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SEMISTABILITY OF FIRST-ORDER EVOLUTION VARIATIONAL INEQUALITIES

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Abstract. Semistability is the property whereby the solutions of a dynamical system converge to a Lyapunov stable equilibrium point determined by the system initial conditions. We extend the theory of semistability to a class of first-order evolution variational inequalities, and study the finite-time semistability. These results are Lyapunov-based and are obtained without any assumptions of sign definiteness on the Lyapunov function. Our results are supported by some examples from unilateral mechanics and electrical circuits involving nonsmooth elements such as Coulomb's friction forces and diodes.

1. Introduction

Stability analysis of dynamical systems constitutes a very important topic in mathematics and engineering. This is the case of mechanical systems subject to unilateral constraints and/or Coulomb friction and/or impacts or electrical circuits with switches, diodes and many other problems. So, it is not surprising that the unilateral dynamical system has played a central role in the understanding of mechanical processes. The mathematical formulation of the unilateral dynamical system involved inequality constraints and necessarily contains natural non-smoothness. The non-smoothness could originate from the discontinuous control term, or from the environment (non-smooth impact), or from the dry friction.

A large class of unilateral dynamical systems can be represented under the formalism of evolution variational inequalities. Recently, new Lyapunov stability results have been developed for these inequalities (see. [2, 10, 12]). In [12], the authors developed a Lyapunov approach to study the stability of stationary solutions of first-order evolution variational inequalities in Hilbert spaces. This approach was efficient for giving sufficient conditions of stability for the problem in the form of a variational inequality. In [2], the authors has developed a LaSalle's invariance theory applicable to a general class of first order non-linear evolution variational inequalities. This approach was applied to the study of the stability and the asymptotic properties of second order dynamical systems involving friction forces. Equally important, is the study of the attractivity properties of the set of

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stationary solutions. In addition, the authors in [1] give a sufficient and a necessary conditions for the finite-time stability of the evolution variational inequalities.

The aim of this paper is to consider an alternative notion of stability called semistability which, in certain sense, lies between stability and asymptotic stability. More precisely, an equilibrium is semistable if it is Lyapunov stable, and every trajectory starting in a neighborhood of the equilibrium converges to a (possibly different) Lyapunov stable equilibrium. It can be seen that, for an equilibrium, asymptotic stability implies semistability, while semistability implies Lyapunov stability. In addition to semistability, it is desirable that a dynamical system that exhibits semistability also possesses the property that trajectories that converge to a Lyapunov stable system state must do so in finite time rather than merely asymptotically. This is the so called finite-time semistability. In a recent series of papers [4, 5, 15, 16], the authors developed a general mathematical approach to study the notions of semistability and finite-time semistability of nonlinear dynamical systems. Here, we will try to extend these notions to the case of first-order evolution variational inequalities.

In [5], the authors describe the relationship between Lyapunov stability, semistability, and asymptotic stability. This relationship can be understood by considering the motion of a body translating along a fixed direction. Such a body, when moving under the action of a linear elastic spring, possesses a unique equilibrium, which is Lyapunov stable. In the additional presence of viscous damping, all motions of the body converge to the unique equilibrium state, which is thus asymptotically stable. On the other hand, a particle moving under the action of viscous damping in the absence of a position-dependent restoring force can remain at rest in any position and thus exhibits a continuum of equilibria, each of which is Lyapunov stable. All motions of such a body converge to rest, and the equilibrium that the body converges to is determined by the initial position and velocity of the body. The motion of the body is thus convergent, while every equilibrium of the dynamics is semistable. Moreover, the authors cite many examples where the semistability theory is applicable, like the lateral dynamics of an aircraft in level trimmed flight and the kinetics of chemical reactions.

In this paper, we are concerned with the study of semistability and finite-time semistability of first-order nonsmooth dynamical systems. More precisely, let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. For a given $x_0 \in \mathrm{Dom}(\partial\varphi)$, we consider the following problem: Find $x(t) \in C^0([0,+\infty);\mathbb{R}^n)$ and $\dot{x} \in L^\infty_{loc}([0,+\infty);\mathbb{R}^n)$ such that $x(t) \in \mathrm{Dom}(\partial\varphi)$ for all $t \geq 0$,

$$\langle \dot{x}(t) + f(x(t)), y - x(t)\rangle + \varphi(y) - \varphi(x(t)) \geq 0, \forall y \in \mathbb{R}^n, \quad \text{a.e. } t \geq 0,$$
$$x(0) = x_0.$$

This variational inequality can equivalently be written as the following differential inclusion:

$$\dot{x}(t) + f(x(t)) \in -\partial\varphi(x(t)), \quad \text{a.e. } t \geq 0, \tag{1.1}$$

where $\partial\varphi$ denotes the subdifferential of φ. We remark that (1.1) involves two particular cases. The first one is when the function φ is of class C^1, then (1.1) becomes an ordinary differential equation. The second one is when the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex continuous and hence $\partial\varphi(x)$ is a nonempty, compact and convex set for every $x \in \mathbb{R}^n$. In this case (1.1) is reduced to well known Filippov's differential inclusion. Finally, a special case is obtained when the function φ is the indicator function of a closed convex subset of $\mathbb{R}^n$. In this case, (1.1) reduces to the complementarity problem introduced in the next section.

The contents of this article is as follows. In Section 2, we recall some definitions, notations and we review some basic Lyapunov stability results for the first-order evolution variational inequalities. Next, in Section 3, we introduce the notions of semistability and finite-time semistability for the evolution variational inequalities in order to give the main results of this work. Note that, these results do not make any assumptions about the sign definiteness of the Lyapunov function. Instead, they require only that the Lyapunov function derivative be nonpositive and the equilibrium be a local minimizer of the Lyapunov function on the set of points at which the orbital derivative along the trajectory of Lyapunov function is a negative definite function. Finally, in Section 4, we illustrate the main results of this article by applying them to examples from unilateral mechanics and electronics.

2. Preliminaries

Throughout this article $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function (l.s.c., for short). Here, we denote by $\partial\varphi$ the subdifferential of φ. In addition, the sets $\mathrm{Dom}(\varphi)$ and $\mathrm{Dom}(\partial\varphi)$ stand for the domains of φ and the subdifferential $\partial\varphi$ of φ respectively, i.e.

$$\mathrm{Dom}(\varphi) = \{x \in \mathbb{R}^n : \varphi(x) < +\infty\}, \quad \mathrm{Dom}(\partial\varphi) = \{x \in \mathbb{R}^n : \partial\varphi(x) \neq \emptyset\}.$$

The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field. Furthermore, We denote by $\|\cdot\|, \langle\cdot,\cdot\rangle, B$ and $\overline{B}$, the Euclidean norm, the usual inner product $\langle\cdot,\cdot\rangle$, the open unit ball, and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, $x + \rho B$ and $x + \rho\overline{B}$ are the open and the closed balls of center x and radius ρ respectively. For a subset S of $\mathbb{R}^n$, $bd\ S$ stands for the boundary of S and $d_S(x)$ denotes the distance from a point x to S; that is, $d_S(x) := \inf_{s\in S} \|x - s\|$. Finally, We introduce the set $\mathcal{K}_\infty$ as follows,

$$\mathcal{K}_\infty := \{g : \mathbb{R}^+ \to \mathbb{R}^+; g(0) = 0, g \text{ is strictly increasing and } g(x) \xrightarrow{x\to+\infty} +\infty\}.$$

Given $x_0 \in \mathrm{Dom}(\partial\varphi)$, we are interested in the first-order differential inclusion: Find $x : [0, +\infty) \to \mathbb{R}^n$ with $x \in C^0([0, +\infty); \mathbb{R}^n)$ and $\dot{x} \in L^\infty_{loc}([0, +\infty); \mathbb{R}^n)$, and

$$\begin{gathered} \dot{x}(t) + f(x(t)) \in -\partial\varphi(x(t)), \quad \text{a.e. } t \geq 0 \\ x(0) = x_0 \end{gathered} \tag{2.1}$$

System (2.1) possesses a solution which is not unique. When the field f is continuous and $f + kI$ is monotone for some $k > 0$, system (2.1) has a unique solution. For more details, we refer the readers to [12] and [2]. In this article, we are interested in the case where the field f is continuous. Letting $\psi(\cdot, x_0)$ a solution of (2.1) that exists on $[0, +\infty)$ and satisfies the initial condition $x(0) = x_0$ and where the map $\psi : [0, +\infty) \times \mathrm{Dom}(\partial\varphi) \to \mathrm{Dom}(\partial\varphi)$ is continuous and satisfies $\psi(0, x_0) = x_0$. Moreover, the set of equilibrium points of (2.1) is given by the set

$$\mathcal{E} := \{z \in \mathrm{Dom}(\partial\varphi) : f(z) \in -\partial\varphi(z)\}.$$

We denote by $\psi_t := \psi(t, \cdot) : \mathrm{Dom}(\partial\varphi) \to \mathrm{Dom}(\partial\varphi)$ and by $\psi_{x_0} := \psi(\cdot, x_0) : [0, +\infty) \to \mathrm{Dom}(\partial\varphi)$. For x_0 fixed, $\psi_{x_0}(t)$ is a maximal solution of (2.1). For a qualitative study of (2.1), it is important to consider the map $\psi_t(\cdot)$, where for all $\tau \geq 0$ fixed $\psi_\tau(\cdot)$ is uniformly continuous on $\mathrm{Dom}(\partial\varphi)$ (see. [2]).

An important and interesting case is obtained when the function φ is replaced by the indicator function of a closed convex set C of $\mathbb{R}^n$. Recall that the indicator function I_C is defined as $I_C = 0$ if $x \in C$ and $I_C = +\infty$ if $x \notin C$. The subdifferential of ∂I_C of I_C at a point x is the normal cone of C at the point $x \in C$, i.e.

$$\partial I_C(x) = N_C(x) := \{z \in \mathbb{R}^n; \langle z, y - x\rangle \le 0,\ \forall y \in C\}.$$

In this case, problem (2.1) reduces to the well-known complementarity problem defined by, for $x_0 \in C$, find $x : [0, +\infty) \to \mathbb{R}^n$ with $x \in C^0([0, +\infty); \mathbb{R}^n)$ and $\dot{x} \in L^\infty_{loc}([0, +\infty); \mathbb{R}^n)$, and

$$\begin{gathered} \dot{x}(t) + f(x(t)) \in -N_C(x(t)), \quad \text{a.e. } t \ge 0 \\ x(0) = x_0 \end{gathered} \tag{2.2}$$

The set of equilibrium points of problem (2.2) is

$$\mathcal{E}_C := \{z \in C; f(z) \in -N_C(z)\}.$$

A set $\mathcal{M} \subseteq \mathbb{R}^n$ is *weakly invariant* with respect to (2.1) if $\psi_t(\mathcal{M}) \subseteq \mathcal{M}$ and is *invariant* with respect to (2.1) if $\psi_t(\mathcal{M}) = \mathcal{M}$. The orbit of a point $x_0 \in \mathrm{Dom}(\partial\varphi)$ is $\mathcal{O}_{x_0} := \psi_{x_0}([0, +\infty))$. The orbit of x_0 is bounded if $\mathcal{O}_{x_0}$ is contained in a compact set. Finally, we define the ω-limit set as follows,

$$\omega - \mathcal{L}^\infty_{x_0} := \{z \in \mathrm{Dom}(\partial\varphi); \exists (t_i)_i \in [0, +\infty), t_i \to +\infty \text{ and } \psi(t_i, x_0) \to z\}.$$

The following proposition gives some important properties for the ω-limit set (see. [17, 2]),

Proposition 2.1. *If, for $x_0 \in \mathrm{Dom}(\partial\varphi)$, the orbit $\mathcal{O}_{x_0}$ of x_0 is bounded, then the set $\omega - \mathcal{L}^\infty_{x_0}$ is a nonempty compact and weakly invariant set. Moreover,*

$$\lim_{\tau \to +\infty} d_{\omega - \mathcal{L}^\infty_{x_0}}(\psi_\tau(x_0)) = 0.$$

We remark that the set of the equilibrium points $\mathcal{E}$ is invariant and for $x_0 \in \mathrm{Dom}(\partial\varphi)$, the $\omega - \mathcal{L}^\infty_{x_0} \subseteq \mathrm{cl}\,\mathcal{O}_{x_0}$. We recall now the definition of Lyapunov stability of an equilibrium point of (2.1).

Definition 2.2. An equilibrium point $x_e \in \mathrm{Dom}(\partial\varphi)$ of (2.1) is said to be *Lyapunov stable* if, for all $\epsilon > 0$, there exists $\eta := \eta(\epsilon) > 0$ such that, for all $x_0 \in x_e + \eta\overline{B} \cap \mathrm{Dom}(\partial\varphi)$, we have $\psi_t(x_e + \eta\overline{B}) \subseteq x_e + \epsilon\overline{B}$.

As Definition 2.2 shows, to prove the Lyapunov stability of an equilibrium point, it is necessary to find a solution of the problem for a given initial condition $x_0 \in x_e + \eta\overline{B} \cap \mathrm{Dom}(\partial\varphi)$. This is not an easy task for most problems (even for ODE's). Lyapunov's direct method (known also as the second method of Lyapunov) allows us to determine the stability of an equilibrium point by studying the behavior of special functions called Lyapunov function. This method avoids the calculation of an explicit solution of the problem. But, it requires to find good Lyapunov candidate functions compatible with the problem, and the disadvantage is that there is no straightforward construction of Lyapunov function. In fact, the method consists to find a positive definite function V of class C^1 such that its orbital derivative along the trajectory, given by $\frac{d}{dt}V(x(t)) = \langle \dot{x}(t), \nabla V(x(t))\rangle$, is a negative semidefinite function.

In [12], the authors extend this method to the case of problems (2.1) and (2.2) as show the following results.

Theorem 2.3. *An equilibrium point x_e of* (2.1) *is Lyapunov stable, if there exists $\alpha > 0$ and a C^1 positive definite function V, such that for $x \in x_e + \alpha\overline{B} \cap \mathrm{Dom}(\partial\varphi)$,*

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq 0.$$

Corollary 2.4. *Let C be a closed convex subset of $\mathbb{R}^n$ such that $0 \in C$. An equilibrium point x_e of* (2.2) *is Lyapunov stable, if there exists $\alpha > 0$ and a C^1 positive definite function V, such that*

1. *$\langle f(x), \nabla V(x)\rangle \geq 0$, for all $x \in x_e + \alpha\overline{B} \cap C$, and*
2. *$(x - \nabla V(x)) \in C$, for all $x \in x_e + \alpha\overline{B} \cap boundary C$.*

We finish this section by recalling the finite-time stability of an equilibrium point of (2.1).

Definition 2.5. An equilibrium point $x_e \in \mathrm{Dom}(\partial\varphi)$ of (2.1) is said to be *finite-time stable* if,

1. it is Lyapunov stable, and
2. it is *finite-time convergent* i.e. there exists $\delta > 0$ such that $x_e + \delta\overline{B} \subseteq \mathrm{Dom}(\partial\varphi)$ and a function $T : x_e + \delta\overline{B} \setminus \mathcal{E} \to]0, +\infty[$ (called *settling-time function*) such that for all $x_0 \in x_e + \delta\overline{B} \setminus \mathcal{E}$, we have $\psi_t(x_0)$ converges to a point in $x_e + \delta\overline{B} \cap \mathcal{E}$ when t tends to $T(x_0)$.

As for the Lypapunov stability, the authors in [1] give a sufficient and necessary conditions for the finite-time stability for both problems (2.1) and (2.2).

3. Semistability Result

We begin this section by introducing the definition of the semistability of an equilibrium point of the system (2.1). Then, we recall a stability result for (2.1) given in [12].

Definition 3.1. An equilibrium point $x_e \in \mathrm{Dom}(\partial\varphi)$ of (2.1) is said to be *semistable*, if

(i) it is Lyapunov stable, and
(ii) there exists $\rho > 0$, with $x_e + \rho\overline{B} \subseteq \mathrm{Dom}(\partial\varphi)$ such that, for $x_0 \in x_e + \rho\overline{B}$, we have $\lim_{t\to+\infty} \psi_t(x_0) = z$, where $z \in \mathrm{Dom}(\partial\varphi)$ and is a Lyapunov stable point.

Definition 3.2.
(1) System (2.1) is called *semistable* if every equilibrium point in $\mathcal{E}$ is semistable.
(2) For a function $V \in C^1$, we define the set $\dot{V}^f_\varphi$ for (2.1) as
$$\dot{V}^f_\varphi := \{x \in \mathrm{Dom}(\partial\varphi); \langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = 0\}.$$
(3) For a function $V \in C^1$, we define the set $\dot{V}^f_C$ for (2.2) as
$$\dot{V}^f_C := \{x \in C; \langle f(x), \nabla V(x)\rangle = 0\}.$$

Theorem 3.3. *Suppose that $\mathcal{O}_{x_0}$ is bounded for all $x_0 \in \mathrm{Dom}(\partial\varphi)$ and there exists a function $V \in C^1$ such that for all $x \in \mathrm{Dom}(\partial\varphi)$,*

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq 0.$$

Let $\mathcal{M}$ be the largest weakly invariant set contained in $\dot{V}^f_\varphi$. For $x \in \mathcal{M}$, if x is Lyapunov stable equilibrium, then it is semistable.

Proof. According to Proposition 2.1 we have, $\omega - \mathcal{L}_{x_0}^{\infty}$ is nonempty and $\omega - \mathcal{L}_{x_0}^{\infty} \subseteq \mathcal{M}$. Every point of $\mathcal{M}$ is Lyapunov stable equilibrium with respect to $\mathrm{Dom}(\partial\varphi)$, we have $\omega - \mathcal{L}_{x_0}^{\infty}$ is reduced to a single point. Let z_{x_0} be that point. Let us show that $z_{x_0} = \lim_{t\to+\infty} \psi_t(x_0)$.

Indeed, as $z_{x_0} \in \mathcal{M}$ then it is a Lyapunov stable equilibrium. So that, for all $\epsilon > 0$ there exists $\eta := \eta(\epsilon)$ such that, for $x_0 \in z_{x_0} + \eta\overline{B}$ we have

$$\psi_t(z_{x_0} + \eta\overline{B}) \subseteq z_{x_0} + \epsilon\overline{B}, \quad \forall t \geq 0.$$

On the other hand, there exists $\tau \geq 0$ such that $\psi_\tau(x_0) \in z_{x_0} + \eta\overline{B}$. So, by the continuity of ψ_t and the properties of semigroup, we get

$$\psi_{t+\tau}(x_0) = \psi_t(\psi_\tau(x_0)) \in \psi_t(z_{x_0} + \eta\overline{B}) \subseteq z_{x_0} + \epsilon\overline{B}, \quad \forall t \geq 0.$$

Then, $z_{x_0} = \lim_{t\to+\infty} \psi_t(x_0)$. □

Corollary 3.4. *Assume that for every $x_e \in \mathcal{E}$, there exist $\alpha > 0$ and a function $V \in C^1$ such that, for all $x \in (x_e + \alpha\overline{B} \cap \mathrm{Dom}(\partial\varphi)) \setminus \mathcal{E}$, we have*

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) > 0. \tag{3.1}$$

If the system (2.1) *is Lyapunov stable, then it is semistable.*

Proof. Since (2.1) is Lyapunov stable, then for all $z \in \mathcal{E}$ and for all $\epsilon \geq 0$ there exists $\eta \geq 0$ such that for all $x_0 \in z + \eta\overline{B}$ we have

$$\psi_t(z + \eta\overline{B}) \subseteq z + \epsilon\overline{B}, \quad \forall t \geq 0.$$

Consider now the set $\mathcal{W} := \cup_{z\in\mathcal{E}}(z + \eta\overline{B}) \subseteq \mathrm{Dom}(\partial\varphi)$. For $x \in \mathcal{W}$, there exists $z \in \mathcal{E}$ such that $x \in z + \eta\overline{B}$ and $\psi_t(x) \in \mathrm{Dom}(\partial\varphi), \forall t \geq 0$ that means $\psi_t(z + \eta\overline{B})$ is bounded. According to Proposition 2.1, we can deduce that the set $\omega - \mathcal{L}_{x_0}^{\infty}$ is weakly invariant.

Let $\mathcal{M}$ be the largest weakly invariant invariant set contained in $\dot{V}_\varphi^f$. For every $x \in \mathcal{E}$, we have $\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = 0$. As $\mathcal{E}$ is weakly invariant and contained in $\mathcal{W}$, it follows that $\mathcal{E} \subseteq \mathcal{M}$. Finally, from (3.1) we can see that $\dot{V}_\varphi^f \subseteq \mathcal{E} \subseteq \mathcal{M} \subseteq \dot{V}_\varphi^f$, thus $\dot{V}_\varphi^f = \mathcal{M} = \mathcal{E}$ and by applying Theorem 3.3, we obtain that (2.1) is semistable. □

Applying Corollary 3.4 to problem (2.2), we obtain the following corollary.

Corollary 3.5. *Let x_e be an equilibrium point of* (2.2)*. Suppose that there exists $\alpha > 0$ and a C^1 function V, such that*

(1) *$\langle f(x), \nabla V(x)\rangle > 0$, for all $x \in (x_e + \alpha\overline{B} \cap C) \setminus \mathcal{E}$, and*
(2) *$(x - \nabla V(x)) \in C$, for all $x \in x_e + \alpha\overline{B} \cap bd\, C$.*

If x_e is Lyapunov stable, then it is semistable.

3.1. **Finite-Time Semistability.**

Definition 3.6. *An equilibrium point $x_e \in \mathrm{Dom}(\partial\varphi)$ of* (2.1) *is said to be* finite-time semistable *if it is semistable and it is finite-time convergent.*

Theorem 3.7. *Assume that for every $x_e \in \mathcal{E}$, there exist $\alpha > 0$ and a function V of class C^1. Suppose also that there exists a function $g \in \mathcal{K}_\infty$ for which the integral $\int_0^\varepsilon \frac{dz}{g(z)}$ converges for all $\varepsilon > 0$. Moreover, assume that for every $x \in (x_e + \delta\overline{B} \cap \mathrm{Dom}(\partial\varphi)) \setminus \mathcal{E}$, we have*

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq g(V(x)). \tag{3.2}$$

Then, if the system (2.1) *is finite-time stable, then it is finite-time semistable.*

Proof. The proof of Theorem 3.7, is a direct consequence of Theorem 3.3 and [1, Theorem 3.5]. Indeed, Theorem 3.3 guarantees the semistability and [1, Theorem 3.5] ensures the existence of the settling-time function T. □

Remark 3.8. Note that in Theorem 3.7, we can take $g(x) = x^\alpha$, with $\alpha \in]0,1[$. Thus, for some $c > 0$, condition (3.2) can be equivalently replaced by

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) \geq c(V(x))^\alpha.$$

This notation is used in [15] in order to study the finite-time semistability.

Now we consider the case of the complementarity problem introduced in Section 2. By applying Theorem 3.7 to problem (2.2), we obtain the following result concerning finite-time semistability.

Corollary 3.9. *Assume that for every x_e of* (2.2)*, there exist $\alpha > 0$ and a function V of class C^1. Suppose that there exist a function $g \in \mathcal{K}_\infty$ for which the integral $\int_0^\varepsilon \frac{dz}{g(z)}$ converges for all $\varepsilon > 0$. Moreover, assume that for every $x \in (x_e + \delta\overline{B} \cap C) \setminus \mathcal{E}$, we have*

$$\langle f(x), \nabla V(x)\rangle \geq g(V(x)).$$

Then, if system (2.2) *is finite-time stable, then it is finite-time semistable.*

4. Applications

In this section, we give some applications of Theorem 3.3 and Corollary 3.4 in electrical circuits containing nonsmooth devices like diodes and in unilateral mechanics problems submitted to a Coulomb's friction force type.

Example 4.1. Consider the system (2.1) where $f(x) \equiv 0$ and where $\varphi(x) = |x|$. Then

$$\partial\varphi(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1,1] & \text{if } x = 0 \\ 1 & \text{if } x > 0\,. \end{cases}$$

Consider the function $V(x) = x^2/2$, the set of the equilibrium points is reduced to $\{0\}$, thus, for all $x \in \mathbb{R} \setminus \{0\}$ we have

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = |x| > 0.$$

Then, the equilibrium point $x_e = 0$ is Lyapunov stable and according to Corollary 3.4, we can deduce that is semistable. Moreover, the equilibrium $x_e = 0$ is finite-time semistable. Indeed, consider the function $g(x) = \sqrt{x}$. It is easy to see that $g \in \mathcal{K}_\infty$ and the condition (3.2) is satisfied,

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = |x| \geq \frac{\sqrt{2}}{2}|x| = g(V(x)).$$

From the proof of [1, Theorem 3.5], and for an initial condition x_0, we can deduce that $T(x_0) \leq 2\sqrt{V(x_0)} < +\infty$.

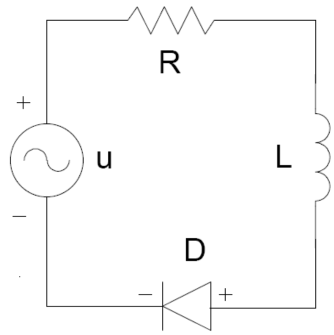

FIGURE 1. Circuit RLD

Example 4.2. *Let us consider the electric circuit of Figure 1., involving a load resistance $R > 0$, an input-signal source u and corresponding instantaneous current i,an inductor $L > 0$ and a diode D. A diode is a device that constitutes a rectifier which permits the easy flow of charges in one direction and restrains the flow in the opposite direction. The electrical superpotential of the diode is*

$$\varphi_D(i) := |i|.$$

Figure 2 illustrates the ampere-volt characteristic of the diode used in this example.

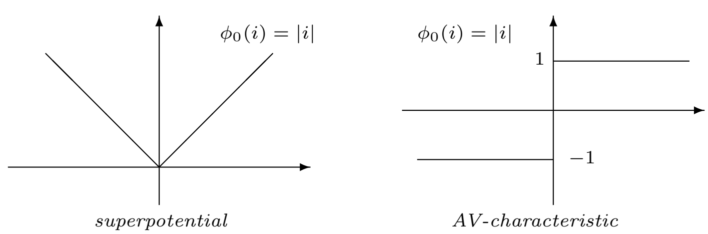

FIGURE 2. Diode Model

The Kirchoff's voltage law gives

$$u - U_R - U_L = U_D,$$

where $U_R = Ri$ denotes the difference of potential across the resistor, $U_L = L\frac{di}{dt}$ and $U_D \in \partial\varphi_D(i)$ is the difference of potential across diode. Thus

$$L\frac{di}{dt} + Ri - u \in -\partial\varphi_D(i).$$

To simplify the work, we take the input-signal $u = 0$. Then we obtain

$$\frac{di}{dt} + \frac{R}{L}i \in -\frac{1}{L}\partial\varphi_D(i). \tag{4.1}$$

Here, $f(i) = \frac{R}{L}i$ and the set of the equilibrium points of (4.1) contains only the equilibrium point $i_e = 0$. If we consider the function $V(i) = \frac{1}{2}i^2$, then its gradient is $\nabla V(i) = i$. For all $i \neq 0$, the inequality (3.1) occurs because

$$\langle f(i), \nabla V(i)\rangle + \varphi_D(i) - \varphi_D(i - \nabla V(i)) = \frac{R}{L}i^2 + |i| > 0.$$

Then, the equilibrium $i_e = 0$ is Lyapunov stable and by applying Corollary 3.4, we deduce that the system is semistable. If we consider the same function $g(x)$ in the previous example, we can remark that

$$\langle f(i), \nabla V(i)\rangle + \varphi_D(i) - \varphi_D(i - \nabla V(i)) = \frac{R}{L} i^2 + |i| \geq \frac{\sqrt{2}}{2}|i| = g(V(i)).$$

Hence, the equilibrium $i_e = 0$ is finite-time semistable.

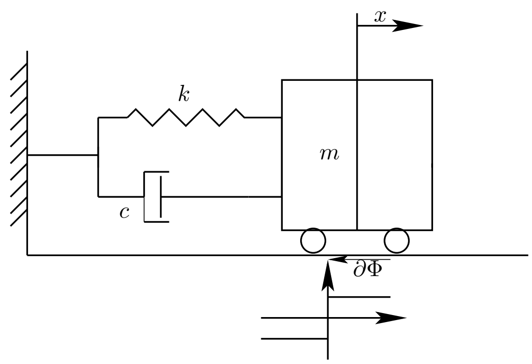

FIGURE 3. Mass-Spring

Example 4.3. (Unilateral Mechanics) Figure 3 describes the motion of a mass $m > 0$ restrained by a spring with stiffness constant $k > 0$ and a damper with viscous damping coefficient $c > 0$. The motion is submitted to a Coulomb's friction force and the system is modeled by the second-order differential inclusion (see. [2])

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) \in \partial\Phi(\dot{q}(t)). \tag{4.2}$$

With $\Phi(x) = \lambda|x|$, where $\lambda > 0$ denotes the coefficient of friction. Then,

$$\partial\Phi(x) = \begin{cases} -\lambda & \text{if } x < 0 \\ [-\lambda, \lambda] & \text{if } x = 0 \\ \lambda & \text{if } x > 0. \end{cases}$$

The set of equilibrium points is given by the interval $\mathcal{E} = [-\frac{\lambda}{k}, \frac{\lambda}{k}]$.

For $x_e \in \mathcal{E}$ and for $x = [x_1\ x_2]^T$, system (4.2) is equivalent to the first-order differential inclusion (2.1),

$$\dot{x}(t) + f(x(t)) \in -\partial\varphi(x(t)).$$

Where, $f(x) = Ax + B$, with $A = \begin{bmatrix} 0 & -1 \\ \frac{k}{m} & \frac{c}{m} \end{bmatrix}$, $B = \frac{k}{m}\begin{bmatrix} 0 \\ x_e \end{bmatrix}$ and $\varphi(x) = \Phi(x_2) = |x_2|$.

If we consider the function $V(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$, the inequality (3.1) holds, i.e.

$$\langle f(x), \nabla V(x)\rangle + \varphi(x) - \varphi(x - \nabla V(x)) = cx_2^2 + \frac{k}{m}x_e x_2 + \frac{1}{m}|x_2| > 0.$$

Then, every equilibrium point is Lyapunov stable, hence it is semistable.

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