Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 266, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR CONCAVE-CONVEX ELLIPTIC PROBLEM WITH NONLINEAR BOUNDARY CONDITION

LI WANG, PEIHAO ZHAO

Abstract. In this article, we study the existence of sign-changing solutions to

$$\begin{gathered} -\Delta u+u=|u|^{p-1}u \quad \text{in } \Omega \\ \frac{\partial u}{\partial n}=\lambda|u|^{q-1}u \quad \text{on } \partial\Omega \end{gathered}$$

with $0<q<1<p\leq\frac{N+2}{N-2}$ and $\lambda>0$. By using a combination of invariant sets and Ljusternik-Schnirelman type minimax method, we obtain two sequences of sign-changing solutions when p is subcritical and one sequence of sign-changing solutions when p is critical.

1. Introduction

In this article we study the existence of infinitely many sign-changing solutions to the nonlinear Neumann problem

$$\begin{gathered} -\Delta u+u=|u|^{p-1}u, \quad \text{in } \Omega \\ \frac{\partial u}{\partial n}=\lambda|u|^{q-1}u, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N>2$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative and $0<q<1<p$, $\lambda>0$.

The existence of sign-changing solutions has been studied extensively in recent years. For the Dirichlet problem

$$\begin{gathered} -\Delta u=f(u) \quad \text{in } \Omega \\ u=0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

the authors in [3] considered that for $f\in C^1(\mathbb{R})$, $f(0)=0$ and $\lim_{u\to\infty}f'(u)<\lambda_1<\lambda_2<f'(0)$, in which λ_i is the eigenvalue of $-\Delta$ on Ω, then problem (1.2) has at least one sign-changing solution. If $f(u)$ is odd about u, superlinear and subcrtical, Bartsch [4] Showed that problem (1.2) has a sequence of unbounded sign-changing solutions. In this case, the positive cone is a invariant set of gradient

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2010 *Mathematics Subject Classification.* 35J60, 47J30, 58E05.
Key words and phrases. Nonlinear boundary condition; concave-convex; invariant sets; sign-changing solutions.

Submitted August 30, 2015. Published October 16, 2015.

flow. For $f(u) = \lambda u + |u|^{2^*-2}u$, $N \geq 7$, $\lambda > 0$, the authors in [18, 19] proved that (1.2) has also infinitely many sign-changing solutions. We can look for more examples in [3, 4, 5, 6] and references therein. Problems with nonlinear boundary condition of form (1.1) appear in a nature way when one considers the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ and conformal deformations on Riemannian manifolds with boundary, see [7, 8]. In [9], Garcia et al considered problem (1.1). For subcritical case, $0 < q < 1 < p < \frac{N+2}{N-2}$, there exists a $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, equation (1.1) has infinitely many solutions with negative energy; and for $0 < q < 1 < p \leq \frac{N+2}{N-2}$, there exists $\Lambda > 0$, such that for $0 < \lambda < \Lambda$, there exists at least two positive solutions for (1.1), for $\lambda = \Lambda$, at least one positive solution, and no positive solution for $\lambda > \Lambda$. Kajikiya et al [10] studied the problem

$$\begin{aligned} -\Delta u + u &= f(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g(x,u), \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

and proved that the problem has two sequences of solutions, if $f(x,u)$ and $g(x,u)$ satisfying that in a neighborhood of $u = 0$, one of $f(x,u)$ and $g(x,u)$ is locally sublinear, and at infinity, one of them is locally superlinear, and they showed that one sequence of the solutions converges to 0, the other diverges to infinity.

Inspired by [9, 10], we consider the existence of sign-changing solutions of problem (1.1). The main part of our work is that (1.1) has two sequence of sign-changing solutions under the subcritical and concave case. In this sense, the work of the present paper extends the results of [9, 10] partially. In section 2, we give the main results of the paper. In section 3, we establish the invariant sets of pseudo gradient. In section 4, we proof the theorems.

2. Main results

In this section, we state the main results and some preliminaries. We call u a weak solution of (1.1) if $u \in H^1(\Omega)$ and it satisfies (1.1) in the distribution sense, i.e.

$$\int_\Omega (\nabla u \nabla v + uv)dx = \int_\Omega |u|^{p-1}uv dx + \lambda \int_{\partial\Omega} |u|^{q-1}uv d\sigma,$$

for any $v \in H^1(\Omega)$. Here $d\sigma$ denotes the surface measure on $\partial\Omega$.

Throughout this paper, the norm of $H^1(\Omega) = W^{1,2}(\Omega)$ is defined by

$$\|u\| := \Big(\int_\Omega (|\nabla u|^2 + u^2)dx \Big)^{1/2},$$

and the $H^1(\Omega)$ inner product of u and v by

$$(u,v) := \int_\Omega (\nabla u \nabla v + uv)dx.$$

We state the main result as follows.

Theorem 2.1. *For $0 < q < 1 < p < \frac{N+2}{N-2}$, $\lambda > 0$, there exist at least two sequences of sign-changing solutions of (1.1), one converges to 0 in $H^1(\Omega)$, and the other diverges to infinity.*

Theorem 2.2. *For $0 < q < 1$, $p = \frac{N+2}{N-2}$, $\lambda > 0$, there exists at least one sequence of sign-changing solutions of (1.1) which converges to 0 in $H^1(\Omega)$.*

Now, we state some results we will need in the following sections. Next lemma is Lemma 6.1 in [9].

Lemma 2.3. *For $0 < q < 1 < p \le \frac{N+2}{N-2}$, there exists a $\Lambda > 0$ such that for $\lambda \le \Lambda$, then* (1.1) *has a minimal positive solution u^+ and a maximal negative solution u_-.*

If $\lambda > \Lambda$, (1.1) has no positive and negative solutions, by the results in [10] as we mentioned above, we know the existence of two sequences of solutions that are sign-changing solutions. Hence, we only need to prove the results in Theorem 2.1 under the condition $0 < \lambda \le \Lambda$. Throughout the paper, we assume that $\lambda \le \Lambda$. For (1.1) the minimal positive solution and the maximal negative solution satisfying $u^+ = -u^-$.

The following lemma is a variant of [14, Lemma 3.2] and we can also look for [15, Lemma 2.4].

Lemma 2.4. *Let H be a Hilbert space, D_1 and D_2 be two closed convex subsets of H, and $I \in C^1(H, \mathbb{R})$. Suppose $I'(u) = u - A(u)$ and $A(D_i) \subset D_i$ for $i = 1, 2$. Then there exists a pseudo gradient vector field V of I in the form $V(u) = u - B(u)$ with B satisfying $B(D_i) \subset \operatorname{int}(D_i)$ if $A(D_i) \subset \operatorname{int}(D_i)$ for $i = 1, 2$, and V is odd if I is even and $D_1 = -D_2$.*

We refer to [10] for the following priori estimates.

Lemma 2.5. *Let $f(x, s)$ and $g(x, s)$ satisfy:*

(1) $|f(x, s)| \le C(|s|^p + 1)$,
(2) $|g(x, s)| \le C(|s|^q + 1)$ *with* $0 < q < 1 < p < \frac{N+2}{N-2}$

Then for every $H^1(\Omega)$-solution u of (1.3) *belongs to $W^{1,r}(\Omega)$ for all $r < \infty$, and satisfies*
$$\|u\|_{W^{1,r}(\Omega)} \le C_r \|u\|_{H^1(\Omega)}^{dp} + C_r \|u\|_{H^1(\Omega)}^{dq} + C_r,$$
where C_r is a constant depends only on r and d is independent of u and r.

Here we call that V is a pseudo gradient vector field of I if $V \in C(H, H)$, $V|_{H \setminus K}$ is locally Lipschitz continuous with $K := \{u \in H : I'(u) = 0\}$, and $(I'(u), V(u)) \ge \frac{1}{2}\|I'(u)\|^2$ and $\|V(u)\| \le 2\|I'(u)\|$ for all $u \in H$.

3. Invariant sets of the gradient flow

To construct nodal solutions we need to isolate the signed solutions into certain invariant sets. We know that problem (1.1) has a minimal positive solution and a maximal negative solution, by this results we can build the invariant sets.

Define $v := A(u)$, $u \in H^1(\Omega)$ if
$$\begin{gathered} -\Delta v + v = |u|^{p-1}u \quad \text{in } \Omega \\ \frac{\partial v}{\partial n} = \lambda |u|^{q-1}u \quad \text{on } \partial\Omega, \end{gathered}$$
and
$$\begin{gathered} \frac{d}{dt}\eta^t(u) = -\eta^t(u) + B(\eta^t(u)) \\ \eta^0(u) = u, \end{gathered}$$
where B is related to A via Lemma 2.4 in which D_1 and D_2 will be constructed in Theorem 3.1. This section is concerned with the construction of these sets which

are invariant under the flow $\eta^t(u)$ such that all positive and negative solutions are contained in these invariant sets. Recall that a subset $W \subset H$ is an invariant set with respect to η if, for any $u \in W$, $\eta^t(u) \in W$ for all $t > 0$.

We first note that because of the sublinear term on the boundary, any neighborhoods of the positive (and negative) cones are no longer invariant sets of the gradient flow. We give a construction inspired by [15]. Let $e_1 \in H^1(\Omega)$ be the first eigenfunction associated with the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{gathered} -\Delta u + u = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u \quad \text{on } \partial\Omega \end{gathered} \tag{3.1}$$

such that $\max_\Omega e_1(x) \le s_0$, in which $0 < s_0 \le \min_\Omega u^+(x)$ and s_0 to be determined later. Then we have $u^+(x) \ge e_1(x)$ and $u^-(x) \le -e_1(x)$ for all $x \in \Omega$, $u^\pm$ is the minimal positive solution and maximal negative solution to (1.1), respectively. Define:

$$D^\pm := \{u \in H^1(\Omega) : \pm u \ge e_1\}.$$

From above, we know that all positive and negative solutions to (1.1) are contained in D^+ and D^-, respectively. Define $(D^\pm)_\epsilon = \{u \in H^1(\Omega) : \operatorname{dist}(u, D^\pm) < \epsilon\}$.

Theorem 3.1. *Assume* $0 < q < 1 < p < \frac{N+2}{N-2}$, *and* $0 < \lambda \le \Lambda$. *Then there exists* $\epsilon_0 > 0$ *such that*

$$\begin{gathered} A((D^\pm)_\epsilon) \subset \operatorname{int}((D^\pm)_\epsilon) \quad \textit{for all } 0 < \epsilon < \epsilon_0, \\ \eta^t((D^\pm)_\epsilon) \subset \operatorname{int}((D^\pm)_\epsilon) \quad \textit{for all } t \ge 0,\ 0 < \epsilon < \epsilon_0, \end{gathered}$$

Proof. We only prove the result for the positive one, the other case follows analogously. For $u \in H^1(\Omega)$, we denote

$$v = Au, \quad v_1 = \max\{e_1, v\}.$$

Then $\operatorname{dist}(v, D^+) \le \|v - v_1\|$ which implies $\operatorname{dist}(v, D^+) \cdot \|v - v_1\| \le \|v - v_1\|^2$ and

$$\begin{aligned} \|v - v_1\|^2 &= (v - e_1, v - v_1) \\ &= \int_\Omega \nabla(v - e_1) \cdot \nabla(v - v_1) + (v - e_1)(v - v_1) dx \\ &= \int_\Omega (-\Delta(v - e_1) + v - e_1)(v - v_1) dx + \int_{\partial\Omega} (\lambda |u|^{q-1} u - \lambda_1 e_1)(v - v_1) d\sigma \\ &= \int_\Omega (|u|^{p-1} u)(v - v_1) dx + \int_{\partial\Omega} (\lambda |u|^{q-1} u - \lambda_1 e_1)(v - v_1) d\sigma \\ &=: I_1 + I_2. \end{aligned}$$

Note that

$$\begin{aligned} I_1 &\le \int_{\{u<0\}\cap\Omega} (v_1 - v)(-|u|^{p-1} u) dx \le \int_{\{u<0\}\cap\Omega} (v_1 - v)(e_1 - |u|^{p-1} u) dx \\ &\le C_p \int_{\{u<0\}\cap\Omega} (v_1 - v)(e_1 - u)^p dx. \end{aligned}$$

On $\{u < 0\} \cap \Omega$, we have $u \le e_1$, hence

$$\begin{aligned} \|e_1 - u\|^p_{L^{p+1}((u<0)\cap\Omega)} &= \inf_{w \in D^+} \|w - u\|^p_{L^{p+1}((u<0)\cap\Omega)} \\ &\le \inf_{w \in D^+} \|w - u\|^p_{L^{p+1}(\Omega)} \le C_p dist^p(u, D^+), \end{aligned}$$

and $I_1 \le C\|v - v_1\| \operatorname{dist}^p(u, D^+)$. Here C_p and C are constants which are relevant to p and e_1, and may change from line to line. Note that

$$\begin{aligned} I_2 &\le \int_{\partial\Omega \cap \{\lambda|u|^{q-1}u < \lambda_1 e_1\}} (\lambda_1 e_1 - \lambda|u|^{q-1}u)(v_1 - v) d\sigma \\ &= \Big(\int_{\partial\Omega \cap \{\lambda(\frac{e_1}{2})^q < \lambda|u|^{q-1}u < \lambda_1 e_1\}} + \int_{\partial\Omega \cap \{u \le \frac{e_1}{2}\}} \Big)(\lambda_1 e_1 - \lambda|u|^{q-1}u)(v_1 - v) d\sigma. \end{aligned}$$

If $\lambda(e_1/2)^q > \lambda_1 e_1$, the first term above vanishing, this can be done by choose s_0 small enough such that $s_0^{1-q} \le \frac{\lambda}{\lambda_1 2^q}$. On $\partial\Omega \cap \{u \le \frac{e_1}{2}\}$, we have

$$\lambda_1 e_1 - \lambda|u|^{q-1}u \le C_r(e_1 - u)^r,$$

where $r \in (1, \frac{N}{N-2})$.

$$\begin{aligned} \|e_1 - u\|^r_{L^{r+1}(\{u<\frac{e_1}{2}\}\cap\partial\Omega)} &= \inf_{w \in D^+} \|w - u\|^r_{L^{r+1}(\{u<\frac{e_1}{2}\}\cap\partial\Omega)} \\ &\le \inf_{w \in D^+} \|w - u\|^r_{L^{r+1}(\partial\Omega)} \le C_r \operatorname{dist}^r(u, D^+). \end{aligned}$$

Hence, $I_2 \le C\|v - v_1\| \operatorname{dist}^r(u, D^+)$.

$$\operatorname{dist}(v, D^+) \cdot \|v - v_1\| \le C\|v - v_1\|(\operatorname{dist}^r(u, D^+) + \operatorname{dist}^p(u, D^+))$$

Then we can choose ϵ_0 small, such that for $\epsilon < \epsilon_0$,

$$\operatorname{dist}(v, D^+) < \operatorname{dist}(u, D^+) \quad \text{for } u \in D^+_\epsilon.$$

The first conclusion in Theorem 3.1 is proved, the second part is a consequence of the first one as shown in [14] via Lemma 2.4 above. ☐

4. Proof of main results

Let us start with a more abstract setting. Consider $I \in C^1(X, \mathbb{R})$ where X is a Banach space. V is a pseudo gradient vector field of I such that V is odd if I is even, and consider

$$\begin{gathered} \frac{d}{dt}\sigma(t, u) = -V(\sigma), \\ \sigma(0, u) = u \in X. \end{gathered}$$

To construct nodal solution by using the combination of invariant sets and minimax method, we need a deformation lemma in the presence of invariant sets. We have the following deformation lemma which follows from [15, Lemma 5.1] (see also [13, Lemma 2.4]).

Lemma 4.1. *Assume I satisfies the (PS)-condition, and $c \in \mathbb{R}$ is fixed, $W = \partial W \cup \operatorname{int}(W)$ is an invariant subset such that $\sigma(t, \partial W) \subset int(W)$ for $t > 0$. Define $K_c^1 := K_c \cap W$, $K_c^2 := K_c \cap (X \backslash W)$, where $K_c := \{u \in X : I'(u) = 0, I(u) = c\}$. Let $\delta > 0$, be such that $(K_c^1)_\delta \subset W$ where $(K_c^1)_\delta = \{u \in X : \operatorname{dist}(u, K_c^1) < \delta\}$. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists $\eta \in C([0, 1] \times X, X)$ satisfying:*

1. *$\eta(t, u) = u$ for $t = 0$ or $u \notin I^{-1}(c - \varepsilon_0, c + \varepsilon_0) \backslash (K_c^2)_\delta$.*
2. *$\eta(1, I^{c+\varepsilon} \cup W \backslash (K_c^2)_{3\delta}) \subset I^{c-\varepsilon} \cup W$ and $\eta(1, I^{c+\varepsilon} \cup W) \subset I^{c-\varepsilon} \cup W$ if $K_c^2 = \emptyset$.*
3. *$\eta(t, \cdot)$ is a homeomorphism of X for $t \in [0, 1]$.*
4. *$\|\eta(t, u) - u\| \le \delta$, for any $(t, u) \in [0, 1] \times X$.*
5. *$I(\eta(t, \cdot))$ is non-increasing.*

(6) $\eta(t, W) \subset W$ *for any* $t \in [0,1]$.
(7) $\eta(t, \cdot)$ *is odd if* I *is even and if* W *is symmetric with respect to* 0.

Set

$$\Sigma := \{A \subset H^1(\Omega)\backslash 0 : A \text{ is closed and } A = -A\},$$
$$\Gamma_k := \{A \subset H^1(\Omega)\backslash 0 : A \text{is closed, symmetric, } \gamma(A) \geq k\}$$

where $\gamma(A)$ denotes the Krasnoselskii's genus of the set A. We refer to [17] for the following properties of genus.

Lemma 4.2. *Let* $A, B \in \Gamma_k$*, and* $h \in C(H^1(\Omega), H^1(\Omega))$ *be an odd map. Then*

(1) $A \subset B \Rightarrow \ \gamma(A) \leq \gamma(B)$;
(2) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;
(3) $\gamma(A) \leq \gamma(h(A))$;
(4) *If* A *is compact, there exists an* $N \in \Gamma_k$ *such that* $A \subset \operatorname{int}(N) \subset N$ *and* $\gamma(A) = \gamma(N)$;
(5) *If* F *is a linear subspace of* $H^1(\Omega)$ *with* $\dim F = n$*,* $A \subset F$ *is bounded, open and symmetric, and* $0 \in A$*, then* $\gamma(\partial_F A) = n$;
(6) *Let* W *be a closed linear subspace of* $H^1(\Omega)$ *whose codimension is finite. If* $\gamma(A)$ *is greater than the codimension of* W*, then* $A \cap W \neq \emptyset$.

We choose an even function $h \in C_0^\infty(\mathbb{R})$ such that $h(s) = 1$ for $|s| \leq 1$, $h(s) = 0$ for$|s| \geq 2$, $0 \leq h \leq 1$; defining

$$f(s) := s|s|^{p-1}h(s),\ g(s) = s|s|^{q-1}h(s); \tag{4.1}$$
$$\widetilde{I}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u)dx - \int_{\partial\Omega} G(u)d\sigma,$$

in which $F(u) = \int_0^u f(s)ds$, $G(u) = \int_0^u g(s)ds$, both of them are bounded. Assume (λ_i, e_i) is the eigenvalue and corresponding eigenfunction of (3.1), and $E_m = \operatorname{span}\{e_1, \cdots, e_m\}$. Then the following lemma is obvious.

Lemma 4.3. $\widetilde{I} \in C^1(H^1(\Omega), \mathbb{R})$,

(1) *for all* $m \in \mathbb{N}$*, there exists a* $\rho > 0$*, such that* $\sup_{E_m \cap \partial B_\rho} \widetilde{I}(u) < 0$*, where* $\partial B_\rho := \{u \in H^1(\Omega) : \|u\| = \rho\}$,
(2) $\widetilde{I}$ *is even, bounded from blow, and the (PS)-condition holds,* $\widetilde{I}(0) = 0$;

The following lemma is similar to [15, Lemma 5.3].

Lemma 4.4. *For any* $\rho > 0$*, let* $B_\rho = \{u \in H^1(\Omega), \|u\| \leq \rho\}$*. Then*

$$\operatorname{dist}(\partial B_\rho \cap E_1^\perp, D^+ \cup D^-) > 0.$$

Proof. Assume on the contrary, that there exists $(u_n) \in D^+$, $v_n \in \partial B_\rho \cap E_1^\perp$, such that $\|u_n - v_n\| \to 0$. Then $(u_n, e_1) = (u_n - v_n, e_1) + (v_n, e_1) \to 0$, as $n \to \infty$. But, since $u_n \geq e_1$, we have

$$(u_n, e_1) = \lambda_1 \int_{\partial\Omega} u_n e_1 \geq \lambda_1 \int_{\partial\Omega} e_1^2 d\sigma \neq 0,$$

a contradiction. ∎

Proof of Theorems. We essentially follow from [15], see also [2] and [16].

Part 1. In this part, we will prove that for $0 < q < 1 < p \leq \frac{N+2}{N-2}$, (1.1) has a sequence of sign-changing solutions which converge to 0. This is a conclusion of [11] and [9]. By Lemma 4.3 above we have taht for each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} \widetilde{I}(u) < 0$. With the help of [11, Theorem 1], there exists a sequence $\{u_k\}$ satisfying

$$\widetilde{I}'(u_k) = 0, \quad \widetilde{I}(u_k) < 0, \quad u_k \to 0 \text{ in } H^1(\Omega).$$

By Lemma 2.5, u_k converges to zero in $C(\overline{\Omega})$. Hence for large k, we have $\|u_k\|_{C(\overline{\Omega})} < 1$, $\widetilde{I}(u_k) = I(u_k)$ and $\widetilde{I}'(u_k) = I'(u_k)$. But from Lemma 2.3 we know that (1.1) has a minimal positive solution and a maximal negative solution, thus, for large j, u_j must change signs. Theorem 2.2 and the first part of Theorem 2.1 follows from the above argument.

Part 2. In this part, we prove the existence of a sequence of sign-changing solutions which tends to infinity under the case $0 < q < 1 < p < \frac{N+2}{N-2}$. The functional

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}dx - \frac{\lambda}{q+1}\int_{\partial\Omega} |u|^{q+1}d\sigma,$$

is well defined on $H^1(\Omega)$ and $I \in C^1(H^1(\Omega), \mathbb{R})$, I satisfies the (PS) condition for $0 < q < 1 < p < \frac{N+2}{N-2}$.

Lemma 4.5. *Assume $m \geq 2$, then there exists $R = R(m) > 0$ such that for all $\lambda > 0$,*

$$\sup_{B_R^c \cap E_m} I(u) < 0.$$

where $B_R^c := H^1(\Omega)\backslash B_R$.

From Theorem 3.1 we can choose an $\epsilon > 0$ small enough such that $(D^\pm)_\epsilon$ are invariant sets. Set W=$\overline{(D^+)_\epsilon} \cup \overline{(D^-)_\epsilon}$, $S = H^1(\Omega)\backslash W$ contains only sign-changing solutions. Set

$$G_m = \{h \in C(B_R \cap E_m, H^1(\Omega)) : h \text{ is odd and } h = \text{id on } \partial B_R \cap E_m\},$$

in which R is determined in Lemma 4.5.

$$\widetilde{\Gamma}_j = \{h(\overline{B_R \cap E_m\backslash Y}) : h \in G_m, \forall m \geq j, Y = -Y, \text{ closed}, \gamma(Y) \leq m-j\}, \quad j \geq 2.$$

From [1] and[15], we know that $\widetilde{\Gamma}_j$ satisfying the following properties:

- (1') $\widetilde{\Gamma}_j \neq \emptyset$ for all $j \geq 2$.
- (2') $\widetilde{\Gamma}_{j+1} \subset \widetilde{\Gamma}_j$ for all $j \geq 2$.
- (3') if $\sigma \in C(H^1(\Omega), H^1(\Omega))$ is odd and $\sigma = id$ on $\partial B_R \cap E_m$, then $\sigma(A) \in \widetilde{\Gamma}_j$ if $A \in \widetilde{\Gamma}_j$.
- (4') if $A \in \widetilde{\Gamma}_j$, $Z = -Z$, closed, and $\gamma(Z) \leq s < j$ and $j - s \geq 2$, then $\overline{A\backslash Z} \in \widetilde{\Gamma}_{j-s}$.

For $j \geq 2$, we define

$$\widetilde{c}_j := \inf_{A\in\widetilde{\Gamma}_j} \sup_{u\in A\cap S} I(u).$$

If $A \in \widetilde{\Gamma}_j$ with $j \geq 2$, then $A \cap \partial B_\rho \cap (E_1)^\perp \neq \emptyset$. By Lemma 4.4, $\partial B_\rho \cap (E_1)^\perp \subset S$. Thus, for $j \geq 2$, and $A \in \widetilde{\Gamma}_j$, $A \cap S \neq \emptyset$, we conclude that

$$\widetilde{c}_j \geq \inf_{\partial B_\rho \cap (E_1)^\perp} I(u) > -\infty.$$

Then from the definition of $\widetilde{c}_j$ and (2') we have $-\infty < \widetilde{c}_2 \leq \widetilde{c}_3 \leq \cdots \leq \widetilde{c}_j \leq \cdots < \infty$. We claim that if $c := \widetilde{c}_j = \cdots = \widetilde{c}_{j+k}$ for some $2 \leq j \leq j+k$ with $k \geq 0$, then $\gamma(K_c \cap S) \geq k+1$. Before we prove this claim, we first show that $\widetilde{c}_j \to \infty$, as $j \to \infty$. We need the following lemma.

Lemma 4.6. *The constant $\widetilde{c}_j$ is independent of the choice of $R(m)$ as long as $R(m)$ is chosen to satisfy Lemma 4.5 for which $m \geq j$.*

The above lemma is well known, see for instance [12, Lemma 4.9]. And we can choose $R(m)$ such that $R(m) \to \infty$, as $m \to \infty$. This part follows by [10]. Let $W_m := \{\sum_{i=m}^\infty t_i w_i : \sum_{i=m}^\infty t_i^2 < \infty\}$ and w_m is the eigenfunction of the Neumann Laplacian equation:

$$-\Delta w = \mu w \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

W_j is a closed linear subspace of $H^1(\Omega)$ whose codimension is equal to $j-1$, we have:

$$h(\overline{B_R \cap E_m \backslash Y}) \cap \partial B_r \cap W_j \neq \emptyset,$$

for $h \in G_m$, $\gamma(Y) \leq m-j$, and $0 < r < R$(since that $\gamma(h(\overline{B_R \cap E_m \backslash Y})) \geq j$, and the codimension of W_j is $j-1$. This implies

$$\sup_{u \in B_R \cap E_m \backslash Y} I(h(u)) \geq \inf\{I(u) : u \in \partial B_r \cap W_j\},$$

for $h \in G_m$, $\gamma(Y) \leq m-j$, taking the infimum of both sides over $h \in G_m$, we have

$$\widetilde{c}_j \geq inf\{I(u) : \ u \in \partial B_r \cap W_j\},$$

for $0 < r < R$. Next we can have $\inf\{I(u) : u \in \partial B_r \cap W_j\}$ diverges to ∞, the rest of the proof is similarly with [10, Lemma 5.14], we omit it here.

Now we give the proof the claim. Denote $K_c \cap S$ by K_c^2. If the claim is false, $\gamma(K_c \cap S) \leq k$, because of $\widetilde{c}_j \to \infty$, we can assume that $0 \notin K_c$ and $K_c^2 = K_c \cap S$ is compact, there exists N such that $K_c^2 \subset \operatorname{int}(N)$ and $\gamma(N) = \gamma(K_c^2)$. Then by Lemma 4.1, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists an $\eta \in C([0,1] \times H^1(\Omega), H^1(\Omega))$ satisfying (1)-(7) of Lemma 4.1. Then

$$\eta(1, I^{c+\epsilon} \cup W \backslash N) \subset (I^{c-\epsilon} \cup W).$$

Choose $A \in \widetilde{\Gamma}_{j+k}$ such that

$$\sup_{A \cap S} I(u) \leq c + \epsilon,$$

Then by (4') above $\overline{A \backslash N} \in \widetilde{\Gamma}_j$ hence $\eta(1, \overline{A \backslash N}) \in \widetilde{\Gamma}_j$. Then

$$c \leq \sup_{\eta(1, \overline{A \backslash N})} I(u) \leq \sup_{(I^{c-\epsilon} \cup W) \cap S} \leq c - \epsilon,$$

contradiction. Hence $\gamma(K_c \cap S) \geq k+1$. Now we finish the proof.

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Li Wang
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
E-mail address: lwang10@lzu.edu.cn

Peihao Zhao
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
E-mail address: zhaoph@lzu.edu.cn