Q(x)|u_n|^{q+1}dx \leq |u_n|_6^{q+1}\Big(\int_E Q^{\frac{6}{5-q}}(x)dx\Big)^{\frac{5-q}{6}} < \varepsilon,$$

where the last inequality is from the absolutely-continuity of $\int_\Omega Q^{\frac{6}{5-q}}(x)dx$. Thus, our claim is proved. Moreover, one also has

$$\int_\Omega |\nabla u_n|^2dx = \int_\Omega |\nabla w_n|^2dx + \int_\Omega |\nabla u|^2dx + o(1), \tag{2.4}$$

$$\Big(\int_\Omega |\nabla u_n|^2dx\Big)^2 = \|w_n\|^4 + \|u\|^4 + 2\|w_n\|^2\|u\|^2 + o(1). \tag{2.5}$$

Since $I'(u_n) \to 0$, one obtains

$$a\|u_n\|^2 + b\|u_n\|^4 - \nu\int_\Omega |u_n|^4dx - \int_\Omega Q(x)|u_n|^{q+1}dx = o(1),$$

consequently, from (2.2)-(2.5), we deduce that

$$a\|w_n\|^2 + a\|u\|^2 + b\|w_n\|^4 + 2b\|w_n\|^2\|u\|^2 + b\|u\|^4 - \nu|u|_4^4 - \int_\Omega Q(x)|u|^{q+1}dx = o(1). \tag{2.6}$$

From (2.1) it follows that

$$\lim_{n\to\infty}\langle I'(u_n), u\rangle = a\|u\|^2 + bl^2\|u\|^2 + b\|u\|^4 - \nu|u|_4^4 - \int_\Omega Q(x)|u|^{q+1}dx = 0, \tag{2.7}$$

where $l = \lim_{n\to\infty}\|w_n\|$. According to (2.6) and (2.7), we have

$$a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 = o(1),$$

consequently, one has $al^2 + bl^4 + bl^2\|u\|^2 = 0$. Thus $l = 0$; that is, $u_n \to u$ as $n \to \infty$ in $H_0^1(\Omega)$. This completes the proof. $\square$

Proof of Theorem 1.1. The main idea is to construct a suitable geometry of mountain pass lemma (see[2]). Then obtain a critical point of I in $H_0^1(\Omega)$. We claim that I has the geometry of mountain pass lemma in $H_0^1(\Omega)$. Indeed, since

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\nu}{4}\int_\Omega |u|^4 dx - \frac{1}{q+1}\int_\Omega Q(x)|u|^{q+1}dx,$$

then $I(0) = 0$, and for every $u \in H_0^1(\Omega)\backslash\{0\}$ one has

$$\lim_{t\to 0+} \frac{I(tu)}{t^2} = \frac{a}{2}\|u\|^2, \quad \lim_{t\to+\infty} \frac{I(tu)}{t^{q+1}} = -\frac{1}{q+1}\int_\Omega Q(x)|u|^{q+1}dx.$$

Since $a > 0$ and $\int_\Omega Q(x)|u|^{q+1}dx > 0$, then there exist $R, \alpha > 0$ and $e \in H_0^1(\Omega)$ with $\|e\| > R$ such that $I|_{\partial B_R} \geq \alpha$ and $I(e) < 0$, where $\partial B_R = \{u \in H_0^1(\Omega) \mid \|u\| = R\}$. Thus, I satisfies the geometry of the mountain-pass lemma.

Let

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}$. Then $c \geq \alpha$. According to Lemma 2.1, I satisfies the conditions of the mountain pass lemma. Applying the mountain-pass lema, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$, such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. Then c is a critical value of I and $c > \alpha > 0$. Moreover, $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to u^*$ in $H_0^1(\Omega)$ as $n \to \infty$. Thus $I(u^*) = c > 0$ and u^* is a nonzero solution of (1.1). Since $I(|u|) = I(u)$, by a result due to Brézis and Nirenberg [4, Theorem 10], we conclude that $u^* \geq 0$. By the strong maximum principle, one has $u^* > 0$ in Ω. Therefore, u^* is a positive solution of problem (1.1) with $I(u^*) > 0$. This completes the proof. □

3. The case q=5

In this part, assume that $Q \in C(\overline{\Omega})$ is a positive function and satisfies (A1). We study the case of $q = 5$. This case is more delicate, because of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is not compact. Thus the functional I does not satisfy the $(PS)_c$ condition. When $Q(x)$ is not constant, the analysis of (PS) sequences becomes complicated, which results in much difficulty. We will complete the proof of Theorem 1.3 by the mountain pass lemma. Now, we prove that I satisfies the local $(PS)_c$ condition.

Lemma 3.1. *Assume $a, b > 0$ and the positive function $Q \in C(\overline{\Omega})$ satisfies (A1), then I satisfies the $(PS)_c$ condition, where $c \in (0, \Lambda)$ with*

$$\Lambda = \frac{abS^3}{4Q_M} + \frac{b^3S^6}{24Q_M^2} + \frac{aS\sqrt{b^2S^4 + 4aSQ_M}}{6Q_M} + \frac{b^2S^4\sqrt{b^2S^4 + 4aSQ_M}}{24Q_M^2}.$$

Proof. Suppose that $\{u_n\}$ is a $(PS)_c$ sequence for $c \in (0, \Lambda)$; that is,

$$I(u_n) \to c, \quad I'(u_n) \to 0, \tag{3.1}$$

as $n \to +\infty$. According to Lemma 2.1, we can easy obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, there exists $u \in H_0^1(\Omega)$ such that (2.2) holds. As usually, letting $w_n = u_n - u$, we need prove that $\|w_n\| \to 0$ as

$n \to \infty$. We denote $\lim_{n\to\infty} \|w_n\| = l$. As in Lemma 2.1, we have (2.4) and (2.5). By Brézis-Lieb's Lemma [5], one has

$$\int_\Omega Q(x)|u_n|^6 dx = \int_\Omega Q(x)|w_n|^6 dx + \int_\Omega Q(x)|u|^6 dx + o(1). \tag{3.2}$$

From (3.1) and (2.2), one obtains

$$a\|u_n\|^2 + b\|u_n\|^4 - \nu \int_\Omega |u|^4 dx - \int_\Omega Q(x)|u_n|^6 dx = o(1),$$

consequently, from (2.4)-(2.5) and (3.2) it follows that

$$\begin{aligned} &a\|u\|^2 + a\|w_n\|^2 + b\|u\|^4 + b\|w_n\|^4 + 2b\|w_n\|^2\|u\|^2 \\ &- \int_\Omega Q(x)|w_n|^6 dx - \int_\Omega Q(x)|u|^6 dx - \nu \int_\Omega |u|^4 dx = o(1). \end{aligned} \tag{3.3}$$

From (3.1) it follows that

$$\lim_{n\to\infty} \langle I'(u_n), u\rangle = a\|u\|^2 + b\|u\|^4 + bl^2\|u\|^2 - \int_\Omega Q(x)|u|^6 dx - \nu \int_\Omega |u|^4 dx = 0. \tag{3.4}$$

On the one hand, from (3.4), we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\nu}{4}\int_\Omega |u|^4 dx - \frac{1}{6}\int_\Omega Q(x)|u|^6 dx \\ &= \frac{a}{4}\|u\|^2 + \frac{1}{12}\int_\Omega Q(x)|u|^6 dx - \frac{bl^2}{4}\|u\|^2 \\ &\ge -\frac{bl^2}{4}\|u\|^2. \end{aligned} \tag{3.5}$$

On the other hand, from (3.3) and (3.4) it follows that

$$a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \int_\Omega Q(x)|w_n|^6 dx = o(1), \tag{3.6}$$

and

$$I(u_n) = I(u) + \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2 - \frac{1}{6}\int_\Omega Q(x)|w_n|^6 dx + o(1). \tag{3.7}$$

From (A1) and (1.7), one has

$$\int_\Omega Q(x)|w_n|^6 dx \le Q_M \int_\Omega |w_n|^6 dx \le Q_M \frac{\|w_n\|^6}{S^3},$$

consequently, it follows from (3.6) that

$$al^2 + bl^4 + bl^2\|u\|^2 \le Q_M \frac{l^6}{S^3},$$

which implies that

$$l^2 \ge \frac{1}{2}\Big[\frac{bS^3}{Q_M} + \frac{\sqrt{b^2S^6 + 4S^3Q_M(a + b\|u\|^2)}}{Q_M}\Big]. \tag{3.8}$$

Thus, from (3.6)-(3.8), we obtain

$$\begin{aligned} I(u) &= \lim_{n\to\infty}\Big[I(u_n) - \frac{a}{2}\|w_n\|^2 - \frac{b}{4}\|w_n\|^4 - \frac{b}{2}\|w_n\|^2\|u\|^2 + \frac{1}{6}\int_\Omega Q(x)|w_n|^6 dx\Big] \\ &= c - \Big(\frac{a}{3}l^2 + \frac{b}{12}l^4 + \frac{b}{3}l^2\|u\|^2\Big) \end{aligned}$$

$$
\begin{aligned}
&\le c-\Big[\frac{a}{6}\Big(\frac{bS^3}{Q_M}+\frac{\sqrt{b^2S^6+4S^3Q_M(a+b\|u\|^2)}}{Q_M}\Big)\\
&\quad+\frac{b}{48}\Big(\frac{bS^3}{Q_M}+\frac{\sqrt{b^2S^6+4S^3Q_M(a+b\|u\|^2)}}{Q_M}\Big)^2\\
&\quad+\frac{b\|u\|^2}{24}\Big(\frac{bS^3}{Q_M}+\frac{\sqrt{b^2S^6+4S^3Q_M(a+b\|u\|^2)}}{Q_M}\Big)\Big]-\frac{bl^2}{4}\|u\|^2\\
&\le c-\Big(\frac{abS^3}{4Q_M}+\frac{b^3S^6}{24Q_M^2}+\frac{aS\sqrt{b^2S^4+4aSQ_M}}{6Q_M}+\frac{b^2S^4\sqrt{b^2S^4+4aSQ_M}}{24Q_M^2}\Big)\\
&\quad-\frac{bl^2}{4}\|u\|^2\\
&<-\frac{bl^2}{4}\|u\|^2,
\end{aligned}
$$

which contradicts (3.5). Hence, $l\equiv 0$; that is, $u_n\to u$ in $H_0^1(\Omega)$ as $n\to\infty$. Therefore, I satisfies the $(PS)_c$ condition for all $c<\Lambda$. This completes the proof. $\square$

Next, we estimate the level value of functional I and obtain the following lemma.

Lemma 3.2. *Assume that $a,b>0$ and the positive function $Q\in C(\overline{\Omega})$ satisfies* (A1). *Then there exists $u_0\in H_0^1(\Omega)$, such that $\sup_{t\ge0}I(tu_0)<\Lambda$ for all $\nu>\nu^*$, where Λ is defined by Lemma 3.1 and ν^* independent of u_0 is a positive constant.*

Proof. Define a cut-off function $\eta\in C_0^\infty(\Omega)$ such that $0\le\eta\le1$, $|\nabla\eta|\le C_1$. For some $\tilde\delta>0$, we define

$$
\eta(x)=\begin{cases}1, & |x-x_0|\le\frac{\tilde\delta}{2},\\ 0, & |x-x_0|\ge\tilde\delta,\end{cases}
$$

where x_0 is defined by (A1). Set $u_\varepsilon=\eta(x)U(x-x_0)$. As well known(see [6, 29]), one has

$$
\|u_\varepsilon\|^2=\|U_\varepsilon\|^2+O(\varepsilon)=S^{3/2}+O(\varepsilon), \tag{3.9}
$$

$$
|u_\varepsilon|_6^6=|U_\varepsilon|_6^6+O(\varepsilon^3)=S^{3/2}+O(\varepsilon^3), \tag{3.10}
$$

and

$$
\begin{gathered}
C_2\varepsilon^{\frac{s}{2}}\le\int_\Omega u_\varepsilon^s dx\le C_3\varepsilon^{\frac{s}{2}},\quad 1\le s<3,\\
C_4\varepsilon^{\frac{s}{2}}|\ln\varepsilon|\le\int_\Omega u_\varepsilon^s dx\le C_5\varepsilon^{\frac{s}{2}}|\ln\varepsilon|,\quad s=3,\\
C_6\varepsilon^{\frac{6-s}{2}}\le\int_\Omega u_\varepsilon^s dx\le C_7\varepsilon^{\frac{6-s}{2}},\quad 3<s<6.
\end{gathered} \tag{3.11}
$$

Moreover, from [30], we have

$$
\begin{gathered}
\|u_\varepsilon\|^4=S^3+O(\varepsilon),\quad \|u_\varepsilon\|^6=S^{\frac{9}{2}}+O(\varepsilon),\\
\|u_\varepsilon\|^8=S^6+O(\varepsilon),\quad \|u_\varepsilon\|^{12}=S^9+O(\varepsilon).
\end{gathered} \tag{3.12}
$$

For all $t\ge0$, we define $I(tu_\varepsilon)$ by

$$
I(tu_\varepsilon)=\frac{a}{2}t^2\|u_\varepsilon\|^2+\frac{b}{4}t^4\|u_\varepsilon\|^4-\frac{\nu}{4}t^4\int_\Omega|u_\varepsilon|^4dx-\frac{t^6}{6}\int_\Omega Q(x)|u_\varepsilon|^6dx,
$$

then we have

$$\lim_{t\to+0} I(tu_\varepsilon)=0,\quad \text{uniformly for all } 0<\varepsilon<\varepsilon_0,$$
$$\lim_{t\to+\infty} I(tu_\varepsilon)=-\infty,\quad \text{uniformly for all } 0<\varepsilon<\varepsilon_0,$$

where $\varepsilon_0>0$ is a small constant. Thus $\sup_{t\geq 0} I(tu_\varepsilon)$ attains for some $t_\varepsilon>0$. Moreover, we can claim that there exist two constants $t_0, T_0>0$, which independent of ε, such that $t_0<t_\varepsilon<T_0$. In fact, from $\lim_{t\to+0} I(tu_\varepsilon)=0$ uniformly for all ε, we choose $\epsilon=\frac{I(t_\varepsilon u_\varepsilon)}{4}>0$, then there exists $t_0>0$ such that $|I(t_0u_\varepsilon)|=|I(t_0u_\varepsilon)-I(0)|<\epsilon$. Then according to the monotonicity of $I(tu_\varepsilon)$ near $t=0$, we have $t_\varepsilon>t_0$. Similarly, we can obtain that $t_\varepsilon<T_0$. Therefore, our claim is proved.

Set $I(tu_\varepsilon)=I_{\varepsilon,1}(t)-\nu I_{\varepsilon,2}(t)$, where

$$I_{\varepsilon,1}(t)=\frac{a}{2}t^2\|u_\varepsilon\|^2+\frac{b}{4}t^4\|u_\varepsilon\|^4-\frac{t^6}{6}\int_\Omega Q(x)u_\varepsilon^6dx,$$

and

$$I_{\varepsilon,2}(t)=\frac{t^4}{4}\int_\Omega u_\varepsilon^4dx.$$

First, we estimate the value $I_{\varepsilon,1}$. Since $I'_{\varepsilon,1}(t)=at\|u_\varepsilon\|^2+bt^3\|u_\varepsilon\|^4-t^5\int_\Omega Q(x)u_\varepsilon^6dx$, letting $I'_{\varepsilon,1}(t)=0$; that is,

$$a\|u_\varepsilon\|^2+bt^2\|u_\varepsilon\|^4-t^4\int_\Omega Q(x)u_\varepsilon^6dx=0, \tag{3.13}$$

one obtains

$$T_\varepsilon^2=\frac{b\|u_\varepsilon\|^4+\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega Q(x)u_\varepsilon^6dx}}{2\int_\Omega Q(x)u_\varepsilon^6dx}.$$

Then $I'_{\varepsilon,1}(t)>0$ for all $0<t<T_\varepsilon$ and $I'_{\varepsilon,1}(t)<0$ for all $t>T_\varepsilon$, so $I_{\varepsilon,1}(t)$ attains its maximum at T_ε. From (A1), let $\varepsilon\to 0^+$, we claim that

$$\Big(\int_\Omega Q(x)u_\varepsilon^6dx\Big)^{1/3}=Q_M^{1/3}|u_\varepsilon|_6^2+o(\varepsilon). \tag{3.14}$$

In fact, for all $\varepsilon>0$, it follows that

$$\begin{aligned}\Big|\int_\Omega Q(x)u_\varepsilon^6dx-\int_\Omega Q_Mu_\varepsilon^6dx\Big| &\leq \int_\Omega |Q(x)-Q(x_0)|u_\varepsilon^6dx\\ &\leq \int_{\{x\in\Omega:\ |x-x_0|\leq\tilde\delta\}} |Q(x)-Q(x_0)|u_\varepsilon^6dx.\end{aligned} \tag{3.15}$$

From (A1), for all $\eta>0$, there exists $\delta>0$ such that

$$|Q(x)-Q(x_0)|<\eta|x-x_0|,\quad \text{for all } 0<|x-x_0|<\delta.$$

When $\varepsilon>0$ small enough, for $\delta>\varepsilon^{1/2}$, it follows from (3.15) and (A1) that

$$\begin{aligned}&\Big|\int_\Omega Q(x)u_\varepsilon^6dx-\int_\Omega Q_Mu_\varepsilon^6dx\Big|\\ &\leq \int_{\{x\in\Omega:|x-x_0|\leq\tilde\delta\}} |Q(x)-Q(x_0)|u_\varepsilon^6dx\\ &< \int_{\{x\in\Omega:|x-x_0|\leq\delta\}} \eta|x-x_0|\frac{(3\varepsilon^2)^{3/2}}{[\varepsilon^2+|x-x_0|^2]^3}dx\end{aligned}$$

$$
\begin{aligned}
&+\int_{\{x\in\Omega:\ \delta<|x-x_0|\le\tilde{\delta}\}}\frac{(3\varepsilon^2)^{3/2}}{[\varepsilon^2+|x-x_0|^2]^3}dx\\
&=\sqrt{27}\eta\int_0^{\delta}r^3\frac{\varepsilon^3}{(\varepsilon^2+r^2)^3}dr+\sqrt{27}\int_{\delta}^{\tilde{\delta}}\frac{\varepsilon^3r^2}{(\varepsilon^2+r^2)^3}dr\\
&=\sqrt{27}\eta\varepsilon\int_0^{\frac{\delta}{\varepsilon}}\frac{r^3}{(1+r^2)^3}dr+\sqrt{27}\int_{\frac{\delta}{\varepsilon}}^{\frac{\tilde{\delta}}{\varepsilon}}\frac{r^2}{(1+r^2)^3}dr\\
&\le C_8\eta\varepsilon+C_9\varepsilon^3.
\end{aligned}
$$

Consequently, one has

$$
\frac{\left|\int_\Omega Q(x)u_\varepsilon^6dx-\int_\Omega Q_Mu_\varepsilon^6dx\right|}{\varepsilon}\le C_8\eta+C_9\varepsilon^2,
$$

which implies

$$
\limsup_{\varepsilon\to0^+}\frac{\left|\int_\Omega Q(x)u_\varepsilon^6dx-\int_\Omega Q_Mu_\varepsilon^6dx\right|}{\varepsilon}\le C_8\eta.
$$

Then from the arbitrariness of η, we obtain (3.14). Thus, from (3.10) and (3.14), one gets

$$
\int_\Omega Q(x)u_\varepsilon^6dx=Q_M|u_\varepsilon|_6^6+o(\varepsilon)=Q_MS^{3/2}+o(\varepsilon). \tag{3.16}
$$

Thus from (3.9),(3.12),(3.13) and (3.16), we have

$$
\begin{aligned}
I_{\varepsilon,1}(t)&\le I_{\varepsilon,1}(T_\varepsilon)\\
&=T_\varepsilon^2\Big(\frac{a}{2}\|u_\varepsilon\|^2+\frac{b}{4}T_\varepsilon^2\|u_\varepsilon\|^4-\frac{T_\varepsilon^4}{6}\int_\Omega Q(x)u_\varepsilon^6dx\Big)\\
&=T_\varepsilon^2\Big(\frac{a}{3}\|u_\varepsilon\|^2+\frac{b}{12}T_\varepsilon^2\|u_\varepsilon\|^4\Big)\\
&=\frac{ab\|u_\varepsilon\|^6+a\|u_\varepsilon\|^2\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega Q(x)u_\varepsilon^6dx}}{6\int_\Omega Q(x)u_\varepsilon^6dx}\\
&\quad+\frac{b^3\|u_\varepsilon\|^{12}+2ab\|u_\varepsilon\|^6\int_\Omega Q(x)u_\varepsilon^6dx}{24\big(\int_\Omega Q(x)u_\varepsilon^6dx\big)^2}\\
&\quad+\frac{b^2\|u_\varepsilon\|^4\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega Q(x)u_\varepsilon^6dx}}{24\big(\int_\Omega Q(x)u_\varepsilon^6dx\big)^2}\\
&=\frac{ab\|u_\varepsilon\|^6}{4\int_\Omega Q(x)u_\varepsilon^6dx}+\frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega Q(x)u_\varepsilon^6dx)^2}\\
&\quad+\frac{a\|u_\varepsilon\|^2\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega Q(x)u_\varepsilon^6dx}}{6\int_\Omega Q(x)u_\varepsilon^6dx}\\
&\quad+\frac{b^2\|u_\varepsilon\|^4\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega Q(x)u_\varepsilon^6dx}}{24\big(\int_\Omega Q(x)u_\varepsilon^6dx\big)^2}\\
&=\frac{ab(S^{\frac{9}{2}}+O(\varepsilon))}{4(Q_MS^{3/2}+o(\varepsilon))}+\frac{b^3(S^9+O(\varepsilon))}{24(Q_MS^{3/2}+o(\varepsilon))^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a(S^{3/2}+O(\varepsilon))\sqrt{b^2S^6+4aS^3+O(\varepsilon)}}{6(Q_MS^{3/2}+o(\varepsilon))}\\
&+\frac{b^2(S^6+O(\varepsilon))\sqrt{b^2S^6+4aS^3+O(\varepsilon)}}{24(Q_MS^{3/2}+o(\varepsilon))^2}\\
&=\frac{abS^3}{4Q_M}+\frac{b^3S^6}{24Q_M^2}+\frac{aS\sqrt{b^2S^4+4aSQ_M}}{6Q_M}\\
&+\frac{b^2S^4\sqrt{b^2S^4+4aSQ_M}}{24Q_M^2}+O(\varepsilon)\\
&=\Lambda+O(\varepsilon). \qquad (3.17)
\end{aligned}
$$

Second, we estimate the value of $I_{\varepsilon,2}$. From (3.11), since $0<t_0<t_\varepsilon<T_0$, one has

$$
I_{\varepsilon,2}(t_\varepsilon)=\frac{t_\varepsilon^4}{4}\int_\Omega u_\varepsilon^4dx\geq\frac{t_0^4}{4}\int_\Omega u_\varepsilon^4dx\geq C_{10}\varepsilon. \qquad (3.18)
$$

Thus, from (3.17) and (3.18), one gets

$$
\begin{aligned}
I(tu_\varepsilon)&=I_{\varepsilon,1}(t)-\nu I_{\varepsilon,2}(t)\\
&\leq I_{\varepsilon,1}(t_\varepsilon)-\nu I_{\varepsilon,2}(t_\varepsilon)\\
&\leq I_{\varepsilon,1}(T_\varepsilon)-\frac{t_0^4}{4}\nu\int_\Omega u_\varepsilon^4dx\\
&\leq \Lambda+O(\varepsilon)-C_{10}\nu\varepsilon<\Lambda,
\end{aligned}
$$

provided that ν is large enough. Thus there exists $\nu^*>0$ such that $I(tu_\varepsilon)<\Lambda$ for all $\nu>\nu^*$. This completes the proof. □

Proof of Theorem 1.3. As in the proof of Theorem 1.1, we can obtain that I has the geometry of mountain pass lemma in $H_0^1(\Omega)$. According to Lemmas 3.1 and 3.2, it follows that I satisfies the conditions of the mountain pass lemma. Then as in the proof of Theorem 1.1, we obtain that (1.1) has a positive solution u^* with $I(u^*)>0$ as long as $\nu>\nu^*$. The proof is complete. □

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