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INVERSE SPECTRAL AND INVERSE NODAL PROBLEMS FOR ENERGY-DEPENDENT STURM-LIOUVILLE EQUATIONS WITH δ-INTERACTION

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Abstract. In this article, we study the inverse spectral and inverse nodal problems for energy-dependent Sturm-Liouville equations with δ-interaction. We obtain uniqueness, reconstruction and stability using the nodal set of eigenfunctions for the given problem.

1. Introduction

We consider the boundary value problem (BVP) generated by the differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \tag{1.1}$$

with the boundary conditions

$$U(y) := y(0) = 0, \quad V(y) := y'(\pi) = 0 \tag{1.2}$$

and at the point $x = \frac{\pi}{2}$ satisfying

$$\begin{gathered} y(\frac{\pi}{2}+0) = y(\frac{\pi}{2}-0) = y(\frac{\pi}{2}), \\ y'(\frac{\pi}{2}+0) - y'(\frac{\pi}{2}-0) = 2\alpha\lambda y(\frac{\pi}{2}) \end{gathered} \tag{1.3}$$

where $q(x)$ is a nonnegative real valued function in $L_2(0,\pi)$, $\alpha \neq \pm 1$ is real number and λ is spectral parameter. Without loss of generality we assume that

$$\int_0^\pi q(x)dx = 0. \tag{1.4}$$

We denote the BVP (1.1), (1.2) and (1.3) by $L = L(q, \alpha)$.

Notice that, we can understand problem (1.1) and (1.3) as studying the equation

$$y'' + (\lambda^2 - 2\lambda p(x) - q(x))y = 0, \quad x \in (0, \pi) \tag{1.5}$$

when $p(x) = \alpha\delta(x - \frac{\pi}{2})$, where $\delta(x)$ is the Dirac function (see [2]).

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We consider the inverse problems of recovering $q(x)$ and α from the given spectral and nodal characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, monographs [7, 16, 19, 24]). Inverse nodal problems consist in constructing operators from the given nodes (zeros) of eigenfunctions (see [5, 12, 15, 20, 27]). Discontinuous inverse problems (in various formulations) have been considered in [3, 8, 14, 26, 28, 29, 30].

Sturm-Liouville spectral problems with potentials depending on the spectral parameter arise in various models quantum and classical mechanics. There λ^2 is related to the energy of the system, this explaining the term "energy-dependent" in (1.5). The non-linear dependence of equation (1.5) on the spectral parameter λ should be regarded as a spectral problem for a quadratic operator pencil. The inverse spectral and nodal problems for energy-dependent Schrödinger operators with $p(x) \in W_2^1(0,1)$ and $q(x) \in L_2[0,1]$ and with Robin boundary conditions was discussed in [4], [10]. Such problems for separated and nonseparated boundary conditions were considered (see [1, 9, 32] and the references therein). The inverse scattering problem for equation (1.5) with eigenparameter-dependent boundary condition on the half line solved in [17].

In this article we obtain some results on inverse spectral and inverse nodal problems and establish connections between them.

2. Inverse spectral problems

In this section we study so-called incomplete inverse problem of recovering the potential $q(x)$ from a part of the spectrum BVP L. The technique employed is similar to those used in [11, 25]. Similar problems for the Sturm-Liouville and Dirac operators were formulated and studied in [22, 23].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on the intervals $(0,\pi/2)$ and $(\pi/2,\pi)$. Denote $\langle y,z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (1.3), then

$$\langle y,z\rangle_{x=\frac{\pi}{2}-0} = \langle y,z\rangle_{x=\frac{\pi}{2}+0} \tag{2.1}$$

i.e. the function $\langle y,z\rangle$ is continuous on $(0,\pi)$.

Let $\varphi(x,\lambda)$ be solution of equation (1.1) satisfying the initial conditions $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = 1$ and the matching condition (1.3). Then $U(\varphi) = 0$. Denote

$$\Delta(\lambda) := -V(\varphi) = -\varphi'(\pi,\lambda). \tag{2.2}$$

By (2.1) and the Liouville's formula (see [6, p.83]), $\Delta(\lambda)$ does not depend on x. The function $\Delta(\lambda)$ is called characteristic function on L.

Lemma 2.1. *The eigenvalues of the BVP L are real, nonzero and simple.*

Proof. Suppose that λ is an eigenvalue BVP L and that $y(x,\lambda)$ is a corresponding eigenfunction such that $\int_0^\pi |y(x,\lambda)|^2dx = 1$. Multiplying both sides of (1.1) by $\overline{y(x,\lambda)}$ and integrate the result with respect to x from 0 to π:

$$-\int_0^\pi y''(x,\lambda)\overline{y(x,\lambda)}dx + \int_0^\pi q(x)|y(x,\lambda)|^2dx = \lambda^2\int_0^\pi |y(x,\lambda)|^2dx \tag{2.3}$$

Using the formula of integration by parts and the conditions (1.2) and (1.3) we obtain

$$\int_0^\pi y''(x,\lambda)\overline{y(x,\lambda)}dx = -2\alpha\lambda|y(0,\lambda)|^2 - \int_0^\pi |y'(x,\lambda)|^2dx.$$

It follows from this and (2.3) that

$$\lambda^2 + B(\lambda)\lambda + C(\lambda) = 0, \tag{2.4}$$

where

$$B(\lambda) = -2\alpha.|y(0,\lambda)|^2,$$
$$C(\lambda) = -\int_0^\pi q(x)|y(x,\lambda)|^2dx - \int_0^\pi |y'(x,\lambda)|^2dx.$$

Thus the eigenvalue λ of the BVP L is a root of the quadratic equation (2.4). Therefore, $B^2(\lambda) - 4C(\lambda) > 0$. Consequently, the equation (2.4) has only real roots.

Let us show that λ_0 is a simple eigenvalue. Assume that this is not true. Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent eigenfunctions corresponding to the eigenvalue λ_0. Then for a given value of λ_0, each solution $y_0(x)$ of (1.5) will be given as linear combination of solutions $y_1(x)$ and $y_2(x)$. Moreover it will satisfy boundary conditions (1.2) and conditions (1.3) at the point $x = \pi/2$. However it is impossible. □

Lemma 2.2. *The BVP L has a countable set of eigenvalues $\{\lambda_n\}_{n\geq 1}$. Moreover, as $n \to \infty$,*

$$\lambda_n := n - \frac{\theta}{\pi} + \frac{1}{2(\pi n - \theta)}(w_0 + (-1)^{n-1}w_1) + o(\frac{1}{n}), \tag{2.5}$$

where

$$\tan\theta = \frac{1}{\alpha}, \quad w_0 = \int_0^\pi q(t)dt, \quad w_1 = \frac{\alpha}{\sqrt{1+\alpha^2}}\Big(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^\pi q(t)dt\Big). \tag{2.6}$$

Proof. Let $\tau := \operatorname{Im}\lambda$. For $|\lambda| \to \infty$ uniformly in x one has (see [31, Chapter 1])

$$\varphi(x,\lambda) = \frac{\sin\lambda x}{\lambda} - \frac{\cos\lambda x}{2\lambda^2}\int_0^x q(t)dt + o\Big(\frac{1}{\lambda^2}\exp(|\tau|x)\Big), \quad x < \frac{\pi}{2}, \tag{2.7}$$

$$\begin{aligned}
&\varphi(x,\lambda)\\
&= \frac{1}{\lambda}\Big(\sqrt{1+\alpha^2}\cos(\lambda x+\theta) + \alpha\cos\lambda(\pi-x)\Big) + \sqrt{1+\alpha^2}\frac{\sin(\lambda x+\theta)}{2\lambda^2}\int_0^x q(t)dt\\
&\quad + \alpha\frac{\sin\lambda(\pi-x)}{2\lambda^2}\Big(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^x q(t)dt\Big) + o\Big(\frac{1}{\lambda^2}\exp(|\tau|x)\Big), \quad x > \frac{\pi}{2}
\end{aligned} \tag{2.8}$$

$$\varphi'(x,\lambda) = \cos\lambda x + \frac{\sin\lambda x}{2\lambda}\int_0^x q(t)dt + o\Big(\frac{1}{\lambda}\exp(|\tau|x)\Big), \quad x < \frac{\pi}{2} \tag{2.9}$$

$$\begin{aligned}
&\varphi'(x,\lambda)\\
&= -\sqrt{1+\alpha^2}\sin(\lambda x+\theta) + \alpha\sin\lambda(\pi-x) + \sqrt{1+\alpha^2}\frac{\cos(\lambda x+\theta)}{2\lambda}\int_0^x q(t)dt\\
&\quad - \alpha\frac{\cos\lambda(\pi-x)}{2\lambda}\Big(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^x q(t)dt\Big) + o\Big(\frac{1}{\lambda}\exp(|\tau|x)\Big), \quad x > \frac{\pi}{2}
\end{aligned} \tag{2.10}$$

It follows from (2.10) that as $|\lambda| \to \infty$

$$\begin{aligned} \Delta(\lambda) &= \sqrt{1+\alpha^2}\sin(\lambda\pi+\theta) - \sqrt{1+\alpha^2}\frac{\cos(\lambda\pi+\theta)}{2\lambda}\int_0^\pi q(t)dt \\ &\quad + \frac{\alpha}{2\lambda}\Big(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^{\pi} q(t)dt\Big) + o\Big(\frac{1}{\lambda}\exp(|\tau|x)\Big). \end{aligned} \tag{2.11}$$

Using (2.11) and Rouché's theorem, by the well-known method (see [7]) one has that as $n \to \infty$,

$$\lambda_n := n - \frac{\theta}{\pi} + \frac{1}{2(\pi n - \theta)}(w_0 + (-1)^{n-1}w_1) + o(\frac{1}{n}).$$

□

Together with L we consider a BVP $\tilde{L} = \tilde{L}(\tilde{q}, \alpha)$ of the same form but with different coefficient $\tilde{q}$. The following theorem has been proved in [13] for the Sturm-Liouville equation. We show it also holds for (1.1)-(1.3).

Theorem 2.3. *If for any* $n \in \mathbb{N} \cup \{0\}$,

$$\lambda_n = \tilde{\lambda}_n, \quad \langle y_n, \tilde{y}_n \rangle_{x=\frac{\pi}{2}-0} = 0,$$

then $q(x) = \tilde{q}(x)$ *almost everywhere (a.e) on* $(0, \pi)$.

Proof. Since

$$\begin{gathered} -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda^2 y(x,\lambda), \quad -\tilde{y}''(x,\lambda) + \tilde{q}(x)\tilde{y}(x,\lambda) = \lambda^2\tilde{y}(x,\lambda), \\ y(0,\lambda) = 0, \quad y'(0,\lambda) = 1, \quad \tilde{y}(0,\lambda) = 0, \quad \tilde{y}'(0,\lambda) = 1, \end{gathered}$$

it follows from (2.1) that

$$\int_0^{\pi/2} r(x)y(x,\lambda)\tilde{y}(x,\lambda)dx = \langle y, \tilde{y} \rangle_{x=\frac{\pi}{2}-0} \tag{2.12}$$

where $r(x) = q(x) - \tilde{q}(x)$. Since $\langle y_n, \tilde{y}_n \rangle_{x=\frac{\pi}{2}-0} = 0$ for $n \in \mathbb{N} \cup \{0\}$, it follows from (2.12) that

$$\int_0^{\pi/2} r(x)y(x,\lambda_n)\tilde{y}(x,\lambda_n)dx = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.13}$$

For $x \le \pi/2$ the following representation holds (see [16, 19]);

$$y(x,\lambda) = \frac{\sin\lambda x}{\lambda} + \int_0^x K(x,t)\frac{\sin\lambda x}{\lambda}dt,$$

where $K(x,t)$ is a continuous function which does not depend on λ. Hence

$$2\lambda^2 y(x,\lambda)\tilde{y}(x,\lambda) = 1 - \cos 2\lambda x - \int_0^x V(x,t)\cos 2\lambda t dt, \tag{2.14}$$

where $V(x,t)$ is a continuous function which does not depend on λ. Substituting (2.14) into (2.13) and taking the relation (1.4) into account, we calculate

$$\int_0^{\pi/2}\Big(r(x) + \int_x^{\pi/2} V(t,x)r(x)dt\Big)\cos 2\lambda_n x dx = 0, \quad n \in \mathbb{N} \cup \{0\},$$

which implies from the completeness of the function cosine, that

$$r(x) + \int_x^{\pi/2} V(t,x)r(x)dt = 0 \quad \text{a.e. on } [0, \frac{\pi}{2}].$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution, it follows that $r(x) = 0$ a.e. on $[0, \frac{\pi}{2}]$. To prove that $q(x) = \tilde{q}(x)$ a.e. on $[\pi/2, \pi]$ we will consider the supplementary problem $\hat{L}$;

$$-y''(x,\lambda) + q_1(x)y(x,\lambda) = \lambda^2 y(x,\lambda), \quad q_1(x) = q(\pi - x), \quad 0 < x < \frac{\pi}{2},$$
$$U(y) := y(0,\lambda) = 0,$$
$$y(\frac{\pi}{2}+0,\lambda) = y(\frac{\pi}{2}-0,\lambda), \quad y'(\frac{\pi}{2}+0,\lambda) - y'(\frac{\pi}{2}-0) = 2\alpha\lambda y(\frac{\pi}{2}-0,\lambda).$$

It follows from (2.1) that $\langle y_n, \tilde{y}_n\rangle_{x=\frac{\pi}{2}+0} = 0$. A direct calculation implies that $\tilde{y}_n(x) := y_n(\pi - x)$ is the solution to the supplementary problem $\hat{L}$, the $\hat{L}$ and $\tilde{y}_n(\frac{\pi}{2} - 0) = y_n(\frac{\pi}{2} + 0)$. Thus for the supplementary problem $\hat{L}$ the assumption conditions in Theorem 2.3 are still satisfied. If we repeat the above arguments then yields $r(\pi - x) = 0$ and $0 < x < \pi/2$, that is $q(x) = \tilde{q}(x)$ a.e. on $[\pi/2, \pi]$. $\square$

3. Inverse nodal problems

In this section, we obtain uniqueness theorems and a procedure of recovering the potential $q(x)$ on the whole interval $(0, \pi)$ from a dense subset of nodal points.

The eigenfunctions of the BVP L have the form $y_n(x) = \varphi(x, \lambda_n)$. We note that $y_n(x)$ are real-valued functions. Substituting (2.5) into (2.7) and (2.8) we obtain the following asymptotic formulae for $n \to \infty$ uniformly in x:

$$\begin{aligned} \lambda_n y_n(x) &= \sin(n - \frac{\theta}{\pi})x + \frac{1}{2(\pi n - \theta)}\Big(-\pi\int_0^x q(t)dt + (w_0 + (-1)^{n-1}w_1)x\Big) \\ &\quad \times \cos(n - \frac{\theta}{\pi})x + o(\frac{1}{n}), \quad x < \frac{\pi}{2} \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\lambda_n y_n(x) \\ &= \cos((n - \frac{\theta}{\pi})x + \theta)[\sqrt{1+\alpha^2} + (-1)^n\alpha] \\ &\quad + \frac{1}{2(\pi n - \theta)}\Big[\pi\sqrt{1+\alpha^2}\int_0^x q(t)dt + (-1)^{n-1}\alpha\pi\Big(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^x q(t)dt\Big) \\ &\quad - (\sqrt{1+\alpha^2}x + (-1)^{n-1}\alpha(\pi - x))(w_0 + (-1)^{n-1}w_1)\Big] \\ &\quad \times \sin((n - \frac{\theta}{\pi})x + \theta) + o(\frac{1}{n}), \quad x > \frac{\pi}{2}. \end{aligned} \tag{3.2}$$

For the BVP L an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunction $y_n(x)$ has exactly $(n-1)$ (simple) zeros inside the interval $(0, \pi)$: $0 < x_n^1 < x_n^2 < \dots < x_n^{n-1} < \pi$. The set $X_L := \{x_n^j\}_{n\geq 2,\ j=\overline{1,n-1}}$ is called the set of nodal points of the BVP L. Denote $X_L^k := \{x_{2m-k}^j\}_{m\geq 1, j=1,2m-k-1}$, $k = 0, 1$. Clearly, $X_L^0 \cup X_L^1 = X_L$. Denote $\mu_n^0 := 0$, $\mu_n^n := 1$, $\mu_n^j := \frac{j}{\pi n - \theta}\pi^2$, $\gamma_n^j := \mu_n^j - \frac{\pi^2 + 2\theta\pi}{2(\pi n - \theta)}$, $j = \overline{1, n-1}$.

Inverse nodal problems consist in recovering the problem $q(x)$ from the given set X_L of nodal points or from a certain part.

Taking (3.1)-(3.2) into account, we obtain the following asymptotic formulae for nodal points as $n \to \infty$ uniformly in j:

for $x_n^j \in (0, \frac{\pi}{2})$:

$$x_n^j = \mu_n^j + \frac{\pi}{2(\pi n-\theta)^2}\Big(\pi\int_0^{\mu_n^j} q(t)dt - (w_0+(-1)^n w_1)\mu_n^j\Big) + o(\frac{1}{n^2}), \tag{3.3}$$

for $x_n^j \in (\frac{\pi}{2}, \pi)$:

$$x_n^j = \gamma_n^j + \frac{\pi}{2(\pi n-\theta)^2}\Big[\pi\int_0^{\gamma_n^j} q(t)dt - ((w_0+(-1)^{n-1}w_1)\gamma_n^j + d_k)\Big] + o(\frac{1}{n^2}), \tag{3.4}$$

where $k=0$ when n is odd and $k=1$ when n is even in d_k, and

$$d_k = (\sqrt{1+\alpha^2}+(-1)^{n-1}\alpha)\Big[2(-1)^{n-1}\alpha\pi\int_0^{\pi/2} q(t)dt+(-1)^n\alpha\pi(w_0+(-1)^{n-1}w_1)\Big]. \tag{3.5}$$

Using these formulae we arrive at the following assertion.

Theorem 3.1. *Fix $k \in \{0,1\}$ and $x \in [0,\pi]$. Let $\{x_n^j\} \subset X_L^k$ be chosen such that $\lim_{n\to\infty} x_n^j = x$. Then there exists a finite limit*

$$g_k(x) := \lim_{n\to\infty}\frac{2(\pi n-\theta)}{\pi}\Big[(\pi n-\theta)x_n^j - \begin{cases} j\pi, & \text{if } x_n^j \in (0,\frac{\pi}{2}) \\ (j+\frac{1}{2})\pi+\theta, & \text{if } x_n^j \in (\frac{\pi}{2},\pi) \end{cases}\Big], \tag{3.6}$$

and

$$\begin{gathered} g_k(x) = \int_0^x q(t)dt - \frac{w_0+(-1)^{k-1}w_1}{\pi}x, \quad x \le \frac{\pi}{2}, \\ g_k(x) = \int_0^x q(t)dt - \frac{w_0+(-1)^{k-1}w_1}{\pi}x + d_k, \quad x \ge \frac{\pi}{2} \end{gathered} \tag{3.7}$$

where d_0 and d_1 are defined by (3.5).

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 3.2. *Fix $k = 0 \vee 1$. Let $X \subset X_L^k$ be a subset of nodal points which is dense on $(0,\pi)$. Let $X = \tilde{X}$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0,\pi)$, $\alpha = \tilde{\alpha}$. Thus the specification of X uniquely determines the potential $q(x)$ on $(0,\pi)$ and the number α. The function $q(x)$ and the number α can be constructed via the formulae*

$$q(x) = g_k'(x) + \frac{1}{\pi}(g_k(\pi)-g_k(0)), \tag{3.8}$$

$$\alpha = \Big[\Big(\frac{2g_k(\pi)+4g_k(\frac{\pi}{2})-6g_k(0)}{\pi(g_0'(x)-g_1'(x))}\Big)^2 - 1\Big]^{-2} \tag{3.9}$$

where $g_k(x)$ is calculated by (3.7).

Proof. Formulae (3.8), (3.9) follow from (3.7), (1.4) and (2.6). Note that by (3.7), we have

$$g_k'(x) = q(x) - \frac{w_0+(-1)^k w_1}{\pi}, \quad x \in (0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi), \tag{3.10}$$

hence

$$g_k(\pi)-g_k(0) = \int_0^\pi q(x)dx - (w_0+(-1)^{n-1}w_1), \quad w_1 = \frac{\pi}{2}\left[g_0'(x)-g_1'(x)\right]. \tag{3.11}$$

Then (3.8) can be derived directly from (3.10) and (3.11). Similarly, we can derive (3.9). Note that if $X = \tilde{X}$, then (3.6) yields $q_k(x) \equiv \tilde{q}_k(x)$, $x \in [0,\pi]$. By (3.8) (3.9), we obtain $q_k(x) = \tilde{q}_k(x)$ a.e. on $(0,\pi)$, $\alpha = \tilde{\alpha}$. □

4. Stability of inverse problem for operator L

Finally, we also solve the stability problem. Stability is about a continuity between two metric spaces. To show this continuity, we use a homeomorphism between these two spaces. These type stability problems were studied in [15, 18, 21, 30].

Definition 4.1. (i) Let $\mathbb{N}' = \mathbb{N}\backslash\{1\}$. We denote

$$\Omega := \big\{q \in L_1(0,\pi) : \int_0^\pi q(x)dx = 0\big\},$$

$\Sigma :=$ the collection of all double sequences X, where

$$X := \big\{x_n^j : j = \overline{1, n-1}; n \in \mathbb{N}'\big\}$$

such that $0 < x_n^1 < x_n^2 < \dots < x_n^{k-1} < x_n^k < \frac{\pi}{2} < x_n^{k+1} < \dots < x_n^{n-1} < \pi$ for each n.

We call Ω the space of discontinuous Sturm-Liouville operators and Σ the space of all admissible sequences. Hence, when $\overline{X}$ is the nodal set associated with $(\overline{q},\alpha)$ and $\overline{X}$ is close to X in Σ, then $(\overline{q},\alpha)$ is close to (q,α).

(ii) Let $X \in \Sigma$ and define $x_n^0 = 0$, $x_n^n = 1$, $L_n^j = x_n^{j+1} - x_n^j$ and $I_n^j = (x_n^j, x_n^{j+1})$ for $j = \overline{0, n-1}$. Note that, $L_n^0 = x_n^1$ and $L_n^{n-1} = \pi - x_n^{n-1}$. We say X is quasinodal to some $q \in \Omega$ if X is an admissible sequence and satisfies the conditions:

(I) As $n \to \infty$ the limit of

$$(\pi n - \theta)\Big[(\pi n - \theta)x_n^j - \begin{cases} j\pi, & \text{if } x_n^j \in (0, \frac{\pi}{2}) \\ (j + \frac{1}{2})\pi + \theta, & \text{if } x_n^j \in (\frac{\pi}{2}, \pi) \end{cases}\Big]$$

exists in $\mathbb{R}$ for all $j = \overline{1, n-1}$;

(II) X has the following asymptotic uniformity for j as $n \to \infty$,

$$x_n^j = \begin{cases} \mu_n^j + O(\frac{1}{n^2}), & \text{if } x_n^j \in (0, \frac{\pi}{2}) \\ \gamma_n^j + O(\frac{1}{n^2}), & \text{if } x_n^j \in (\frac{\pi}{2}, \pi) \end{cases}$$

for $j = \overline{1, n-1}$.

Definition 4.2. Suppose that X, $\overline{X} \in \Sigma$ with L_k^n and $\overline{L}_k^n$ as their respective grid lengths. Let

$$S_n(X, \overline{X}) = (\pi n - \theta)^2 \sum_{k=1}^{n-1} |L_k^n - \overline{L}_k^n|$$

and $d_0(X,\overline{X}) = \limsup_{n\to\infty} S_n(X,\overline{X})$ and $d_\Sigma(X,\overline{X}) = \limsup_{n\to\infty} \frac{S_n(X,\overline{X})}{1+S_n(X,\overline{X})}$.

Since the function $f(x) = \frac{x}{1+x}$ is monotonic, we have

$$d_\Sigma(X,\overline{X}) = \frac{d_0(X,\overline{X})}{1 + d_0(X,\overline{X})} \in [0,\pi],$$

admitting that if $d_0(X,\overline{X}) = \infty$, then $d_\Sigma(X,\overline{X}) = 1$. Conversely,

$$d_0(X,\overline{X}) = \frac{d_\Sigma(X,\overline{X})}{1 - d_\Sigma(X,\overline{X})}.$$

After the following theorem, we can say that inverse nodal problem for operator L is stable.

Theorem 4.3. *The matric spaces $(\Omega, \|\cdot\|_1)$ and $(\Sigma/\sim, d_\Sigma)$ are homeomorphic to each other. Here, $\sim$ is the equivalence relation induced by d_Σ. Furthermore*

$$\|q-\overline{q}\|_1 = \frac{2d_\Sigma(X,\overline{X})}{1-d_\Sigma(X,\overline{X})},$$

where $d_\Sigma(X,\overline{X}) < 1$.

Proof. According to Theorem 3.2, using the definition of norm on L_1 for the potential functions, we obtain

$$\begin{aligned} \|q-\overline{q}\|_1 &\le 2(n-\frac{\theta}{\pi})^3 \int_0^\pi |L_n^j - \overline{L}_n^{\overline{j}}|dx + o(1) \\ &\le 2(n-\frac{\theta}{\pi})^3 \int_0^\pi |L_n^j - \overline{L}_n^j|dx + 2(n-\frac{\theta}{\pi})^3 \int_0^\pi |\overline{L}_n^j - \overline{L}_n^{\overline{j}}|dx + o(1) \end{aligned} \tag{4.1}$$

Here, the integrals in the second and first terms can be written as

$$\int_0^\pi |\overline{L}_n^j - \overline{L}_n^{\overline{j}}|dx = o(\frac{1}{n^3})$$

and

$$\int_0^\pi |L_n^j - \overline{L}_n^j|dx = \frac{1}{(\pi n-\theta)} \sum_{k=1}^{n-1} |L_k^n - \overline{L}_k^n|,$$

respectively. If we consider these equalities in (4.1), we obtain

$$\|q-\overline{q}\|_1 \le 2(\pi n-\theta)^2 \sum_{k=1}^{n-1} |L_k^n - \overline{L}_k^n| + o(1) = 2S_n(X,\overline{X}) + o(1). \tag{4.2}$$

Similarly, we can easily obtain

$$\|q-\overline{q}\|_1 \ge 2S_n(X,\overline{X}) + o(1) \tag{4.3}$$

The proof is complete after by taking limits in (4.2) and (4.3) as $n \to \infty$. $\square$

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