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NUMERICAL SOLUTIONS FOR VOLTERRA INTEGRO-DIFFERENTIAL FORMS OF LANE-EMDEN EQUATIONS OF FIRST AND SECOND KIND USING LEGENDRE MULTI-WAVELETS

PRAKASH KUMAR SAHU, SANTANU SAHA RAY

Abstract. A numerical method based on Legendre multi-wavelets is applied for solving Lane-Emden equations which form Volterra integro-differential equations. The Lane-Emden equations are converted to Volterra integro-differential equations and then are solved by the Legendre multi-wavelet method. The properties of Legendre multi-wavelets are first presented. The properties of Legendre multi-wavelets are used to reduce the system of integral equations to a system of algebraic equations which can be solved by any numerical method. Illustrative examples are discussed to show the validity and applicability of the present method.

1. Introduction

In this article, we discuss a Lane-Emden equation of first kind [5, 11, 12, 13, 14] of the form

$$y'' + \frac{\kappa}{x}y' + y^m = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad \kappa > 1 \tag{1.1}$$

and Lane-Emden equation of second kind [4, 9, 15] of the form

$$y'' + \frac{\kappa}{x}y' + e^y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad \kappa \geq 1 \tag{1.2}$$

where κ is the shape factor.

Equation (1.1) is a basic equation in the theory of stellar structure [2]. It is used in astrophysics for computing the structure of interiors of polytropic stars. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics [12]. The Lane-Emden equation of the first kind appears also in other contexts such as radiative cooling, self-gravitating gas clouds, mean-field treatment of a phase transition in critical adsorption, and modeling of clusters of galaxies.

Equation (1.2) is the Lane-Emden equation of the second kind that models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere [10]. In the

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study of stellar structures one considers the star as a gaseous sphere in thermodynamic and hydrostatic equilibrium for a certain equation of state [3].

The well-known Lane-Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, the theory of thermionic currents, and in the modeling of clusters of galaxies. A substantial amount of work has been done on these types of problems for various structures. The singular behavior that occurs at $x = 0$ is the main difficulty of eqs. (1.1)–(1.2).

In this article, our main work is to establish Volterra integro-differential equation equivalent to the Lane-Emden equation of first and second kind. The newly established Volterra integro-differential equation will be solved by using the Legendre multi-wavelet method (LMWM). Legendre multi-wavelet method has been applied to solve the integral equations and integro-differential equations of different forms [7, 8, 17, 1, 18]. The Legendre multi-wavelet method converts the Volterra integro-differential equation to a system of algebraic equations and that algebraic equations system again can be solved by any of the usual numerical methods.

2. Volterra integro-differential form of the Lane-Emden equation

Let us consider the Lane-Emden equation

$$y''(x) + \frac{\kappa}{x} y'(x) + f(y) = 0, \quad y(0) = \alpha, \quad y'(0) = 0, \quad \kappa \geq 1. \tag{2.1}$$

Multiplying by x^κ and integrating on $[0, x]$ we have

$$y'(x) = -\int_0^x \big(\frac{t^\kappa}{x^\kappa}\big) f(y(t)) dt \quad \kappa \geq 1, \quad y(0) = \alpha. \tag{2.2}$$

Integrating again on $[0, x]$, (2.1) becomes

$$y(x) = \alpha - \frac{1}{\kappa - 1} \int_0^x t \Big(1 - \frac{t^{\kappa-1}}{x^{\kappa-1}}\Big) f(y(t)) dt. \tag{2.3}$$

3. Properties of Legendre multi-wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as

$$\Psi_{a,b}(x) = |a|^{-1/2} \Psi\big(\frac{x - b}{a}\big), \quad a, b \in \mathbb{R}, \ a \neq 0 \tag{3.1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, and k are positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(x) = |a_0|^{-k/2} \psi(a_0^k x - nb_0),$$

where $\psi_{k,n}(x)$ forms a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ form an orthonormal basis.

Legendre multi-wavelets $\psi_{n,m}(x) = \psi(k, n, m, x)$ have four arguments. $n = 0, 1, 2, \dots, 2^k - 1$, $k \in \mathbb{Z}^+$, where m is the order of Legendre polynomials and x is normalized time. These functions are defined on $[0,T)$ as (see [16])

$$\psi_{n,m}(x)=\begin{cases}\sqrt{2m+1}\big(\frac{2^{k/2}}{\sqrt{T}}\big)P_m\big(\frac{2^k x}{T}-n\big), & \frac{nT}{2^k}\le x<\frac{(n+1)T}{2^k}\\ 0, & \text{otherwise},\end{cases} \tag{3.2}$$

where $m=0,1,\dots,M-1$ and $n=0,1,2,\dots,2^k-1$. The dilation parameter is $a=2^{-k}T$ and translation parameter is $b=n2^{-k}T$.

Here $P_m(x)$ are the well-known shifted Legendre polynomials of order m, which are defined on the interval $[0,1]$, and can be determined with the aid of the following recurrence formulae

$$P_0(x)=1,\quad P_1(x)=2x-1,$$
$$P_{m+1}(x)=\big(\frac{2m+1}{m+1}\big)(2x-1)P_m(x)-\big(\frac{m}{m+1}\big)P_{m-1}(x),\quad m=1,2,3,\dots$$

4. Function approximation by Legendre multi-wavelets

A function $f(x)$ defined over $[0,T)$ can be expressed by the Legendre multi-wavelets as

$$f(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(x) \tag{4.1}$$

where $c_{n,m}=\langle f(x),\psi_{n,m}(x)\rangle$, in which $\langle\cdot,\cdot\rangle$ denotes the inner product. If the infinite series in (4.1) is truncated, then (4.1) can be written as

$$f(x)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}c_{n,m}\psi_{n,m}(x)=C^T\Psi(x) \tag{4.2}$$

where C and $\Psi(x)$ are $(2^k(M+1)\times 1)$ matrices given by

$$C=[c_{0,0},c_{0,1},\dots,c_{0,M},c_{1,0},\dots,c_{1,M},\dots,c_{2^k-1,0},\dots,c_{2^k-1,M}]^T, \tag{4.3}$$

$$\Psi(x)=[\psi_{0,0}(x),\psi_{0,1}(x),\dots,\psi_{0,M}(x),\dots,\psi_{2^k-1,0}(x),\dots,\psi_{2^k-1,M}(x)]^T. \tag{4.4}$$

5. Legendre multi-wavelet method for Volterra integro-differential equation form of Lane-Emden equation

Consider the Volterra integro-differential equation given in (2.2) which is the form of Lane-Emden equation defined in (2.1). To apply the Legendre multi-wavelets, we first approximate the unknown function $y(x)$ as

$$y(x)=C^T\Psi(x), \tag{5.1}$$

where C is defined similar to (4.3).

Integrating (2.2) and using the initial condition $y(0)=\alpha$, we have

$$y(x)=\alpha-\int_0^x\Big[\int_0^z\Big(\frac{t^\kappa}{z^\kappa}\Big)f(y(t))dt\Big]dz,\quad \kappa\ge 1 \tag{5.2}$$

Then from (5.1) and (5.2), we have

$$\begin{aligned}C^T\Psi(x)&=\alpha-\int_0^x\Big[\int_0^z\Big(\frac{t^\kappa}{z^\kappa}\Big)f(C^T\Psi(t))dt\Big]dz,\quad \kappa\ge 1\\ &=\alpha-\int_0^x H(z)dz,\end{aligned} \tag{5.3}$$

where

$$H(z)=\int_0^z\Big(\frac{t^\kappa}{z^\kappa}\Big)f(C^T\Psi(t))dt.$$

Now we collocate (5.3) at $x_i=\frac{(2i-1)T}{2^{k+1}(M+1)}$, $i=1,2,\dots,2^k(M+1)$ as

$$C^T\Psi(x_i)=\alpha-\int_0^{x_i}H(z)dz \tag{5.4}$$

To use the Gaussian integration formula for (5.4), we transfer the interval $[0,x_i]$ into the interval $[-1,1]$ by means of the transformation

$$\tau=\frac{2}{x_i}z-1$$

Equation (5.4) can be written as

$$C^T\Psi(x_i)=\alpha-\frac{x_i}{2}\int_{-1}^1 H\big(\frac{x_i}{2}(\tau+1)\big)d\tau. \tag{5.5}$$

Using the Gaussian integration formula, we obtain

$$C^T\Psi(x_i)\cong\alpha-\frac{x_i}{2}\sum_{j=1}^s w_jH\big(\frac{x_i}{2}(\tau_j+1)\big), \tag{5.6}$$

where τ_j are s zeros of Legendre polynomials P_{s+1} and w_j are the corresponding weights. The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding $2s+1$. Equation (5.6) gives a system of $2^k(M+1)$ nonlinear algebraic equations with same number of unknowns for coefficient matrix C. Solving this system numerically by Newton's method, we can get the values of unknowns for C and hence we obtain the solution $y(x)=C^T\Psi(x)$.

6. Convergence analysis

Theorem 6.1. *The series solution* $y(x)=\sum_{n=0}^\infty\sum_{m=0}^\infty c_{n,m}\psi_{n,m}(x)$ *defined in* (4.1) *using Legendre multi-wavelet method converges to* $y(x)$.

Proof. The set $\{\psi_{n,m};n,m=0,1,\dots\}$ is a complete orthonormal set in the Hilbert space $L^2(\mathbb{R})$. Let $y(x)=\sum_{m=0}^M C_{n,m}\psi_{n,m}(x)$ where $C_{n,m}=\langle y(x),\psi_{n,m}(x)\rangle$, for fixed n. Let us denote $\psi_{n,m}(x)=\psi(x)$ and let $\alpha_j=\langle y(x),\psi(x)\rangle$. Now we define the sequence of partial sum $\{S_n\}$ of $\big(\alpha_j\psi(x_j)\big)$. Let $\{S_n\}$ and $\{S_m\}$ be the partial sums with $n\ge m$. We have to prove $\{S_n\}$ is a Cauchy sequence in the Hilbert space. Let $S_n=\sum_{j=1}^n\alpha_j\psi(x_j)$. Now

$$\langle y(x),S_n\rangle=\langle y(x),\sum_{j=1}^n\alpha_j\psi(x_j)\rangle=\sum_{j=1}^n|\alpha_j|^2.$$

We claim that

$$\|S_n-S_m\|^2=\sum_{j=m+1}^n|\alpha_j|^2,\quad n>m.$$

Now

$$\|\sum_{j=m+1}^n\alpha_j\psi(x_j)\|^2=\langle\sum_{j=m+1}^n\alpha_j\psi(x_j),\sum_{j=m+1}^n\alpha_j\psi(x_j)\rangle=\sum_{j=m+1}^n|\alpha_j|^2,$$

for $n > m$. Therefore,

$$\|\sum_{j=m+1}^{n}\alpha_j\psi(x_j)\|^2=\sum_{j=m+1}^{n}|\alpha_j|^2,\quad \text{for } n>m.$$

From Bessel's inequality, we have $\sum_{j=1}^{\infty}|\alpha_j|^2$ is convergent and hence

$$\|\sum_{j=m+1}^{n}\alpha_j\psi(x_j)\|^2\to 0\quad \text{as } n\to\infty.$$

So,

$$\|\sum_{j=m+1}^{n}\alpha_j\psi(x_j)\|\to 0$$

and $\{S_n\}$ is a Cauchy sequence and it converges to s (say).

We assert that $y(x)=s$. In fact,

$$\begin{aligned}\langle s-y(x),\psi(x_j)\rangle &= \langle s,\psi(x_j)\rangle-\langle y(x),\psi(x_j)\rangle\\ &= \langle \lim_{n\to\infty}S_n,\psi(x_j)\rangle-\alpha_j\\ &= \alpha_j-\alpha_j.\end{aligned}$$

This implies $\langle s-y(x),\psi(x_j)\rangle=0$, which gives $y(x)=s$ and $\sum_{j=1}^{n}\alpha_j\psi(x_j)$ converges to $y(x)$ as $n\to\infty$ and completes the proof. □

7. Illustrative examples

Example 7.1. Consider the generalized form of Lane-Emden equation of first kind

$$y''(x)+\frac{\kappa}{x}y'(x)+y^m(x)=0,\quad \kappa\ge 1,\quad y(0)=1,\quad y'(0)=0.$$

This equation is equivalent to the integro-differential equation

$$y'(x)=-\int_0^x\big(\frac{t^\kappa}{x^\kappa}\big)y^m(t)dt,\quad y(0)=1,\quad \kappa\ge 1.$$

The exact solutions of this problem for $\kappa=2$ and $m=0,1,5$ respectively are

$$y(x)=1-\frac{1}{3!}x^2$$
$$y(x)=\frac{\sin x}{x}$$
$$y(x)=\big(1+\frac{x^2}{3}\big)^{-1/2}$$

The approximate solutions obtained by Legendre multi-wavelet method ($M=7,k=1$) for shape factor $\kappa=2$ and $m=0,1,5$ with their corresponding exact solutions and absolute errors have been shown in Tables 1–3 respectively.

Example 7.2. Consider the Lane-Emden equation of second kind

$$y''(x)+\frac{\kappa}{x}y'(x)+e^{y(x)}=0,\quad y(0)=y'(0)=0,\quad \kappa>1.$$

This equation is equivalent to

$$y'(x)=-\int_0^x\big(\frac{t^\kappa}{x^\kappa}\big)e^{y(t)}dt,\quad y(0)=1,\quad \kappa>1.$$

x	LMWM solution	Exact solution	Absolute error
0.2	0.993333	0.993333	2.66664E-12
0.4	0.973333	0.973333	2.13333E-11
0.6	0.940000	0.940000	7.20001E-11
0.8	0.893333	0.893333	1.70667E-10
1	0.833333	0.833333	3.33333E-10

Table 1. Numerical solutions for Example 7.1 when $\kappa = 2$, $m = 0$

x	LMWM solution	Exact solution	Absolute error
0.2	0.993347	0.993347	2.45593E-9
0.4	0.973546	0.973546	5.46664E-10
0.6	0.941071	0.941071	2.45289E-10
0.8	0.896695	0.896695	1.94895E-10
1	0.841471	0.841471	2.45936E-10

Table 2. Numerical solutions for Example 7.1 when $\kappa = 2$, $m = 1$

x	LMWM solution	Exact solution	Absolute error
0	1	1	2.66055E-9
0.2	0.993399	0.993399	1.07934E-11
0.4	0.974355	0.974355	1.17952E-11
0.6	0.944911	0.944911	1.64531E-11
0.8	0.907841	0.907841	2.17233E-11

Table 3. Numerical solutions for Example 7.1 when $\kappa = 2$, $m = 5$

The approximate solutions obtained by Legendre multi-wavelet method ($M = 7, k = 1$) for shape factor $\kappa = 2, 3, 4$ have been compared with the solutions obtained by a variational iteration method (VIM) [14] cited in Table 4.

x	$\kappa = 2$		$\kappa = 3$		$\kappa = 4$	
	LMWM	VIM	LMWM	VIM	LMWM	VIM
0	-5.7433E-11	0	-2.484E-11	0	-1.2637E-11	0
0.2	-0.006653	-0.006653	-0.004992	-0.004992	-0.003994	-0.003994
0.4	-0.026456	-0.026456	-0.019868	-0.019868	-0.015909	-0.015909
0.6	-0.058944	-0.058944	-0.044337	-0.044337	-0.035544	-0.035544
0.8	-0.103386	-0.103386	-0.077935	-0.077935	-0.062578	-0.062578

Table 4. Numerical solutions for Example 7.2

Example 7.3. Next, consider the Lane-Emden type equation given by

$$y''(x) + \frac{8}{x}y'(x) + (18y(x) + 4y(x)\ln(y(x))) = 0, \quad y(0) = 1, \quad y'(0) = 0$$

The Volterra integro-differential form of this equation is given by

$$y'(x) + \int_0^x \frac{t^8}{x^8}(18y(t) + 4y(t)\ln y(t))dt = 0, \quad y(0) = 1$$

with exact solution e^{-x^2} . The Legendre multi-wavelets solutions for $M = 7, k = 1$ along with their corresponding exact solutions and absolute errors have been shown in Table 5.

x	LMWM solution	Exact solution	Absolute error
0	1	1	3.95615E-8
0.1	0.990050	0.990050	2.96242E-10
0.2	0.960789	0.960789	3.82808E-10
0.3	0.913931	0.913931	2.95619E-8
0.4	0.852143	0.852143	4.68592E-7
0.5	0.778797	0.778797	3.64064E-6

Table 5. Numerical solutions for Example 7.3

Example 7.4. Consider the Lane-Emden type equation given by

$$y''(x) + \frac{1}{x}y'(x) + (3y^5(x) - y^3(x)) = 0, \quad y(0) = 1, \quad y'(0) = 0$$

The Volterra integro-differential form of this equation is given by

$$y'(x) + \int_0^x \frac{t}{x}(3y^5(t) - y^3(t))dt = 0, \quad y(0) = 1$$

with exact solution $\frac{1}{\sqrt{1+x^2}}$. The Legendre multi-wavelets solutions for $M = 7, k = 1$ along with their corresponding exact solutions and absolute errors have been shown in Table 6.

x	LMWM solution	Exact solution	Absolute error
0	1	1	9.41731E-8
0.2	0.980581	0.980581	8.91026E-10
0.4	0.928477	0.928477	1.53517E-9
0.6	0.857493	0.857493	1.16852E-9
0.8	0.780869	0.780869	1.55470E-9

Table 6. Numerical solutions for Example 7.4

Example 7.5. Consider the Lane-Emden type equation given by

$$y''(x) + \frac{2}{x}y'(x) + 4\Big(2e^{y(x)} + e^{\frac{y(x)}{2}}\Big) = 0, \quad y(0) = y'(0) = 0.$$

The Volterra integro-differential form of this equation is given by

$$y'(x) + \int_0^x \frac{t^2}{x^2}\Big(4\Big(2e^{y(t)} + e^{\frac{y(t)}{2}}\Big)\Big)dt = 0, \quad y(0) = 0$$

with exact solution $-2\ln(1+x^2)$. The Legendre multi-wavelets solutions for $M = 7, k = 1$ along with their corresponding exact solutions and absolute errors have been shown in Table 7.

x	LMWM solution	Exact solution	Absolute error
0	1.1743E-7	0	1.17430E-7
0.2	-0.078441	-0.078441	1.25003E-9
0.4	-0.296840	-0.296840	1.65908E-7
0.6	-0.614985	-0.614969	1.52712E-5
0.8	-0.989704	-0.989392	3.11348E-4

Table 7. Numerical solutions for Example 7.5

Example 7.6. Consider the system of nonlinear Lane-Emden type equations given by

$$y_1''(x) + \frac{8}{x}y_1'(x) + (18y_1(x) - 4y_1(x)\ln y_2(x)) = 0$$
$$y_2''(x) + \frac{4}{x}y_2'(x) + (4y_2(x)\ln y_1(x) - 10y_2(x)) = 0$$

with initial conditions

$$y_1(0) = 1, \quad y_1'(0) = 0,$$
$$y_2(0) = 1, \quad y_2'(0) = 0$$

The system of nonlinear Volterra integro-differential form of the above system is given by

$$y_1'(x) + \int_0^x \frac{t^8}{x^8}(18y_1(t) - 4y_1(t)\ln y_2(t))dt = 0,$$
$$y_2'(x) + \int_0^x \frac{t^4}{x^4}(4y_2(t)\ln y_1(t) - 10y_2(t))dt = 0,$$

with initial conditions $y_1(0) = 1, \quad y_2(0) = 1$. The corresponding exact solutions of this system are

$$y_1(x) = e^{-x^2}, \quad y_2(x) = e^{x^2}$$

The approximate solutions obtained by Legendre multi-wavelet method for $M = 7, k = 1$ along with their corresponding exact solutions and absolute errors have been shown in Table 8.

Example 7.7. To verify the accuracy of the presented method, we have considered a fractional order integro-differential equation [19] as

$$D^\alpha y(x) - \int_0^1 xt[y(t)]^2 dt = 1 - \frac{x}{4}, \quad 0 \le x < 1,\ 0 < \alpha \le 1,$$

with initial condition $y(0) = 0$ and the exact solution $y(x) = x$ when $\alpha = 1$. This problem has been solved by Chebyshev wavelet method (CWM) in [19] for $\alpha = 1$. The results obtained by the Chebyshev wavelet method [19] have been compared with the results obtained by presented method and the root mean square errors (RMSE) of these two methods have been cited in Table 9.

x	LMWM solution		Exact solution		Absolute error	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0	1	1	1	1	7.15876E-8	8.44232E-8
0.1	0.99005	1.01005	0.99005	1.01005	5.61584E-10	6.59049E-10
0.2	0.960789	1.04081	0.960789	1.04081	9.69923E-10	3.34747E-10
0.3	0.913931	1.09417	0.913931	1.09417	3.5286E-8	4.47131E-8
0.4	0.852144	1.17351	0.852144	1.17351	6.22823E-7	8.00388E-7
0.5	0.778805	1.28402	0.778801	1.28403	4.48153E-6	7.03964E-6

TABLE 8. Numerical solutions for Example 7.6

Error	LMWM	CWM [19]		
	$k=3, M=2$	$k=3, M=2$	$k=4, M=2$	$k=5, M=2$
RMSE	3.92041E-10	2.9700E-7	1.8610E-8	1.1645E-9

TABLE 9. Root mean square errors for Example 7.7

Example 7.8. Again to verify the accuracy of the method here presented, we consider the nonlinear Volterra-Fredholm integro-differential equation (see [6])

$$y'(x) + y(x) + \frac{1}{2}\int_0^x xy^2(t)dt - \frac{1}{4}\int_0^1 ty^3(t)dt = g(x),$$

with $g(x) = 2x + x^2 + \frac{1}{10}x^6 - \frac{1}{32}$ and initial condition $y(0) = 0$. The exact solution of this problem is x^2. This problem has been solved by hybrid Legendre polynomials and Block-Pulse functions (HLPBPF) in [6]. The results obtained using HLPBPF [6] are compared with the results obtained by presented method and cited in Table 10. The maximum absolute errors obtained by these two methods has been cited in Table 11.

x	LMWM	HLPBPF [6]			Exact
	$M=8, k=1$	$M=8, n=2$	$M=8, n=4$	$M=8, n=4$	
0	0	0	0	0	0
0.1	0.01	0.010917	0.010256	0.010031	0.01
0.2	0.04	0.041703	0.040487	0.040075	0.04
0.3	0.09	0.092364	0.090698	0.090171	0.09
0.4	0.16	0.162911	0.160866	0.160094	0.16
0.5	0.25	0.253371	0.250997	0.250228	0.25
0.6	0.36	0.364244	0.361061	0.360502	0.36
0.7	0.49	0.493830	0.490969	0.490583	0.49
0.8	0.64	0.642375	0.640830	0.640374	0.64
0.9	0.81	0.810337	0.810183	0.810047	0.81

TABLE 10. Numerical solutions for Example 7.8

Conclusion. Using the equivalence between the Lane-Emden equations of first and second kind and Volterra integro-differential equations a numerical method

Error	LMWM	HLPBPF [6]		
	$M = 8, k = 1$	$M = 8, n = 2$	$M = 8, n = 4$	$M = 8, n = 8$
Max. Abs. Err.	1.85984E-9	4.244E-3	1.0610E-3	5.83E-4

Table 11. Maximum absolute errors for Example 7.8

that overcomes the difficulty of the singular behavior at $x = 0$ is established. The numerical method is reduced to solving a system of algebraic equations. Examples that demonstrate the validity and applicability of the present technique are included. These examples also exhibit the accuracy and efficiency of the proposed method.

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Prakash Kumar Sahu
National Institute of Technology Rourkela, Department of Mathematics, Rourkela, Odisha-769008, India
E-mail address: prakash.2901@gmail.com

Santanu Saha Ray
National Institute of Technology Rourkela, Department of Mathematics, Rourkela, Odisha-769008, India
E-mail address: santanusaharay@yahoo.com