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REMARKS ON THE SHARP CONSTANT FOR THE SCHRÖDINGER STRICHARTZ ESTIMATE AND APPLICATIONS

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Abstract. In this article, we compute the sharp constant for the homogeneous Schrödinger Strichartz inequality, and for the Fourier restriction inequality on the paraboloid in any dimension under the condition conjectured (and proved for dimensions 1 and 2) that the maximizers are Gaussians. We observe also how this would imply a far from optimal, but "cheap" and sufficient, criterion of the global wellposedness in the L^2-critical case $p = 1 + 4/n$.

1. Introduction

Consider the nonlinear Schrödinger equation (NLS for short)

$$i\partial_t u(t,x) + \Delta u(t,x) + \mu|u|^{p-1}u(t,x) = 0 \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{1.1}$$

with initial datum $u(0,x) = u_0(x), \quad x \in \mathbb{R}^n$. Here the space dimension $n \geq 1$, the nonlinearity has $p \geq 1$ and $\mu = -1, 0, 1$ in which cases the equation is said to be *defocusing, linear* and *focusing* respectively.

A lot of research has been done to prove the global wellposedness of the above problem in the scale of Hilbert Spaces $H^s(\mathbb{R}^n)$ (see Section 2 for a precise definition). In the case of regular solutions $s > n/2$, the algebra property of the space $H^s(\mathbb{R}^n)$ makes the proof simpler, while in the case $s \leq n/2$ one needs Strichartz estimates to close the argument (see again Section 2). We refer to [25] for more details and references.

Strichartz estimates were originally proved by Strichartz [22] in the non end-point case and much later for the end-point case by Keel and Tao [17] in the homogeneous case and by Foschi [15] in the inhomogeneous case, following Keel and Tao's approach. After Strichartz's work, a research field opened and Strichartz estimates were proved for a lot of different equations. See [25] and the references therein, for a more complete discussion on Strichartz estimates.

Several mathematicians have then been interested in the problem of the sharpness of Strichartz Inequalities. As far as we know, the first one addressing this problem has been Kunze [20], who proved the existence of a maximizing function for the estimate

$$\|e^{it\partial_x^2}u\|_{L^6_{t,x}(\mathbb{R}^2)} \leq S_h(1)\|u\|_{L^2(\mathbb{R})}$$

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(case of dimension $n = 1$), by means of the concentration compactness principle used in the Fourier space and by means of multilinear estimates due to Bourgain [2]. This method has been first developed by him in relation to a variational problem from nonlinear fiber optics on Strichartz-type estimates [19]. The first author to give explicit values of the sharp Strichartz constants and characterize the maximizers has been Foschi [14], who proved that in dimensions $n = 1$ the sharp constant is $S_h(1) = 12^{-1/12}$, while in dimension $n = 2$ the sharp constant is $S_h(2) = 2^{-1/2}$. He also proved that the maximizer is the Gaussian function $f(x) = e^{-|x|^2}$ (up to symmetries) in both dimensions $n = 1$ and $n = 2$ (see Section 2 below). He moreover conjectured (Conjecture 1.10) that Gaussians are maximizers in every dimension $n \geq 1$. Independently, this result has been reached also by Hundertmark and Zharnitsky in [16] that gave also a conjecture on the value of the Strichartz Constant (Conjecture 1.7). An extension of these results can be found in [4]. A step towards proving Foschi's conjecture has been done by Christ and Quilodán [5], who demonstrated that Gaussians are critical points in any dimension $n \geq 1$. They do not give any conjecture on the explicit value of the sharp Strichartz constant $S_h(n)$ for general dimension n. Duyckaerts, Merle and Roudenko in [11] give an estimate of $S_h(n)$ and also precise asymptotics in the small data regime, but not the explicit value. Here, assuming that Gaussians are actually maximizers, as it is conjectured, and not just critical points, we compute the Strichartz Constant in a setting a little more general than the one of the conjecture of Hundertmark and Zharnitsky [16] and this is the main contribution of the paper.

Theorem 1.1. *Suppose Gaussians maximize Strichartz estimates for any $n \geq 1$. Then, for any $n \geq 1$ and (q, r) admissible pair (see Section 2 below), the* sharp homogeneous Strichartz constant $S_h(n, q, r) = S_h(n, r)$ *defined by*

$$S_h(n,r) := \sup \Big\{ \frac{\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)}}{\|u\|_{L^2_x(\mathbb{R}^n)}} : u \in L^2_x(\mathbb{R}^n), u \neq 0 \Big\}, \tag{1.2}$$

is given by

$$S_h(n,r) = 2^{\frac{n}{4} - \frac{n(r-2)}{2r}} r^{-\frac{n}{2r}}. \tag{1.3}$$

Moreover, if we define $S_h(n) := S_h(n, 2 + 4/n, 2 + 4/n)$ by

$$S_h(n) = \sup \Big\{ \frac{\|u\|_{L^{2+4/n}_{t,x}(\mathbb{R}\times\mathbb{R}^n)}}{\|u\|_{L^2_x(\mathbb{R}^n)}} : u \in L^2(\mathbb{R}^n), u \neq 0 \Big\}, \tag{1.4}$$

then for every $n \geq 1$ we have that

$$S_h(n) = \Big(\frac{1}{2}(1 + \frac{2}{n})^{-n/2}\Big)^{\frac{1}{2+4/n}}; \tag{1.5}$$

$S_h(n)$ is a decreasing function of n and

$$S_h(n) \to \frac{1}{(2e)^{1/2}}, \quad n \to +\infty.$$

For any $n \geq 1$ and $(\tilde{q}, \tilde{r})$ admissible pair, the sharp dual homogeneous Strichartz constant $S_d(q, r, n) = S_d(n, r)$ *is defined by*

$$S_d(n,r) := \sup \Big\{ \| \int_{\mathbb{R}} e^{is\Delta} F(s) ds \|_{L^2_x} \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} : F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n), F \neq 0 \Big\}, \tag{1.6}$$

We have that $S_h(n, r) = S_d(n, r)$.

Remark 1.2. We notice that q and r are not independent since they are an admissible pair. For this reason, q appears in $S(n,r)$ just as a function of r. One could have also expressed the sharp constant as a function of q by

$$S_h(n,q) = 2^{-\frac{1}{q}}\Big(1 - \frac{4}{qn}\Big)^{-1/q+n/4},$$

since $r = \frac{2qn}{nq-4}$ (just plug this expression inside $S_h(n,r)$).

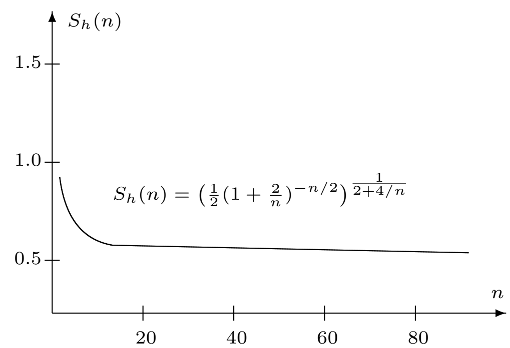

FIGURE 1. Homogeneous Strichartz constant in the case $q = r = 2 + 4/n$, $n \geq 1$.

Remark 1.3. We can see that, for $n = 1$ and $n = 2$, we recover the values of $S_h(n)$ found by Foschi in [14].

Remark 1.4. The asymptotic behavior of $S_h(n)$ basically says that in the non compact case of $\mathbb{R}^n$, the increase of the spatial dimension n allows more dispersion, but the rate of dispersion, measured by the homogeneous Strichartz estimate, does not increase indefinitely. We believe that a similar phenomenon should appear in the case of the Schrödinger equation on the hyperbolic space. We think that it might not be the case for manifolds which become more and more negatively curved with the increase of the dimension, in which case we might observe an indefinitely growing dispersion rate.

The knowledge of the Optimal Strichartz Constant gives a more precise upper bound on the size of the L^2-norm for which the "cheapest argument" (standard Duhamel Principle) gives global wellposedness for (1.1) in the L^2-critical case $p = 1 + 4/n$. From now on we will concentrate on the case $s = 0$ (note $0 < n/2$ for every $n > 0$), namely we will consider just the case in which the initial datum $u_0(x) \in L^2(\mathbb{R}^n)$ and just the case of not *supercritical* nonlinearities $1 < p \leq 1+4/n$. In the *subcritical* case $1 < p < 1 + 4/n$, Tsustsumi [27] proved local wellposedness and also global wellposedness due to the fact that the local time of existence given by his strategy depends just on the L^2-norm of the initial datum and that the NLS have a conservation law at the L^2-regularity ($T_{\rm loc} = T_{\rm loc}(\|u_0\|_{L^2(\mathbb{R}^n)})$). Also in the *critical* case, Tsutsumi proved local wellposedness, thanks to the global bound of the $L^{2(n+2)/n}_{t,x}$ Strichartz Norm (see Section 2), but now the conservation law could not lead to global existence because the local existence time depends on the profile of the solution ($T_{\rm loc} = T_{\rm loc}(u_0)$). The problem of global wellposedness for the NLS, in the L^2-critical case in any dimension, has been solved just recently in a series of papers by Dodson (see [8], [9], [10]). However if the initial datum is "sufficiently small" in L^2_x then one can get global existence with the argument developed in [27], namely by a straight contraction mapping argument. Here, we give a more precise estimate of this "sufficiently small" and so we have the following theorem.

Theorem 1.5. *Consider equation* (1.1) *with initial datum* $u_0(x) \in L^2_x(\mathbb{R}^n)$ *satisfying the following bound*

$$\|u_0(x)\|_{L^2_x} < \frac{1}{S_h(n,r)\alpha}\Big(\frac{1}{S_i(n,r)} - \frac{1}{S_i(n,r)\alpha}\Big)^{n/4} \tag{1.7}$$

with $\alpha = 2$ *if* $n \geq 4$ *and* $\alpha = 1 + n/4$ *for* $1 \leq n \leq 4$*. Here* $S_h(n,r)$ *and* $S_i(n,r)$ *are, respectively, the sharp homogeneous and inhomogeneous Strichartz constants. Then, there is a unique global solution* $u(t,x) \in L^2_x(\mathbb{R}^n)$ *for every* $t \geq 0$.

Remark 1.6. This result reminds a bit what happens in the focusing case, in which there is an upper bound on the size of the L^2-norm of the initial datum for which one can get global well-posedness and condition (1.7) reminds the Gagliardo-Nirenberg Inequality (see [28] and [25]). Anyways, we want to make clear that condition (1.7) is in some sense fictitious and it is not a threshold, since, for example, the results of Dodson [8, 9, 10].

Strichartz inequalities can be set in the more general framework of Fourier restriction inequalities in harmonic analysis. This connection has been made already clear in the original paper of Strichartz [22]. Therefore, Theorem 1.1 can be rephrased in this framework.

Theorem 1.7. *Fix* $n \geq 1$ *and consider the paraboloid* $(\mathbb{P}^n, dP^n)$ *defined in* (5.1) *and* (5.2) *below. Suppose Gaussians maximize the Fourier restriction inequality*

$$\|\widehat{fdP^n}\|_{L_{t,x}^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})} \leq S_h(n)\|f\|_{L^2(\mathbb{P}^n, dP^n)} \tag{1.8}$$

Then, the sharp constant $S_h(n)$ *is*

$$S_h(n) = \Big(\frac{1}{2}\Big(1 + \frac{2}{n}\Big)^{-n/2}\Big)^{\frac{1}{2+4/n}}.$$

The remaining part of the paper is organized as follows. In Section 2, we fix some notation and collect some preliminary results, about the Fourier transform and the fundamental solution for the linear Schrödinger equation, about the Strichartz estimates and their symmetries and the main results in the literature about maximizers for the Strichartz inequality and about the sharp Strichartz constant. In Section 3, we prove Theorem 1.1, while, in Section 4, we prove Theorem 1.5. In Section 5, we discuss the connection between Strichartz and restriction inequalities, proving Theorem 1.7 in Subsection 5. In the appendix, we give some further comments on the inhomogeneous Strichartz estimate and on the wave equation.

2. Preliminaries

By Schwartz functions we mean functions belonging to the function space

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta\},$$

with α and β multi-indices, endowed with the following norm
$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$
Let (X,Σ,μ) be a measure space. For $1\le p\le +\infty$, we define the space $L^p(X)$ of all measurable functions from $f: X\to\mathbb{C}$ such that
$$\|f\|_{L^p(X)} := \Big(\int_X |f|^p \, d\mu\Big)^{1/p} < \infty.$$
Consider $f:\mathbb{R}^n\to\mathbb{C}$ a Schwartz function in space and $F(t,x):\mathbb{R}\times\mathbb{R}^n\to\mathbb{C}$ a Schwartz function in space and time. We will use the following notation (and constants) for the space Fourier transform
$$\hat f(t,\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$
and for the Inverse space Fourier transform
$$f(x) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)d\xi,$$
and the following for the space-time Fourier transform
$$\mathcal{F}(F)(\tau,\xi) := \frac{1}{(2\pi)^{\frac{n+1}{2}}}\int_{\mathbb{R}^n} e^{-it\tau-ix\cdot\xi} f(t,x)\,dx\,dt$$
and the Inverse space-time Fourier transform
$$F(t,x) := \frac{1}{(2\pi)^{\frac{n+1}{2}}}\int_{\mathbb{R}^{n+1}} e^{it\tau+ix\cdot\xi}\mathcal{F}(\tau,\xi)d\xi d\tau.$$
By means of the Fourier transform, we can finally define H^s-spaces as the set of functions such that
$$\|u\|_{H^s(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |\hat u(\xi)|^2(1+|\xi|)^{2s}\Big)^{1/2} < +\infty.$$

2.1. **Fourier transform and fundamental solutions for linear Schrödinger equations.** In this subsection we solve the linear Schrödinger equation
$$i\partial_t u(t,x) = \Delta u(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R}^n, \tag{2.1}$$
with initial datum $u_0(x) = e^{-|x|^2}\in\mathcal{S}(\mathbb{R}^n)$. These computations are well known, but we will rewrite them here in order to clarify what we will compute in the next sections. Since $u_0(x)\in\mathcal{S}(\mathbb{R}^n)$, then also $\partial_t u(t,x)\in\mathcal{S}(\mathbb{R}^n)$ and $\Delta u(t,x)\in\mathcal{S}(\mathbb{R}^n)$. So we can apply the Fourier transform to both sides of (2.1) and get:
$$i\hat u_t = -|\xi|^2\hat u,$$
whose solution is
$$\hat u(\xi,t) = e^{i|\xi|^2 t}\hat u(\xi,0).$$
So we just need to compute the Fourier transform of the initial datum and then the inverse Fourier transform of $\hat u(t,\xi)$ to get the explicit form of the solution.
$$\begin{aligned}\hat u(0,\xi) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(0,x)dx\\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}e^{-|x|^2}dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-(|x|^2 + ix\cdot\xi - |\xi|^2/4)} e^{-|\xi|^2/4} dx \\
&= \frac{e^{-|\xi|^2/4}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-i\xi/2|^2} dx,
\end{aligned}
$$

by using contour integrals. We notice that, with a simple change of variables, we have:

$$
2^{n/2} \int_{\mathbb{R}^n} e^{-|x-i\xi/2|^2} dx = 2^{n/2} \int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_{\mathbb{R}^n} e^{-|x|^2/2} dx = (2\pi)^{n/2}.
$$

Hence

$$
\hat{u}(0,\xi) = \frac{e^{-|\xi|^2/4}}{(2\pi)^{n/2}} \pi^{n/2} = \frac{e^{-|\xi|^2/4}}{2^{n/2}}.
$$

With this we can conclude that

$$
\begin{aligned}
u(t,x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i|\xi|^2 t + ix\cdot\xi} \frac{e^{-|\xi|^2/4}}{2^{n/2}} \\
&= \frac{1}{2^n} \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|\xi|^2(1/4 - it) + ix\dot{\xi}} d\xi \\
&= \frac{1}{2^n} \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-(|\xi|^2(1/4-it) - ix\dot{\xi} - |x|^2/(1-4it))} e^{-|x|^2/(1-4it)} d\xi \\
&= \frac{1}{2^n} \frac{1}{\pi^{n/2}} e^{-|x|^2/(1-4it)} \int_{\mathbb{R}^n} e^{-|\xi\sqrt{1/4-it} + ix/(\sqrt{1-4it})|^2} d\xi.
\end{aligned}
$$

Now we make the change of variables $\eta = \xi\sqrt{1/4 - it} + ix/(\sqrt{1-4it})$ to get

$$
\begin{aligned}
u(t,x) &= \frac{1}{2^n} \frac{1}{\pi^{n/2}} e^{-|x|^2/(1-4it)} \int_{\mathbb{R}^n} e^{-|\eta|^2} (1/4 - it)^{-n/2} d\eta \\
&= \frac{1}{2^n} \frac{1}{\pi^{n/2}} e^{-|x|^2/(1-4it)} (1/4 - it)^{-n/2} \pi^{n/2} \\
&= (1-4it)^{-n/2} e^{-\frac{|x|^2}{1-4it}}
\end{aligned}
$$

Hence

$$
u(t,x) = (1-4it)^{-n/2} e^{-\frac{|x|^2}{1-4it}}. \tag{2.2}
$$

Strichartz estimates and their symmetries. In this subsection, we state the Strichartz estimates for the Schrödinger equation, since they are the main topic of the present paper and it will help to clarify the statement of our main theorems.

Definition 2.1. Fix $n \geq 1$. We call a set of exponents (q,r) admissible if $2 \leq q, r \leq +\infty$ and

$$
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.
$$

Proposition 2.2 ([17, 15, 22])**.** *Suppose $n \geq 1$. Then, for every (q,r) and $(\tilde{q},\tilde{r})$ admissible and for every $u_0 \in L^2_x(\mathbb{R}^n)$ and $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n)$, the following hold:*

- *the* homogeneous Strichartz estimates

$$
||e^{-it\Delta} u_0||_{L^q_t L^r_x} \leq S_h(n,q,r) \|u_0\|_{L^2_x};
$$

- *the* dual homogeneous Strichartz estimates

$$\Big\| \int_{\mathbb{R}} e^{is\Delta} F(s) ds \Big\|_{L^2_x} \le S_d(n,q,r) \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}};$$

- *the* Inhomogeneous Strichartz estimates

$$\Big\| \int_{s<t} e^{-i(t-s)\Delta} F(s) ds \Big\|_{L_t^q L_x^r} \le S_i(n,q,r,\tilde{q},\tilde{r}) \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

As explained for example in [14], Strichartz estimates are invariant by the following set of symmetries.

Lemma 2.3 ([14]). *Let $\mathcal{G}$ be the group of transformations generated by:*

- *space-time translations: $u(t,x) \mapsto u(t+t_0, x+x_0)$, with $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$;*
- *parabolic dilations: $u(t,x) \mapsto u(\lambda^2 t, \lambda x)$, with $\lambda > 0$;*
- *change of scale: $u(t,x) \mapsto \mu u(t,x)$, with $\mu > 0$;*
- *space rotations: $u(t,x) \mapsto u(t, Rx)$, with $R \in SO(n)$;*
- *phase shifts: $u(t,x) \mapsto e^{i\theta} u(t,x)$, with $\theta \in \mathbb{R}$;*
- *Galilean transformations:*

$$u(t,x) \mapsto e^{\frac{i}{4}\Big(|v|^2 t + 2v\cdot x\Big)} u(t, x+tv),$$

with $v \in \mathbb{R}^n$.

Then, if u solves equation (2.1) and $g \in \mathcal{G}$, also $v = g \circ u$ solves equation (2.1). Moreover, the constants $S_h(n,q,r)$, $S_d(n,q,r)$ and $S_i(n,q,r,\tilde{q},\tilde{r})$ are left unchanged by the action of $\mathcal{G}$.

Remark 2.4. For Strichartz estimates for different equations and different regularities, we refer to [25].

Previous results on sharp Strichartz constant and maximizers. Here we collect the results concerning the optimization of Strichartz inequalities that we need for the next sections. For a broader discussion, we refer to [26] and the references therein.

Proposition 2.5 ([20, 5, 14]). *For any $n \ge 1$ and (q,r) admissible pair, we define $S_h(n) := S_h(n, 2+4/n, 2+4/n)$ by*

$$S_h(n) := \sup\Big\{ \frac{\|u\|_{L^{2+4/n}_{t,x}(\mathbb{R}\times\mathbb{R}^n)}}{\|u\|_{L^2(\mathbb{R}^n)}} : u \in L^2(\mathbb{R}^n), u \ne 0 \Big\}. \tag{2.3}$$

Then we have the following results:

- *Radial Gaussians are critical points of the homogeneous Strichartz inequality in any dimension $n \ge 1$ for all admissible pairs $(q,r) \in (0,+\infty) \times (0,+\infty)$;*
- *The explicit sharp Strichartz constants $S_h(n)$ can be computed explicitly in dimension $n = 1$: $S_h(1) = 12^{-1/12}$; and dimension $n = 2$: $S_h(2) = 2^{-1/2}$. Moreover, in both the cases $n = 1$ and $n = 2$, the maximizers are Gaussians.*

3. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. We assume, as conjectured, that radial Gaussians are mazimizers and not just critical points as proved in [5]. So we will take $u_0(x) = e^{-|x|^2}$. By Lemma 2.3, the choice of the Gaussian is done without loss of generality. We start to compute the L^2-norm of the initial datum and so of the solution:

$$\begin{aligned} \|u(t,x)\|_{L^2_x} &= \|u_0(x)\|_{L^2_x} = \Big(\int_{\mathbb{R}^n} e^{-2|x|^2} dx\Big)^{1/2} \\ &= \Big(\int_{\mathbb{R}^n} e^{-2|x|^2/4} 2^{-n} dy\Big)^{1/2} \\ &= 2^{-n/2}\Big(\int_{\mathbb{R}^n} e^{-|x|^2/2} dy\Big)^{1/2} \\ &= 2^{-n/2}(2\pi)^{n/4} = \Big(\frac{\pi}{2}\Big)^{n/4} \end{aligned}$$

by similar computations as in Subsection 2.1.

Now we compute the $L^q_t L^r_x$-norm of the linear solution

$$u(t,x) = (1-4it)^{n/2} e^{-\frac{|x|^2}{1-4it}}.$$

First

$$\begin{aligned} |u(t,x)|^r &= |1-4it|^{-rn/2} |e^{-\frac{|x|^2}{1-4it}}|^r \\ &= |1+16t^2|^{-rn/4} |e^{-\frac{(1+4it)|x|^2}{1+16t^2}}|^r \\ &= |1+16t^2|^{-rn/4} e^{-\frac{r|x|^2}{1+16t^2}}. \end{aligned}$$

Then

$$\|u(t,x)\|^r_{L^r_x} = |1+16t^2|^{-rn/4} \int_{\mathbb{R}^n} e^{-\frac{r|x|^2}{1+16t^2}} dx$$

By the change of variable $y = r^{1/2}(1+16t^2)^{-1/2}$ and hence $dy = r^{n/2}x(1+16t^2)^{-n/2}dx$, we get

$$\|u(t,x)\|^r_{L^r_x} = |1+16t^2|^{n/2-rn/4} r^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2} dy = |1+16t^2|^{n/2-rn/4} r^{-n/2} \pi^{n/2},$$

which implies

$$\|u(t,x)\|_{L^r_x} = |1+16t^2|^{n/(2r)-n/4} r^{-n/(2r)} \pi^{n/(2r)}.$$

Now we have to take the L^q_t-norm of what we obtained:

$$\|u(t,x)\|_{L^q_t L^r_x} = \Big(\int_{\mathbb{R}^n} \|u(t,x)\|^q_{L^r_x}\Big)^{1/q}$$

which means, since (q,r) is an admissible pair (and so $q = 4r/[n(r-2)]$), that

$$\|u(t,x)\|_{L^q_t L^r_x} = \Big(\int_{\mathbb{R}^n} \|u(t,x)\|^{\frac{4r}{n(r-2)}}_{L^r_x}\Big)^{\frac{n(r-2)}{4r}} = \Big[\int_{\mathbb{R}} |1+16t^2|^{-1}\Big]^{\frac{n(r-2)}{4r}} \Big(\frac{\pi}{r}\Big)^{n/(2r)},$$

since $(n/(2r) - n/4)q = -1$. Now by a simple change of variable inside the integral $(4t = s)$ we get:

$$\|u(t,x)\|_{L^q_t L^r_x} = \Big(\frac{\pi}{r}\Big)^{\frac{n}{2r}} \Big(\frac{\pi}{4}\Big)^{\frac{n(r-2)}{4r}}.$$

Putting everything together we get the equation:

$$S(n,r)\Big(\frac{\pi}{2}\Big)^{n/4}=\Big(\frac{\pi}{r}\Big)^{\frac{n}{2r}}\Big(\frac{\pi}{4}\Big)^{\frac{n(r-2)}{4r}}$$

and so

$$S(n,r)=2^{\frac{n}{4}-\frac{n(r-2)}{2r}}r^{-\frac{n}{2r}}.$$

In the case $q=r=2+4/n$ one gets

$$\|u(t,x)\|^q_{L^q_{t,x}}=q^{-n/2}\pi^{n/2}\int_{\mathbb{R}}|1+16t^2|^{-1}=\pi^{n/2}(2+4/n)^{-n/2}\frac{\pi}{4}.$$

Putting all the information together we obtain

$$2^{-2}\pi^{1+n/2}(2+4/n)^{-n/2}=S_h(n)^{2+4/n}(\pi/2)^{1+n/2}$$

and solving for $S_h(n)$ one gets

$$S_h(n)=\Big(\frac{1}{2}\Big(1+\frac{2}{n}\Big)^{-n/2}\Big)^{\frac{1}{2+4/n}}$$

Now we have to prove that $S_h(n)$ is a decreasing function of n, namely we have to prove that

$$\Big(\frac{1}{2}\Big(1+\frac{2}{n+1}\Big)^{-(n+1)/2}\Big)^{\frac{1}{2+4/(n+1)}}=S_h(n+1)\le S_h(n)=\Big(\frac{1}{2}\Big(1+\frac{2}{n}\Big)^{-n/2}\Big)^{\frac{1}{2+4/n}}.$$

Taking the natural logarithm to both sides and using the fact that the logarithm is a monotone increasing function of his argument we obtain

$$\begin{aligned}&\frac{1}{2+4/(n+1)}\Big[-\log(2)-\frac{n+1}{2}\log(1+2/(n+1))\Big]\\&\le\frac{1}{2+4/n}\Big[-\log(2)-\frac{n}{2}\log(1+2/n)\Big].\end{aligned}$$

We can easily see that

$$\frac{-\log(2)}{2+4/(n+1)}\le\frac{-\log(2)}{2+4/n},$$

so it remains to prove that

$$\frac{1}{2+4/(n+1)}\Big[-\frac{n+1}{2}\log(1+2/(n+1))\Big]\le\frac{1}{2+4/n}\Big[-\frac{n}{2}\log(1+2/n)\Big].$$

Changing variables to $x:=(n+1)/2$ and $y:=n/2$ leads to

$$\frac{x\log(1+1/x)}{1+1/x}\ge\frac{y\log(1+1/y)}{1+1/y}$$

and changing variables again $\alpha:=1+1/x>1$ and $\beta:=1+1/y>1$ we remain with

$$\frac{\log(\alpha)}{\alpha(\alpha-1)}\ge\frac{\log(\beta)}{\beta(\beta-1)}.$$

So now it remains to show that the function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(t)=\frac{\log(t)}{t(t-1)},$$

is decreasing in t and this would lead to the conclusion since $\alpha<\beta$. Computing its derivative $f'(t)$ one gets

$$f'(t)=\frac{t-1-\log(t)(2t-1)}{t^2(t-1)^2}.$$

We have to verify the inequality just for $t \geq 1$. We define then

$$g(t) = \log(t) - \frac{t-1}{2t-1}$$

and compute its derivative:

$$g'(t) = \frac{(2t-1)^2 - t}{t(2t-1)^2}$$

and so we can see (remember $t \geq 1$) that $g'(t) \leq 0$ if and only if $t \leq 1$, and $g'(1) = 0$, so $t = 1$ is a minimum. $g(1) = 0$ and then positive. So, going backwards with the computations, the inequality $S_h(n+1) < S_h(n)$ is verified.

Now we have to prove the asymptotic behavior and this is easy:

$$\lim_{n\to+\infty} S(n) = \lim_{n\to+\infty} \Big(\frac{1}{2}(1+\frac{2}{n})^{-n/2}\Big)^{\frac{1}{2+4/n}} = \lim_{n\to+\infty} 2^{-1/2} 1/e^{\frac{1}{2+4/n}} = \frac{1}{\sqrt{2e}}.$$

It remains to prove the equivalence between the homogeneous and the dual constant. It basically comes from a duality argument. Define $Tu := e^{it\Delta}u$. Then for every $f \in L^2_x$ an $F \in L^q_t L^r_x$ we have

$$|\langle f, T^*F\rangle| = |\langle Tf, F\rangle| \leq \|Tf\|_{L^q_t L^r_x} \|F\|_{L^{q'}_t L^{r'}_x} \leq S_h \|f\|_{L^2_x} \|F\|_{L^{q'}_t L^{r'}_x}.$$

So

$$\|T^*F\|_{L^2_x} := \sup_{f\in L^2_x} \frac{|\langle f, T^*F\rangle|}{\|f\|_{L^2_x}} \leq S_h \|F\|_{L^{q'}_t L^{r'}_x},$$

hence $S_d \leq S_h$. Analogously,

$$|\langle Tf, F\rangle| = |\langle f, T^*F\rangle| \leq \|f\|_{L^2_x} \|T^*F\|_{L^2_x} \leq S_d \|f\|_{L^2_x} \|F\|_{L^{q'}_t L^{r'}_x}.$$

So

$$\|Tf\|_{L^q_t L^r_x} := \sup_{F\in L^{q'}_t L^{r'}_x} \frac{|\langle Tf, F\rangle|}{\|F\|_{L^{q'}_t L^{r'}_x}} \leq S_d \|f\|_{L^2_x},$$

hence $S_h \leq S_d$ and so we get $S_h = S_d$. This concludes the proof of the theorem.

4. Proof of Theorem 1.5

Here we will give the proof of Theorem 1.5. We will skip some of the details because standard in the theory of global wellposedness for the NLS. We refer to [25] for some of the details skipped. We consider equation (1.1):

$$i\partial_t u(t,x) + \Delta u(t,x) + \mu|u|^{p-1}u(t,x) = 0 \quad (t,x) \in (0,\infty)\times\mathbb{R}^n, \tag{4.1}$$

with initial datum $u(0,x) = u_0(x)$, space dimension is $n \geq 1$, $p \geq 1$ in both the focusing and defocusing case: $\mu = -1, 1$, since we are dealing with a small data analysis. By Duhamel Principle we define

$$Lu := \chi(t/T)e^{-it\Delta}u_0(x) - i\mu\chi(t/T)\int_0^t e^{-i(t-s)\Delta}|u(s,x)|^{p-1}u(s,x)ds, \tag{4.2}$$

where $T > 0$ and $\chi(r)$ is a smooth cut-off function supported on $-2 \leq r \leq 2$ and such that $\chi(r) = 1$ on $-1 \leq r \leq 1$. Using Duhamel formula, we take the $L^q_t L^r_x$-norm of Lu (from now on, unless specified, $t \in [-T,T]$ in the definition of $L^q_t L^r_x$-norm), and get

$$\begin{aligned}\|Lu\|_{L^q_t L^r_x} &\leq S_h(n,r)\|u\|_{L^2_x} + S_i(n,r)\|u\|^p_{L^{\tilde{q}'p}_t L^r_x}\\ &\leq S_h(n,r)\|u\|_{L^2_x} + S_i(n,r)T^{1/(\tilde{q}')-p/q}\|u\|^p_{L^q_t L^r_x}\end{aligned}$$

choosing $\tilde{r}'p = r$.

Now we need to do numerical considerations. Since (q, r) and $(\tilde{q}, \tilde{r})$ are admissible pairs: $2/q + n/r = n/2$, $2/\tilde{q} + n/\tilde{r} = n/2$. Moreover, since we are in the L^2-critical case we can choose $\tilde{r}'p = r$ and $\tilde{q}'p = q$, having still some freedom on the choice of (q, r) as it can be seen by the following lemma. The conditions on (q, r) and $(\tilde{q}, \tilde{r})$ can be rewritten as a system of linear equations in $(1/q, 1/\tilde{q}, 1/r, 1/\tilde{r})$.

Lemma 4.1. *There exist infinite many solutions to the system $Se = N$, where*

$$S = \begin{pmatrix} 2 & 0 & n & 0 \\ 0 & 2 & 0 & n \\ 0 & 0 & p & 1 \\ p & 1 & 0 & 0 \end{pmatrix},$$

$E = (1/q, 1/\tilde{q}, 1/r, 1/\tilde{r})^T$ *and* $N = (n/2, n/2, 1, 1)^T$, *if and only if* $p = 1 + 4/n$. *If* $p \neq 1 + 4/n$ *the system has no solutions.*

Remark 4.2. Basically this lemma implies that, using the estimates that we have used above in the H^s-scale, we cannot remove a power of T in front of the nonlinear term in the *subcrtical* (good) and *supercritical* (bad) cases.

Proof. We can see that $\det(S) = 0$ and $\operatorname{rank}(S) = 3$, because the upper-left 3×3 matrix is not singular for $p \neq 0$. If $p \neq 1 + 4/n$, then $\operatorname{rank}([S, N]) = 4$, so the system has no solutions, while for $p = 1 + 4/n$, $\operatorname{rank}([S, N]) = 3$ and so the system has infinite solutions. $\square$

Remark 4.3. Similar computations can be done for any regularity s, and with nonlinear exponent $p(s) = 1 + 4/(n - 2s)$. The *critical* case $\tilde{q}'p = q$ is the interesting one for us, because in the *subcritical* case $\tilde{q}'p < q$ one can shrink the interval, since $T^{1/(\tilde{q}') - p/q}$ appear with a positive power, and so does not really need to do a small data theory.

Now we will see how big the datum can be in order to have a "cheap" contraction with only the estimates done above. Define $R := \alpha S_h(n, r)\|u_0\|_{L^2_x}$ and

$$B_R := \{u \in L^q_t L^r_x : \|u\|_{L^q_t L^r_x} \leq R\}.$$

Choose also $\beta > 0$ such that

$$S_i(n, r)R^{p-1} < 1/\beta.$$

With these choices we get

$$\begin{aligned} \|Lu\|_{L^q_t L^r_x} &\leq S_h(n, r)\|u\|_{L^2_x} + S_i(n, r)T^{1/(\tilde{q}') - p/q}\|u\|^p_{L^q_t L^r_x} \\ &\leq R(1/\alpha + 1/\beta) \leq R \end{aligned}$$

for every $1/\alpha + 1/\beta \leq 1$ and with $1/\alpha + 1/\beta = 1$ in the less restrictive case. So the Duhamel operator L sends the balls B_R into themselves if $\|u_0\|_{L^2_x}$ is small enough, more precisely when

$$\|u_0\|_{L^2_x} = \frac{R}{S_h(n)\alpha}.$$

This implies that

$$S_i(n, r)\big(\alpha S_h(n, r)\|u\|_{L^2_x}\big)^{p-1} < 1/\beta,$$

which means

$$\|u\|_{L^2_x} < \frac{1}{S_h(n, r)\alpha}\Big(\frac{1}{\beta S_i(n, r)}\Big)^{1/(p-1)}.$$

Using our hypotheses on p, α, β we obtain

$$\|u\|_{L^2_x} < \frac{1}{S_h(n,r)\alpha}\Big(\frac{1}{S_i(n,r)} - \frac{1}{S_i(n,r)\alpha}\Big)^{n/4}. \tag{4.3}$$

For now, the only restriction on α is $1/\alpha + 1/\beta \leq 1$.

Remark 4.4. The coefficients α and β are almost conjugate exponents, suggesting an orthogonal decomposition of the solution on the linear flow and on the nonlinear one.

Now we check that the operator Lu is a contraction. Let

$$u(t) = e^{-it\Delta}u_0 - i\mu\int_0^t e^{-i(t-s)\Delta}|u(s)|^{p-1}u(s)ds, \tag{4.4}$$

$$v(t) = e^{-it\Delta}u_0 - i\mu\int_0^t e^{-i(t-s)\Delta}|v(s)|^{p-1}v(s)ds. \tag{4.5}$$

be two solutions of (4.1). Then

$$\begin{aligned}
\|Lu - Lv\|_{L^q_tL^r_x} &= \Big\|\int_0^t e^{-i(t-s)\Delta}\left(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\right)ds\Big\|_{L^q_tL^r_x} \\
&\leq S_i(n,r)\||u|^{p-1}u - |v|^{p-1}v\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x} \\
&\leq S_i(n,r)\Big(\|u\|^{p-1}_{L^q_tL^r_x} + \|v\|^{p-1}_{L^q_tL^r_x}\Big)\|u - v\|_{L^q_tL^r_x}
\end{aligned}$$

in the above choice of exponents (q,r) and $(\tilde{q},\tilde{r})$. This implies:

$$\|Lu - Lv\|_{L^q_tL^r_x} \leq 2S_i(n)R^{p-1}\|u - v\|_{L^q_tL^r_x} < 2/\beta\|u - v\|_{L^q_tL^r_x},$$

so we need $2/\beta \leq 1$, namely $\beta \geq 2$ and so $1 \leq \alpha \leq 2$, since $1/\alpha + 1/\beta \leq 1$. This is the last restriction on α that we need to apply to the estimate (4.3). We remark here that (4.3) holds for every $1 \leq \alpha \leq 2$ and so we are allowed to take the maximum on both sides of (4.3). Notice also that the left hand side of (4.3) does not depend on α.

Remark 4.5. To have a contraction the ball needs to be big enough, but not that much namely $S_h(n,r)\|u\|_{L^2_x} \leq R \leq 2S_h(n,r)\|u\|_{L^2_x}$.

Now we want to optimize on $\|u_0\|_{L^2_x}$, namely we want to take it as big as possible, maintaining the property of Lu of being a contraction. In other words we have to find the maximum of the function

$$F_n(\alpha) = \frac{1}{\alpha}\Big(1 - \frac{1}{\alpha}\Big)^{n/4},$$

when $\alpha \in [1,2]$. Taking the derivative, we get

$$F_n'(\alpha) = -\alpha^{-2-n/4}(\alpha - 1)^{n/4-1}\big(-(1 + n/4)(\alpha - 1) + \alpha n/4\big).$$

So $F_n'(\alpha) \geq 0$ if and only if

$$1 \leq \alpha \leq 1 + n/4.$$

In particular when $n \geq 4$, $\alpha_{\max} = 2$ and when $n \leq 4$, $\alpha_{\max} = 1 + n/4$. This concludes the proof of Theorem 1.5.

Remark 4.6. The coefficient $\alpha = 2$ is not always the optimal one, as it is usually used in every exposition on the topic. The optimal α depends on the dimension n. We can compute explicitly the values of $F_n(\alpha_{\max})$ in any dimension: for $n = 1$ $F_n(\alpha_{\max}) = F_1(5/4) = 5^{-5/4}4$, for $n = 2$, $F_n(\alpha_{\max}) = F_2(3/2) = 3^{-3/2}2$, for $n = 3$, $F_n(\alpha_{\max}) = F_3(7/4) = 3^{3/4}7^{-7/4}4$ and for $n \geq 4$, $F_n(\alpha_{\max}) = 2^{-1-n/4}$.

5. Applications to Fourier restriction inequalities

Strichartz inequalities can be set in the more general framework of Fourier restriction inequalities in Harmonic Analysis. This connection has been made clear already in the original paper of Strichartz [22]. In this section we will highlight this relationship in the Schrödinger/paraboloid case and we will see how to prove Theorem 1.7. For the case of different flows and hypersurfaces, like the Wave/Cone or Helmholtz/Sphere cases, we refer to [26] and the references therein for more details.

Consider a function $f \in L^1(\mathbb{R}^n)$, then its Fourier transform $\hat{f}$ is a bounded and continuous function on all $\mathbb{R}^n$ and it vanishes at infinity. So $\hat{f}|_{\mathcal{S}}$, the restriction of $\hat{f}$ to a set $\mathcal{S}$ is well defined even if $\mathcal{S}$ has measure zero, like, for example, if $\mathcal{S}$ is a hypersurface. It becomes then interesting to understand what happens if $f \in L^p(\mathbb{R}^n)$ for $1 < p < 2$. From Hausdorff-Young inequality, we can see that if $f \in L^p(\mathbb{R}^n)$ then $\hat{f} \in L^{p'}(\mathbb{R}^n)$ with $1/p + 1/p' = 1$, so $\hat{f}$ can be naturally restricted to any set $\mathcal{A}$ of positive measure. It turns out that a big role is played by the geometry of the set $\mathcal{S}$. Stein proved that if the set $\mathcal{S}$ is sufficiently smooth and its curvature is big enough (in fact it is not true for hyperplanes), then it makes sense to talk about $\hat{f}|_{\mathcal{S}}$ belonging to L^p-spaces.

Proof of Theorem 1.7. From now on we will focus on the case where the hypersurface is the paraboloid $\mathcal{S} = \mathbb{P}^n$, where $\mathbb{P}^n$ is defined as

$$\mathbb{P}^n := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : -\tau = |\xi|^2\} \tag{5.1}$$

and is endowed with the measure dP^n that is given by

$$\int_{\mathbb{P}^n} h(\tau, \xi) dP^n = \int_{\mathbb{R}^n} h(-|\xi|^2, \xi) d\xi. \tag{5.2}$$

(here h is a Schwartz function) and induced by the embedding $\mathbb{P}^n \hookrightarrow \mathbb{R}^{n+1}$. To prove the theorem, we have just to show the equivalence of Restriction Inequalities and Strichartz Inequalities.

It makes sense to talk about a restriction, if $\hat{f}|_{\mathcal{S}}$ is not infinite almost everywhere and a *restriction estimate* holds:

$$\|\hat{f}|_{\mathbb{P}^n}\||_{L^q(\mathbb{P}^n, dP^n)} \leq \|f\|_{L^p(\mathbb{R}^n)},$$

for some $1 \leq q < \infty$ and for every Schwartz function f. This last estimate is equivalent, by a duality argument and Parseval Identity, to

$$\|\mathcal{F}^{-1}(\hat{F} dP^n)|_{\mathbb{P}^n}\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^{q'}(\mathbb{P}^n, dP^n)},$$

for all Schwartz functions F on $\mathbb{P}^n$ and where

$$\mathcal{F}^{-1}(\hat{F} dP^n)(t, x) = \int_{\mathbb{P}^n} e^{ix\xi + it\tau} \hat{F}(\tau, \xi) d\tau dP^n(\tau, \xi)$$

is the inverse space-time Fourier transform of the measure $\hat{F}dP^n$. The dual formulation connects directly to the fundamental solution (2.2)

$$u(t,x) = (1-4it)^{-n/2} e^{-\frac{|x|^2}{1-4it}}$$

of equation (2.1)

$$i\partial_t u(t,x) = \Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n.$$

Since u can be rewritten in the form

$$u = \mathcal{F}^{-1}(\hat{u}_0 dP^n).$$

In this way the homogeneous Strichartz inequality

$$\|e^{it\Delta} u_0\|_{L^q_t L^r_x} \leq S_h(n,q,r) \|u_0\|_{L^2_x},$$

for $q = r = 2 + 4/n$, as in this present case, can be rewritten as

$$\|\widehat{fdP^n}\|_{L_{t,x}^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})} \leq S_h(n) \|f\|_{L^2(\mathbb{P}^n, dP^n)} \tag{5.3}$$

where

$$S_h(n) = \Big(\frac{1}{2}\Big(1+\frac{2}{n}\Big)^{-n/2}\Big)^{\frac{1}{2+4/n}}.$$

This proves Theorem 1.7.

Remark 5.1. We notice that results for the paraboloid seem easier to obtain than for example for the sphere. For example there is not yet the counterpart of [5] in the wave/sphere case and we do not have a conjecture on the sharp Strichartz constant in general dimension in the case of the wave equation.

Remark 5.2. As we said above, the connection between restriction theorems and PDE links a much broader class of hypersurfaces and PDEs. For more details on the more recent results, we refer to [5, 6, 7, 12, 13, 26] for a survey on restriction theorems.

Remark 5.3. In some of the proof of the existence of maximizers for restriction inequalities it has been crucial the Hilbert structure. See for example [12] and [5]. Here we are in L^2_x and so a Hilbert case, but our analysis is not touched by this problem, because we are interested in the optimal constants and not on the extremizers.

6. Comments on the inhomogeneous case and the wave equation

In this section, we want to share some comments and computations on the inhomogeneous Strichartz estimate and on the case of the wave equation. We will not prove any theorem, but we will highlight some difficulties and make some remarks.

Inhomogeneous Strichartz constant S_i. By the TT^* principle (take $Tu := e^{it\Delta}$) and by duality, the homogeneous Strichartz and the dual Strichartz inequality are equivalent. By the same principle one can prove that the operator $TT^* : L^q_t L^r_x \to L^{\tilde{q}'}_r L^{\tilde{r}'}_x$ is bounded if and only if the operator $T : L^2_x \to L^q_t L^r_x$ is bounded. Unfortunately, the inhomogeneous Strichartz inequality cannot be seen as such a composition because it involves the retarded operator. This does not prevent the retarded operator to keep the boundedness properties of TT^* but it complicates a lot the computation of $S_i(n,r,q,\tilde{r},\tilde{q})$ and the proof of the existence of critical points, that, as far as we know, has not been treated yet in the literature. In the following, we will outline how the integrals become not tractable in the inhomogeneous case already in the case of a Gaussian and so a simple direct computation seems not to be enough to calculate the best Strichartz Constant. We will concentrate also here on the L^2-critical case. See [25] or [18] for more details on the TT^*-method. We now test the inhomogeneous inequality with Gaussians for every dimensions. It is not known yet in the literature if they are maximizers or not, but an explicit computation would lead at least to a lower bound on the constant. We recall that the solutions that we want to test are

$$u(t,x) = (1-4it)^{-n/2} e^{-\frac{|x|^2}{1-4it}},$$

while the inequality we need to test is

$$\Big\| \int_{s<t} e^{-i(t-s)\Delta} F(s)ds \Big\|_{L_t^q L_x^r} \leq S_i(n,q,r\tilde{q},\tilde{r}) \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

with $F(t,x) = |u(t,x)|^{p-1} u(t,x)$.

We start by computing the norm on the right hand side of this inequality. By the choice of the exponents and the criticality of the problem $\tilde{r}'p = r$ and $\tilde{q}'p = q$. So we get

$$\|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} = \||u|^p\|_{L_t^{q/p} L_x^{r/p}} = \|u\|^p_{L_t^q L_x^r}.$$

By the computations done in Section 3, we then obtain

$$\|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} = \Big(\frac{\pi}{4}\Big)^{\frac{np(r-2)}{4r}} \Big(\frac{\pi}{r}\Big)^{\frac{pn}{2r}}.$$

Now we have to compute the left hand side of the inhomogeneous Strichartz inequality:

$$\Big\| \int_{s<t} e^{-i(t-s)\Delta} F(s)ds \Big\|_{L_t^q L_x^r}.$$

We start computing explicitly $e^{-i(t-s)\Delta}F(s)$. By definition of $e^{-i(t-s)\Delta}$, we have

$$e^{-i(t-s)\Delta}F(s) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{e^{-i(t-s)\Delta} F} d\xi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi + i(t-s)|\xi|^2} \hat{F} d\xi.$$

So we have now to compute $\hat{F}(s,\xi)$:

$$\begin{aligned}
&\hat{F}(s,\xi) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(s,x)dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} |u(s,x)|^{p-1} u(s,x)dx \\
&= (2\pi)^{-n/2} |1+16s^2|^{-(p-1)n/4} (1-4is)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-\frac{|x|^2}{1-4is} - \frac{(p-1)|x|^2}{1+16s^2}} \\
&= (2\pi)^{-n/2} |1+16s^2|^{-(p-1)n/4-n/2} (1+4is)^{n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-\frac{(p+4is)|x|^2}{1+16s^2}} \\
&= (2\pi)^{-n/2} |1+16s^2|^{-(p-1)n/4-n/2} (1+4is)^{n/2} \times \\
&\quad \times \int_{\mathbb{R}^n} e^{-\frac{(p+4is)|x|^2}{1+16s^2} - ix\cdot\xi + \frac{(1+16s^2)|\xi|^2}{4(p+4is)}} e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}} \\
&= (2\pi)^{-n/2} |1+16s^2|^{-(p-1)n/4-n/2} (1+4is)^{n/2} \times
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}^n} e^{-|\frac{x(p+4is)^{1/2}}{(1+16s^2)^{1/2}}+i\frac{\xi(1+16s^2)^{1/2}}{2(p+4is)^{1/2}}|^2} e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}} \\
&= \frac{e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}}}{(2\pi)^{n/2}} |1+16s^2|^{-(p-1)n/4-n/2}(1+4is)^{n/2}(1+16s^2)^{n/2}(p+4is)^{-n/2}\pi^{n/2} \\
&= 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}}
\end{aligned}
$$

by completing the square and changing integration variables to

$$
y = \frac{x(p+4is)^{1/2}}{(1+16s^2)^{1/2}} + i\frac{\xi(1+16s^2)^{1/2}}{2(p+4is)^{1/2}},
$$

similarly to the computations done in Section 2. So

$$
\hat{F}(s,\xi) = 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}}.
$$

Notice that this is consistent with what we got in Section 2 in the case $s=0$ and $p=1$. Now, putting everything together, we obtain

$$
\begin{aligned}
&e^{-i(t-s)\Delta}F(s) \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi+i(t-s)|\xi|^2}\hat{F}d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi+i(t-s)|\xi|^2} 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}} \\
&= \frac{1}{(2\pi)^{n/2}} 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{(1+16s^2)|\xi|^2}{4(p+4is)}} e^{ix\cdot\xi+i(t-s)|\xi|^2} \\
&= \frac{1}{(2\pi)^{n/2}} 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} \\
&\quad\times \int_{\mathbb{R}^n} e^{-|\xi|^2\left[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)\right]+ix\xi+\frac{|x|^2}{4\left[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)\right]}} e^{-\frac{|x|^2}{4\left[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)\right]}} \\
&= \frac{1}{(2\pi)^{n/2}} 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} \\
&\quad\times \int_{\mathbb{R}^n} e^{-|\xi[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)]^{1/2}-\frac{ix}{2\left[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)\right]^{1/2}}|^2} e^{-\frac{|x|^2}{4\left[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)\right]}}
\end{aligned}
$$

which by the change of variable

$$
\eta = \xi[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)]^{1/2} - \frac{ix}{2[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)]^{1/2}},
$$

becomes

$$
\begin{aligned}
e^{-i(t-s)\Delta}F(s) &= \frac{1}{(2\pi)^{n/2}} 2^{-n/2}|1+16s^2|^{-(p-1)n/4}\Big(\frac{1+4is}{p+4is}\Big)^{n/2} \\
&\quad\times \int_{\mathbb{R}^n} e^{-|\eta|^2} e^{-\frac{|x|^2}{4[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)]}} \big[\frac{(1+16s^2)}{4(p+4is)}-i(t-s)\big]^{-n/2}.
\end{aligned}
$$

In conclusion,

$$
e^{-i(t-s)\Delta}F(s)
$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{n/2}} 2^{-n/2} |1+16s^2|^{-(p-1)n/4} \Big(\frac{1+4is}{p+4is}\Big)^{n/2} \big[\frac{(1+16s^2)}{4(p+4is)} - i(t-s)\big]^{-n/2} \\
&\quad \times \pi^{n/2} e^{-\frac{|x|^2}{4[\frac{(1+16s^2)}{4(p+4is)} - i(t-s)]}} \\
&= |1+16s^2|^{-(p-1)n/4} \Big(\frac{1+4is}{p+4is}\Big)^{n/2} \big[\frac{(1+16s^2)}{p+4is} - 4i(t-s)\big]^{-n/2} e^{-\frac{|x|^2}{4[\frac{(1+16s^2)}{4(p+4is)} - i(t-s)]}} \\
&= |1+16s^2|^{-(p-1)n/4} \big[\frac{1+4is}{1-4ip(t-s)+16ts}\big]^{n/2}
\end{aligned}
$$

Again, this is consistent with what we got in Section 2 in the case $s = t = 0$ and $p = 1$. At this point the approach of the direct computation seems not good enough anymore, because one should integrate in the variable s and this does not seem to have an explicit expression with elementary functions. We refer to [21] for more details on a possible numerical approach to the problem.

Remark 6.1. If one would be able to compute explicitly $S_i(n, r)$, one could use Theorem 1.5 also as a stability-type result for the solutions of the NLS, in a similar spirit of the stability of solitons in the focusing case. This connection links, in some sense, optimizers and stability, also when the functionals involve both space and time.

The wave equation case. For completeness, we want to mention here that similar studies have been done for several others homogeneous Strichartz estimates, like the wave equation. The complete characterization of critical points done by [5] in the case of the Schrödinger Equation is still not available in the case of the wave equation. We believe that an argument completely similar to the one that we have given in Section 3 would lead to the computation of the possible best homogeneous wave Strichartz constant $W(n)$ for the wave equation, once a complete characterization of the maximizers would be available. For more details on the case of the wave equation we refer to [3, 4, 14].

Remark 6.2. There are well known transformations that send solutions to the Schrödinger equation to solutions of the wave equation, see for example [25]. So one strategy here could be also to transform the maximizers of $S_h(n, r)$ into solutions of the corresponding wave equation and hope that the known transformation sends maximizers to maximizers. Unfortunately, to our knowledge, no known transformation does this job. This technique could be very helpful also for other equations.

Remark 6.3. Note that the functions that optimize the wave Strichartz inequality (see [14]), optimize also the Sobolev embeddings (see [23] and [1]). Let $1 < p < n$ and $p^* = \frac{np}{n-p}$, then

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p) \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

with optimal constant $C(n, p)$ given by

$$C(n,p) = \pi^{1/2} n^{-1/p} \Big(\frac{p-1}{n-p}\Big)^{1-1/p} \Big(\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\Big)$$

and maximizers given by

$$u(x) = (a + b|x|^{\frac{p}{p-1}})^{-\frac{n-p}{p}},$$

with $a, b > 0$. We notice that with $p = 2$ and substituting n with $n+1$ in the above optimizers, we recover the optimizers given in [14]. The correspondence between the constants seems more involved.

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