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MULTIPLE SOLUTIONS FOR AN INDEFINITE KIRCHHOFF-TYPE EQUATION WITH SIGN-CHANGING POTENTIAL

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Abstract. In this article, we study a Kirchhoff-type equation with sign-changing potential on an infinite domain. Using Morse theory and variational methods, we show the existence of two and of infinitely many nontrivial solutions.

1. Introduction and statement of main results

In this article, we study the existence of multiple solutions for the nonlinear Kirchhoff-type equation

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+V(x)u=f(x,u),\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N\geq 2$ and parameters $a>0$, $b\geq 0$ and the potential V satisfies the condition

(A1) $V\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $V(x)\leq \bar V\in(0,\infty)$ for all $x\in\mathbb{R}^N$ and there exists a constant $l_0>0$ such that

$$\int_{\mathbb{R}^N}[|\nabla u|^2+V(x)|u|^2]dx\geq l_0\int_{\mathbb{R}^N}[\bar V-V(x)]|u|^2dx,\quad \forall u\in H^1(\mathbb{R}^N). \tag{1.2}$$

From this condition, we see that $V(x)$ is allowed to be sign-changing and we consider the increasing sequence $\lambda_1\leq\lambda_2\leq\lambda_3\leq\dots$ of minimax values defined by

$$\lambda_n:=\inf_{V\in\mathcal{V}_n}\sup_{u\in V,u\neq 0}\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2+Vu^2\right)dx}{\int_{\mathbb{R}^N}u^2dx}, \tag{1.3}$$

where $\mathcal{V}_n$ denotes the family of n-dimensional subspaces of $C_0^\infty(\mathbb{R}^N)$. Denote $\lambda_\infty=\lim_{n\to\infty}\lambda_n$. Then λ_∞ is the bottom of the essential spectrum of $-\Delta+V$ if it is finite and for every $n\in\mathbb{N}$ the inequality $\lambda_n<\lambda_\infty$ implies that λ_n is an eigenvalue of $-\Delta+V$ of finite multiplicity [23]. Throughout this paper, we assume there exists $k\geq 1$ such that

$$\lambda_k<0<\lambda_{k+1}. \tag{1.4}$$

Problem (1.1) has been widely studied in recent years. For instance, by using a variant version of fountain theorem, Liu and He [14] studied the existence of

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infinitely many high energy solutions of (1.1). Wu [27] investigated the existence of nontrivial solutions and infinitely many high energy solutions of (1.1) via a symmetric mountain pass theorem. Sun and Wu [25] applied variational methods to study the existence and the non-existence of nontrivial solutions of (1.1) and explored the concentration of solutions. Li and Ye [9] considered (1.1) with pure power nonlinearities $f(x,u) = |u|^{p-1}u$ in $\mathbb{R}^3$. By using a monotonicity trick and a new version of global compactness lemma, they verified that the problem has a positive ground state solution which can be viewed as a partial extension of [7] where the authors studied the existence and concentration behavior of positive solutions of (1.1). For other interesting results on the related Kirchhoff equations, we refer to [3, 5, 6, 10, 11, 18, 19, 20, 21, 22, 28, 32, 33] and the references therein.

It is well known that the Morse theory [2] and variational methods [17] are two useful tools in studying the existence and multiplicity of solutions for the variational problem (see, e.g.[8, 12, 13, 24, 26, 31]). However, to the best of our knowledge, there is only one paper [15], in which the authors considered the problem in a domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, dealing with the Kirchhoff-type problem by using Morse theory up to now. Inspired by the above facts, the aim of this paper is to study the multiple solutions of (1.1) with sign-changing potential by using Morse theory and variational methods.

Before stating our main results we need to make some assumptions on the nonlinearity f.

(A2) $f \in C^1(\mathbb{R}^N \times \mathbb{R})$ and there exist $p \in (2, 2^*)$ and $c_1 > 0$ such that

$$|f(x,t)| \le c_1(1 + |t|^{p-1}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.5}$$

(A3) There exists $0 < h < \lambda_\infty$ such that

$$0 < tf(x,t) \le ht^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.6}$$

Our main results read as follows.

Theorem 1.1. *Assume* (A1)–(A3) *hold. Then* (1.1) *has two nontrivial solutions.*

Theorem 1.2. *Assume* (A1)–(A3) *hold and that*

(A4) *$f(x,-u) = -f(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.*

Then (1.1) *has infinitely many nontrivial solutions $\{u_m\}$ with $\|u_m\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $m \to \infty$.*

It should be pointed out that in a large number of the aforementioned references, the authors always employed the variational methods such as mountain pass theorem, fountain theorem, linking theorem or the variant versions of them to study problem (1.1) with positive potential $V(x)$ (see [7, 14, 25, 27, 29, 30]) and they usually obtain that (1.1) has one and infinitely many solutions under some suitable assumptions on f, such as $f(x,t) = o(t)$ as $t \to 0$, $F(x,u)/u^4 \to +\infty$ as $|u| \to \infty$ and (AR) (or variant version (AR)) condition (see, e.g.[10, 22, 27]). However, in this article, we consider the problem (1.1) with sign-changing potential and without the condition $f(x,t) = o(t)$ as $t \to 0$, $F(x,u)/u^4 \to +\infty$ as $|u| \to \infty$ and (AR)-condition, and we can get that the problem (1.1) has two solutions by combining a three points theorem [31] with local linking method. Moreover, when the functional is even we can also prove that the problem (1.1) has infinitely many solutions $\{u_m\}$ with $\|u_m\| \to 0$ as $m \to \infty$ via a variant version of Clark's theorem due to Liu and Wang [16]. This is quite different from the references we cited above.

The remainder of this article is organized as follows. In Section 2, some important preliminaries are presented while the proofs of the main results are given in Section 3.

2. Preliminaries

As usual, let $L^p(\mathbb{R}^N)$ be the standard L^p space for $1 \le p < +\infty$ associated with the norm

$$\|u\|_p = \Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{1/p}, \quad u \in L^p(\mathbb{R}^N),$$

and let $H^1(\mathbb{R}^N)$ be the standard Sobolev space with norm

$$\|u\|_{H^1(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} [|\nabla u|^2 + u^2] dx\Big)^{1/2}, \quad u \in H^1(\mathbb{R}^N). \tag{2.1}$$

Let

$$E = \big\{u | u \in H^1(\mathbb{R}^N), Vu^2 \in L^1(\mathbb{R}^N)\big\}.$$

Corresponding to the eigenvalue λ_k, we let W^- and W^+ be the negative space and positive space of the quadratic form

$$\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2] dx.$$

From (1.4), we deduce that $E = W^- \bigoplus W^+$. For any $u, v \in E$, we define

$$(u, v) = \int_{\mathbb{R}^N} (\nabla \widehat{u}^+ \nabla \widehat{v}^+ + V(x)\widehat{u}^+\widehat{v}^+) dx - \int_{\mathbb{R}^N} (\nabla \widehat{u}^- \nabla \widehat{v}^- + V(x)\widehat{u}^-\widehat{v}^-) dx,$$

where $u = \widehat{u}^+ + \widehat{u}^-$, $v = \widehat{v}^+ + \widehat{v}^-$, $\widehat{u}^+, \widehat{v}^+ \in W^+$ and $\widehat{u}^-, \widehat{v}^- \in W^-$. Then $(\cdot,\cdot)$ is an inner product in E. Therefore, E s a Hilbert space with the norm

$$\|u\| = (u, u)^{1/2} = \big(\|u^+\|^2 - \|u^-\|^2\big)^{1/2}.$$

Furthermore, we have the following result, by Deng, Jin and Peng [4].

Lemma 2.1 ([4]). *Assume that V satisfies* (A1). *Then there exist two positive constants $C_1, C_2 > 0$ such that*

$$C_1\|u\|^2_{H^1(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2] dx \le C_2\|u\|^2_{H^1(\mathbb{R}^N)}, \quad \forall u \in E. \tag{2.2}$$

Recall that $H^1(\mathbb{R}^N)$ is a Hilbert space with the norm (2.1) and is continuous embedded into $L^p(\mathbb{R}^N)$ for any $p \in [2, 2^*]$. By Lemma 2.1, for any $p \in [2, 2^*]$, there exists an imbedding constant $\gamma_s \in (0, \infty)$ such that

$$\|u\|_s \le \gamma_s \|u\|, \quad \forall u \in E. \tag{2.3}$$

From (A3), we can choose $l_0 > 0$ and $\bar{V} \in (h, \lambda_\infty)$ such that $\bar{V} \notin \{\lambda_i | 1 \le i < +\infty\}$ and (1.2) holds. Let E^- be the space spanned by the eigenfunctions with corresponding eigenvalues less than $\bar{V}$. Then, E^- is finite dimensional subspace of E. Let E^+ be the orthogonal complement space of E^- in E. Since E is a Hilbert space, we have $E = E^+ \bigoplus E^-$. So, for every $u \in E$, we have a unique decomposition $u = u^+ + u^-$ with $u^+ \in E^+$ and $u^- \in E^-$.

By $\bar{V} \notin \{\lambda_i | 1 \le i < +\infty\}$ and Lemma 2.1, there exists an equivalent norm of E, still denoted by $\|\cdot\|$, such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(x)u^2 - \bar{V}\int_{\mathbb{R}^N} u^2 = \|u^+\|^2 - \|u^-\|^2. \tag{2.4}$$

Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$.

Definition 2.2 ([2])**.** Let u be an isolated critical point of J with $J(u) = c$, for $c \in \mathbb{R}$, and let U be a neighborhood of u, containing the unique critical point. We call

$$C_q(J,u) := H_q(J^c \cap U, J^c \cap U \setminus \{u\}), \quad q = 0,1,2,\dots,$$

the qth critical group of J at u, where $J^c := \{u \in E : J(u) \le c\}$, $H_q(\cdot,\cdot)$ stands for the qth singular relative homology group with integer coefficients.

We say that u is a homological nontrivial critical point of J if at least one of its critical groups is nontrivial.

Proposition 2.3 ([1])**.** *Let 0 be a critical point of J with $J(0) = 0$. Assume that J has a local linking at 0 with respect to $E = E_1 \oplus E_2$, $m = \dim E_1 < \infty$, that is, there exists $\rho > 0$ small such that*

$$J(u) \le 0, \quad u \in E_1, \|u\| \le \rho, \quad J(u) > 0, \quad u \in E_2,\ 0 < \|u\| \le \rho. \tag{2.5}$$

Then $C_m(J,0) \ncong 0$; that is, 0 is a homological nontrivial critical point of J.

Definition 2.4. We say that $J \in C^1(E,\mathbb{R})$ satisfies (PS)-condition if any sequence $\{u_n\}$ in E such that

$$J(u_n) \to c, \quad J'(u_n) \to 0, \quad \text{as } n \to \infty,$$

has a convergent subsequence.

Proposition 2.5 ([31])**.** *Assume that J satisfies the (PS)-condition and is bounded from below. If J has a critical point that is homological nontrivial and is not the minimizer of J. Then J has at least three critical points.*

Proposition 2.6 ([16])**.** *Let X be a Banach space, $J \in C^1(X,\mathbb{R})$. Assume that J satisfies (PS)-condition, is even and bounded from below, and $J(0) = 0$. If for any $m \in \mathbb{N}$, there exists a k-dimensional subspace X^m of X and $\rho_m > 0$ such that $\sup_{X^m \cap S_{\rho_m}} J < 0$, where $S_{\rho_m} = \{u \in X | \|u\| = \rho_m\}$, then at least one of the following conclusions holds.*

- (i) *There exists a sequence of critical points $\{u_m\}$ satisfying $J(u_m) < 0$ for all m and $\|u_m\| \to 0$ as $m \to \infty$.*
- (ii) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $J(u) = 0$.*

3. Proofs of main results

We begin this section by defining a functional J on E as

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big[a|\nabla u|^2 + V(x)u^2\big] + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 - \int_{\mathbb{R}^N} F(x,u), \tag{3.1}$$

for all $u \in E$, where $F(x,t) = \int_0^t f(x,s)$.

Under assumption (A1), (A2) and (A3), following [27, Lemma 1], it is easy to show that J is a C^1-functional in E, and for all $u,v \in E$, and the derivative of J is given by

$$\langle J'(u), v\rangle = \Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2\Big)\int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} V(x)uv - \int_{\mathbb{R}^N} f(x,u)v. \tag{3.2}$$

Consequently, the critical points of J are the solutions of (1.1).

To complete the proofs, we need the following lemmas.

Lemma 3.1. *Assume that $V(x)$ satisfies* (A1) *and the conditions* (A2) *and* (A3) *hold. Then J is coercive, bounded from below in E.*

Proof. Arguing by contradiction, we suppose that there exists $C > 0$ and $\|u_n\| \to \infty$ such that $J(u_n) \le C$ as $n \to \infty$. For all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, we deduce from (1.6) that

$$\frac{1}{2} h u^2 \ge F(x, u) > 0. \tag{3.3}$$

Now, we choose $h < \bar{V} < \lambda_\infty$ and $l_0 > 0$ such that $\bar{V} \notin \{\lambda_i | 1 \le i < +\infty\}$ and (1.2) holds. Then, applying (2.4), (3.1) and (3.3) yields

$$\begin{aligned} J(u_n) &= \frac{1}{2} \int_{\mathbb{R}^N} [a|\nabla u_n|^2 + V(x) u_n^2 - \bar{V} u_n^2] + \frac{b}{4} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \Big)^2 \\ &\quad + \int_{\mathbb{R}^N} \big[\frac{1}{2} \bar{V} u_n^2 - F(x, u_n) \big] \\ &\ge \frac{1}{2} \min\{a, 1\} \left(\|u_n^+\|^2 - \|u_n^-\|^2 \right) + \frac{b}{4} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \Big)^2 \\ &\ge \frac{1}{2} \min\{a, 1\} \big(\|u_n^+\|^2 - \|u_n^-\|^2 \big). \end{aligned} \tag{3.4}$$

Let $v_n := u_n / \|u_n\|$. By $\|u_n\| \to \infty$, $J(u_n) \le C$ and (3.4), we have

$$\|v_n^+\|^2 \le \|v_n^-\|^2 + o(1). \tag{3.5}$$

Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in E and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. If $v = 0$, then by the finite dimension of E^-, we deduce that $v_n^- \to 0$ in E. This and (3.5) yield $v_n \to 0$ in E. It is a contradiction, because for every n, we have $\|v_n\| = 1$. Therefore, $v^- \ne 0$ and then $v \ne 0$. Then it deduces from Fatou's lemma that

$$\begin{aligned} \liminf_{n \to \infty} \frac{b}{4\|u_n\|^4} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \Big)^2 &= \liminf_{n \to \infty} \frac{b}{4} \Big(\int_{\mathbb{R}^N} |\nabla v_n|^2 \Big)^2 \\ &\ge \frac{b}{4} \Big(\int_{\mathbb{R}^N} |\nabla v|^2 \Big)^2 > 0. \end{aligned} \tag{3.6}$$

Since $\|u_n\| \to \infty$ and $J(u_n) \le C$, we have

$$\|u_n\|^{-4} J(u_n) \to 0, \quad \text{as } n \to \infty. \tag{3.7}$$

Hence, multiplying both sides of the following inequality by $\|u_n\|^{-4}$ and letting $n \to \infty$,

$$J(u_n) \ge \frac{1}{2} \min\{a, 1\} \left(\|u_n^+\|^2 - \|u_n^-\|^2 \right) + \frac{b}{4} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \Big)^2, \tag{3.8}$$

From (3.6) and (3.7) we obtain

$$0 \ge \frac{b}{4} \Big(\int_{\mathbb{R}^N} |\nabla v|^2 \Big)^2 > 0.$$

It is a contradiction. Therefore, we prove that J is coercive in E. Consequently, J is bounded form below in E. The proof is complete. □

Lemma 3.2. *Assume that* (A1)-(A3) *hold. Then J satisfies the* (PS)*-condition.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence, i.e., $J(u_n) \to c$ and $J'(u_n) \to 0$ in E^*, as $n \to \infty$. Lemma 3.1 shows that J is coercive. Then $J(u_n) \to c$ implies that $\{u_n\}$ is bounded. By (3.2) and $J'(u_n) \to 0$, we have

$$\begin{aligned}
&o(\|u_n\|)\\
&= \langle J'(u_n), u_n\rangle\\
&= a\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)u_n^2 + b\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n\\
&\geq \min\{a,1\}\big(\|u_n^+\|^2 - \|u_n^-\|^2\big) + b\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^2\\
&\quad + \int_{\mathbb{R}^N} [hu_n^2 - f(x,u_n)u_n].
\end{aligned} \tag{3.9}$$

Then we deduce from (3.9) that

$$\begin{aligned}
&o(\|u_n\|) + \min\{a,1\}\|u_n^-\|^2\\
&\geq \min\{a,1\}\|u_n^+\|^2 + b\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^2 + \int_{\mathbb{R}^N} [hu_n^2 - f(x,u_n)u_n].
\end{aligned} \tag{3.10}$$

Up to a subsequence, we may assume $u_n \rightharpoonup u$ in E. Then we have that u is a critical point of J. It follows that

$$\begin{aligned}
0 &= \langle J'(u), u\rangle\\
&\geq \min\{a,1\}\left(\|u^+\|^2 - \|u^-\|^2\right) + b\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 + \int_{\mathbb{R}^N} [hu^2 - f(x,u)u],
\end{aligned} \tag{3.11}$$

which implies

$$\begin{aligned}
&\min\{a,1\}\|u^-\|^2\\
&\geq \min\{a,1\}\|u^+\|^2 + b\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 + \int_{\mathbb{R}^N} [hu^2 - f(x,u)u].
\end{aligned} \tag{3.12}$$

Since E^- is a finite dimensional subspace of E, we get $u_n^- \to u^-$, and then $\|u_n^-\|^2 \to \|u^-\|^2$. This together with (3.10) and (3.12) imply

$$\begin{aligned}
&\lim_{n\to\infty} \Big[\min\{a,1\}\|u_n^+\|^2 + b\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^2 + \int_{\mathbb{R}^N} [hu_n^2 - f(x,u_n)u_n]\Big]\\
&= \min\{a,1\}\|u^+\|^2 + b\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 + \int_{\mathbb{R}^N} [hu^2 - f(x,u)u].
\end{aligned} \tag{3.13}$$

An easy calculation, using (A3) and Fatou's lemma, shows that

$$\begin{aligned}
&\liminf_{n\to\infty} \Big[b\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^2 + \int_{\mathbb{R}^N} [hu_n^2 - f(x,u_n)u_n]\Big]\\
&\geq b\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 + \int_{\mathbb{R}^N} [hu^2 - f(x,u)u].
\end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14) gives that $\lim_{n\to\infty} \|u_n^+\|^2 = \|u^+\|^2$. It follows that $u_n \to u$ in E. Thus, we completed the proof. □

Now, we are in a position to calculate the critical groups of J at 0.

Lemma 3.3. *Assume that* (A1)–(A3) *hold. Then there exists* $m \in \mathbb{N}$ *with* $m \geq k$ *such that* $C_m(J,0) \not\cong 0$.

Proof. Let $E_1 = E^-$ and $E_2 = E^+$. Then $m = dim(E^-) \geq k$. On one hand, from (2.3), (3.1), (3.3) and Lemma 2.1, for any $u \in E_1$, we have

$$\begin{aligned} J(u) &\leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 + \frac{1}{2}\bar{V}\int_{\mathbb{R}^N} u^2 \\ &\quad - \int_{\mathbb{R}^N} F(x,u) \\ &\leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2 + \frac{1}{2}\bar{V}\int_{\mathbb{R}^N} u^2 \\ &\leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\Big)^2 + \frac{1}{2}\bar{V}\|u\|_2^2 \\ &\leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + \frac{b}{4}\|u\|^4_{H^1(\mathbb{R}^N)} + \frac{1}{2}\bar{V}\|u\|_2^2 \\ &\leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + C_1\|u\|^4 + C_2\|u\|_2^2. \end{aligned} \tag{3.15}$$

Since E_1 is a finite dimensional subspace and all norms on a finite dimensional space are equivalent, we deduce from (3.15) that

$$J(u) \leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + C_1\|u^-\|^4 - C_2\|u^-\|^2,$$

which implies that $J(u) \leq 0$, if $\|u\|$ small.

On the other hand, for any $u \in E_2$, (3.4) shows that

$$J(u) \geq \frac{1}{2}\min\{a,1\}\|u^+\|^2,$$

which implies that $J(u) > 0$, if $\|u\|$ is small.

The above arguments shows that J has a local linking at 0 with respect to $E = E^- \oplus E^+$. Clearly, it follows from (3.1) that $J(0) = 0$. Therefore, by Proposition 2.3, we get that there exists $m \in \mathbb{N}$ such that $C_m(J,0) \ncong 0$. That is, 0 is a homological nontrivial critical point of J. The proof is complete. $\square$

Proof of Theorem 1.1. From Lemmas 3.1 and 3.2, we know that J is bounded from below and satisfies (PS)-condition. Lemma 3.3 shows that $0 \in E$ is a homologically nontrivial critical point of J but not a minimizer. Then by virtue of Proposition 2.5, we get that problem (1.1) has two nontrivial solutions. The proof is complete. $\square$

Proof of Theorem 1.2. By (A4) and (3.1), one can easily check that functional J is even and satisfies $J(0) = 0$. Lemma 3.1 and Lemma 3.2 show that J is bounded from below in E and satisfies the (PS)-condition. For any $m \in \mathbb{N}$ and $m \geq k$, $\rho_m > 0$, let $S_{\rho_m} = \{u \in X : \|u\| = \rho_m\}$. Then for any $u \in S_{\rho_m}$, it deduces from (A3) that

$$J(u) \leq \frac{1}{2}\max\{a,1\}\left(\|u^-\|^2 - \|u^-\|^2\right) + \frac{b}{4}\|u\|^4_{H^1(\mathbb{R}^N)} + \frac{1}{2}\bar{V}\|u\|_2^2. \tag{3.16}$$

Note that $E^- := X^m$ is a $m-$dimensional subspace of E. Since all norms are equivalent on a finite dimensional space, for $u \in X^m \cap S_{\rho_m}$, it follows from (3.16) that

$$\sup_{X^m \cap S_{\rho_m}} J(u) \leq -\frac{1}{2}\max\{a,1\}\|u^-\|^2 + C_3\|u^-\|^4 - C_4\|u^-\|^2, \tag{3.17}$$

which implies that

$$\sup_{X^m \cap S_{\rho_m}} J(u) < 0,$$

if $\rho_m > 0$ is sufficiently small. Moreover, if there exists $r > 0$ such that for any $0 < a < r$ with $\|u\| = a$, then (3.16) implies that $J(u) \not\equiv 0$. Therefore, by Proposition 2.6, we get that problem (1.1) has infinitely many solutions $\{u_m\}$ such that $\|u_m\| \to 0$, as $m \to \infty$. The proof is complete. $\square$

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