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CONTROLLABILITY OF NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. This article concerns the complete controllability for nonlinear neutral impulsive stochastic integro-differential system in finite dimensional spaces. Sufficient conditions ensuring the complete controllability are formulated and proved under the natural assumption that the associated linear control system is completely controllable. The results are obtained by using the Banach fixed point theorem. A numerical example is provided to illustrate our technique.

1. Introduction

The problem of controllability is one of the fundamental concept in mathematical control theory and engineering. The problem of controllability is to show the existence of a control function, which steers dynamical control systems from its initial state to the final state, where the initial and final states may vary over the entire space. The controllability of nonlinear deterministic systems in a finite dimensional space has been extensively studied, [1, 2].

Stochastic differential equations have been considered extensively through discussion in the finite dimensional spaces. As a matter of fact, there exist broad literature on the related to the topic and it has played an important role in many ways such as option pricing, forecast of the growth of population, etc., and as an applications which cover the generalizations of stochastic differential equations arising in the fields such as electromagnetic theory, population dynamics, and heat conduction in material with memory and stochastic differential equations are obtained by including random fluctuations in ordinary differential equations which have been deduced from phenomenological or physical laws. Random differential and integral equations play an important role in characterizing numerous social, physical, biological and engineering problems. For more details reader may refer [3, 4, 5] and reference therein. For a dynamic system the simplest continuous stochastic perturbation is naturally considered to be a Brownian motion (BM). In general, a continuous stochastic perturbation will be modeled as some stochastic integral with respect to the (BM). However, the (BM) has the strange property that even

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though its trajectory is continuous in t, it is not differentiable for all t . So for a stochastic integral with respect to (BM) one has to use a different approach, Ito approach is used to define it (see [6, 7] for details).

Controllability of non-linear stochastic systems in finite-dimensional spaces has been investigated by many authors. Klamka and Socha [8] derived sufficient conditions for the stochastic controllability of linear and nonlinear systems using a Lyapunov technique. Mahmudov and Zorlu [9] derived sufficient conditions for complete and approximate controllability of semilinear stochastic systems with non-Lipschitz coefficients via Picard-type iterations. Balachandran et al. [10, 11] studied the controllability of semilinear stochastic integrodifferential systems using the Banach fixed point theorem.

The theory of impulsive differential equations has provided a natural framework for mathematical modeling of many real world phenomena, namely in control, biological and medical domains [12, 13, 14]. In these models, the processes are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. The presence of impulses implies that the trajectories of the system do not necessarily preserve the basic properties of the non-impulsive dynamical systems. To this end the theory of impulsive differential systems has emerged as an important area of investigation in applied sciences [15]. Yang, Xu and Xiang [16] established the exponential stability of non-linear impulsive stochastic differential equations with delays. More recently, Liu and Liao [17] studied the existence, uniqueness and stability of stochastic impulsive systems using Lyapunov-like functions.

Many of the physical systems may also contain some information about the derivative of the state component and such systems are called neutral systems. Therefore, the investigation of stochastic impulsive neutral differential equations attracts great attention, especially as regards to controllability [18, 19].

In this article, we consider the impulsive neutral semilinear stochastic integrodifferential system

$$\begin{gathered} d\{x(t)-G(t,x(t),g(\eta x(t))\} \\ = A(t)x(t)dt + B(t)u(t)dt + F_1\left(t,x(t),f_{1,1}(\eta x(t)),f_{1,2}(\delta x(t)),f_{1,3}(\xi x(t))\right)dt \\ + F_2\left(t,x(t),f_{2,1}(\eta x(t)),f_{2,2}(\delta x(t)),f_{2,3}(\xi x(t))\right)dw(t), \\ t\in[0,T],\quad t\neq t_k, \\ \Delta x(t_k)=I_k(x(t_k^-)),\quad t=t_k,\ k=1,2,\dots,r, \\ x(0)=x_0\in\mathbb{R}^n, \end{gathered} \tag{1.1}$$

where, for $i=1,2$:

$$f_{i,1}(\eta x(t))=\int_0^t f_{i,1}(t,s,x(s))ds,\quad f_{i,2}(\delta x(t))=\int_0^T f_{i,2}(t,s,x(s))ds,$$
$$g(\eta x(t))=\int_0^t g(t,s,x(s))ds,\quad f_{i,3}(\xi x(t))=\int_0^t f_{i,3}(t,s,x(s))dw(s).$$

Here $A(t)$ and $B(t)$ are continuous matrices of dimensions $n\times n$, and $n\times m$ respectively

$$F_1:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n,$$
$$F_2:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to R^{n\times n},$$

$$G : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad f_{i,1}, f_{i,2} : [0,T] \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$f_{i,3} : [0,T] \times [0,T] \times \mathbb{R}^n \to R^{n\times n}, \quad g : [0,T] \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n.$$

$I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $u(t)$ is a feedback control and w is a n-dimensional standard (BM). Furthermore, $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Also $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump, the initial value x_0 is $\mathcal{F}_0$-measurable with $\mathbf{E}\|x_0\|^2 < \infty$.

The system (1.1) is in a very general form and it covers many possible models with various definitions of $f_{1,1}$, $f_{1,2}$, $f_{1,3}$, $f_{2,1}$, $f_{2,2}$, $f_{2,3}$ and g. We would like to mention that Balachandran and Karthikeyan [11] studied the case $I_k = g = f_{1,3} = f_{2,3} = 0$. The controllability problem with $g = f_{1,3} = f_{2,3} = 0$ was studied by Sakthivel [20]. The system (1.1) with $f_{1,3} = f_{2,3} = 0$ was investigated by Karthikeyan and Balachandran [21].

Motivated by the above references, we extend the results to obtain the complete controllability for wide class of impulsive neutral integro-differential equations under basic assumptions on the system operators. In particular, we assume the complete controllability of the associated linear system. To prove the main results, Theorem 3.5, we use stochastic analysis and fixed point theorem. Our work is organized as follows. The next section contains definitions, preliminary results and a mathematical model of impulsive stochastic systems with control. Section 3 is devoted to analyzing complete controllability results of the problem (1.1) via a fixed point technique. Section 4 contains an illustrative example.

2. Preliminaries

The problem of controllability of a linear stochastic system of the form

$$\begin{gathered} dx(t) = [A(t)x(t) + B(t)u(t)]dt + \sigma(t)dw(t), \quad t \in [0,T], \\ x(0) = x_0, \end{gathered} \tag{2.1}$$

has been studied by various authors [22, 23, 24] where $\sigma : [0,T] \to R^{n\times n}$. In this article, the following notation is adopted.

- $(\Omega, \mathcal{F}, P)$ is the probability space with probability measure P on Ω.
- $\{\mathcal{F}_t \mid t \in [0,T]\}$ is the filtration generated by $\{w(s) : 0 \le s \le t\}$ and $\mathcal{F} = \mathcal{F}_T$.
- $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ is the Hilbert space of all $\mathcal{F}_T$-measurable square integrable variables with values in $\mathbb{R}^n$.
- $U_{ad} = L_2^{\mathcal{F}}([0,T], R^m)$ is the Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in R^m.
- $PC([0;T];\mathbb{R}^n)$ is the space of function from $[0;T]$ into $\mathbb{R}^n$ such that $x(t)$ is continuous at $t \ne t_k$ and left continuous at $t = t_k$ and the right limit $X(t_k^+)$ exists for $k = 1, 2, \dots r$.
- $\mathbf{H}_2 := PC^b_{\mathcal{F}_t}([0,T], L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n))$ is the Banach space of all bounded $\mathcal{F}_t$ -measurable, $PC([0;T];\mathbb{R}^n)$ valued random variables φ satisfying
$$\|\varphi\|^2 = \sup_{t\in[0,T]} \mathbf{E}\|\varphi(t)\|^2.$$
- $\mathcal{L}(X, Y)$ is the space of all linear bounded operators from a Banach space X to a Banach space Y,
- $\phi(t) = \exp(At)$.

Now we introduce the following operators and sets.

- The operator $L_0^T \in \mathcal{L}\left(L_2^{\mathcal{F}}([0,T],R^m), L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)\right)$ is defined by
$$L_0^T = \int_0^T \phi(T-s)B(s)u(s)ds.$$
Clearly, the adjoint $\left(L_0^T\right)^* : L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n) \to L_2^{\mathcal{F}}([0,T],R^m)$ is defined by
$$(L_0^T)^* z = B^*\phi^*(T-t)\mathbf{E}(z|\mathcal{F}_t).$$
- The controllability operator Π_0^T associated with (2.1) is
$$\Pi_0^T(.) = \int_s^T \phi(T-t)BB^*\phi^*(T-t)\mathbf{E}(. \mid \mathcal{F}_t)dt,$$
which belongs to $\mathcal{L}\left(L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n), L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)\right)$ and the controllability matrix $\Gamma_s^T \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$
$$\Gamma_s^T = \int_s^T \phi(T-t)BB^*\phi^*(T-t)dt, \quad 0 \le s \le t.$$
- The set of all states attainable from x_0 in time $T>0$ is
$$\mathcal{R}_t(x_0) = \{x(t,x_0,u) : u \in U_{ad}\},$$
where $x(t,x_0,u)$ is the solution of (1.1) corresponding to $x_0 \in \mathbb{R}^n$ and $u(.) \in U_{ad}$.

Definition 2.1. System (1.1) is completely controllable on $[0,T]$ if
$$\mathcal{R}_T(x_0) = L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n),$$
that is, if all the points in $L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ can be reached from the point x_0 at time T. See Example (4.1) for Motivated application.

3. Controllability

In this section we derive controllability conditions for the non-linear stochastic system (1.1) using the contraction mapping principle.

We impose the following conditions on data of the problem

(H1) The functions F_i, $f_{i,j}$, G, g, $i=1,2$, $j=1,3$ satisfies the Lipschitz condition: there exist constants L_1, N_1, K_1, C_1, $q_k>0$ for x_h, y_h, v_h, $z_h \in \mathbb{R}^n$, $h=1,2$ and $0 \le s \le t \le T$ such that
$$\begin{gathered}
\|F_i(t,x_1,y_1,v_1,z_1) - F_i(t,x_2,y_2,v_2,z_2)\|^2 \\
\le L_1\left(\|x_1-x_2\|^2 + \|y_1-y_2\|^2 + \|v_1-v_2\|^2 + \|z_1-z_2\|^2\right), \\
\|G(t,x_1,y_1) - G(t,x_2,y_2)\|^2 \le N_1\left(\|x_1-x_2\|^2 + \|y_1-y_2\|^2\right), \\
\|f_{i,j}(t,s,x_1(s)) - f_{i,j}(t,s,x_2(s))\|^2 \le K_1\|x_1-x_2\|^2, \\
\|g(t,s,x_1(s)) - g(t,s,x_2(s))\|^2 \le C_1\|x_1-x_2\|^2, \\
\|I_k(x) - I_k(y)\|^2 \le q_k\|x-y\|^2, \quad k \in \{1,\dots,r\}.
\end{gathered}$$

(H2) The functions F_i, $f_{i,j}$, G, g, $i=1,2$, $j=1,3$ are continuous and there are constants L_2, N_2, K_2, C_2, $d_k>0$ for $x,y,v,z \in \mathbb{R}^n$ and $0 \le t \le T$ such that
$$\|F_i(t,x,y,v,z)\|^2 \le L_2\left(1 + \|x\|^2 + \|y\|^2 + \|v\|^2 + \|z\|^2\right),$$

$$
\begin{gathered}
\|G(t,x,y)\|^2 \le N_2\left(1+\|x\|^2+\|y\|^2\right), \\
\|f_{i,j}(t,s,x(s))\|^2 \le K_2\left(1+\|x\|^2\right), \\
\|g(t,s,x)\|^2 \le C_2\left(1+\|x\|^2\right), \\
\|I_k(x)\|^2 \le d_k\left(1+\|x\|^2\right), \quad k\in\{1,\dots,r\}.
\end{gathered}
$$

(H3) The linear system (2.1) is completely controllable.

Now for our convenience, let us introduce the following notation:

$$
\begin{gathered}
l_1=\max\{\|\phi(t)\|^2,\ t\in[0,T]\}, \quad l_2=\max\{\|A(t)\|^2,\ t\in[0,T]\}, \\
M=\max\{\|\Gamma_s^T\|^2,\ s\in[0,T]\}.
\end{gathered}
$$

The following lemma will play an important role in the proofs of our main results (see [5]).

Lemma 3.1 (Ito isometry)**.** *Let $\Psi : J\times\Omega\to\mathbb{R}^n$ be measurable and $\mathcal{F}_t$-adapted mapping and such that $\mathbf{E}\int_0^T\|\Psi(s,\omega)\|^2ds<\infty$. Then*

$$
\mathbf{E}\|\int_0^t\Psi(s)dw(s)\|^2=\mathbf{E}\Big(\int_0^t\|\Psi(s)\|^2ds\Big), \quad \text{for } t\in[0,T]
$$

Lemma 3.2 ([25])**.** *For every $z\in L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$*

- $\mathbf{E}\|\Pi_0^t z\|^2\le M\mathbf{E}\|z\|^2$.
- *Assume (H3) holds, then there exist $l_3>0$ such that*
$$
\mathbf{E}\|(\Pi_0^T)^{-1}\|^2\le l_3.
$$

We define the operator V from $\mathbf{H}_2$ to $\mathbf{H}_2$ as follows:

$$
\begin{aligned}
(Vx)(t)&=(\widehat{G}x)(t)+\int_0^t A\phi(t-s)(\widehat{G}x)(s)ds+\int_0^t\phi(t-s)(\widehat{F}_1x)(s)ds \\
&\quad+\int_0^t\phi(t-s)(\widehat{F}_2x)(s)dw(s)+\sum_{0<t_k<t}\phi(t-t_k)I_k(x(t_k^-)),
\end{aligned}
$$

where, for $i=1,2$,

$$
\begin{gathered}
(\widehat{F}_ix)(t)=F_i\left(t,x(t),f_{i,1}(\eta x(t)),f_{i,2}(\delta x(t)),f_{i,3}(\xi x(t))\right), \\
(\widehat{G}x)(t)=G\left(t,x(t),g(\eta x(t))\right).
\end{gathered}
$$

The following results will be used throughout this paper.

Lemma 3.3. *Under conditions* (H1) *and* (H2)*, there exist real constants $M_1,M_2>0$ such that for $x,y\in\mathbf{H}_2$, we have*

$$
\mathbf{E}\|(Vx)(t)-(Vy)(t)\|^2\le M_1\Big(\sup_{s\in[0,T]}\mathbf{E}\|x(s)-y(s)\|^2\Big), \tag{3.1}
$$

$$
\mathbf{E}\|(Vx)(t)\|^2\le M_2\Big(1+T\sup_{s\in[0,T]}\mathbf{E}\|x(s)\|^2\Big). \tag{3.2}
$$

Proof. First, we prove inequality (3.1), since (3.2) can be established in a similar way. For $i=1,2$, let $x,y\in\mathbf{H}_2$. It follows from condition (H1), Holder inequality and Ito isometry that

$$
\begin{aligned}
&\|(\widehat{F}_ix)(t)-(\widehat{F}_iy)(t)\|^2 \\
&\le L_1\Big(\|x(t))-y(t))\|^2+\|f_{i,1}(\eta x(t))-f_{i,1}(\eta y(t))\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \|f_{i,2}(\delta x(t)) - f_{i,2}(\delta y(t))\|^2 + \|f_{i,3}(\xi x(t)) - f_{i,3}(\xi y(t))\|_Q^2\Big), \\
&\le L_1(1 + 2T^2K_1 + TK_1) \sup_{s\in[0,T]} \|x(s) - y(s)\|^2,
\end{aligned}
$$

from which it follows that

$$
\mathbf{E}\Big(\int_0^t \|(\widehat{F}_i x)(s) - (\widehat{F}_i y)(s)\|^2 ds\Big) \le L_1T(1+2T^2K_1+TK_1) \sup_{s\in[0,T]} \mathbf{E}\|x(s) - y(s)\|^2.
$$

We have

$$
\begin{aligned}
\|(\widehat{G}x)(t) - (\widehat{G}y)(t)\|^2 &\le N_1\Big(\|x(t) - y(t)\|^2 + \|g(\eta x(t)) - g(\eta y(t))\|^2\Big), \\
&\le N_1(1 + T^2C_1)\Big(\sup_{s\in[0,T]} \|x(s) - y(s)\|^2\Big),
\end{aligned}
$$

Then we obtain

$$
\mathbf{E}\Big(\int_0^t \|(\widehat{G}x)(s) - (\widehat{G}y)(s)\|^2 ds\Big) \le N_1T(1+T^2C_1)\Big(\sup_{s\in[0,T]} \mathbf{E}\|x(s) - y(s)\|^2\Big). \tag{3.3}
$$

It follows from the above inequality, Holder inequality and Ito isometry that

$$
\begin{aligned}
&\mathbf{E}\|(Vx)(t) - (Vy)(t)\|^2 \\
&\le 5\mathbf{E}\|\int_0^t A\phi(t-s)[(\widehat{G}x)(s) - (\widehat{G}y)(s)]ds\|^2 \\
&\quad + 5\mathbf{E}\|\int_0^t \phi(t-s)[(\widehat{F}_1x)(s) - (\widehat{F}_1y)(s)]ds\|^2 \\
&\quad + 5\mathbf{E}\|\int_0^t \phi(t-s)[(\widehat{F}_2x)(s) - (\widehat{F}_2y)(s)]dw(s)\|^2 \\
&\quad + 5\mathbf{E}\|\sum_{0<t_k<t} \phi(t-t_k)[I_k(x(t_k^-)) - I_k(y(t_k^-))]\|^2 + 5\mathbf{E}\|(\widehat{G}x)(t) - (\widehat{G}y)(t)\|^2,
\end{aligned}
$$

then, we have

$$
\begin{aligned}
&\mathbf{E}\|(Vx)(t) - (Vy)(t)\|^2 \\
&\le 5Tl_1l_2\mathbf{E}\int_0^t \|(\widehat{G}x)(s) - (\widehat{G}y)(s)\|^2ds + 5Tl_1\mathbf{E}\int_0^t \|(\hat{F}_1x)(s) - (\hat{F}_1y)(s)\|^2ds \\
&\quad + 5l_1\mathbf{E}\int_0^t \|(\hat{F}_2x)(s) - (\hat{F}_2y)(s)\|^2ds + 5l_1r\sum_{k=1}^r \mathbf{E}\|I_k(x(t_k^-)) - I_k(y(t_k^-))\|^2 \\
&\quad + 5\mathbf{E}\|(\widehat{G}x)(t) - (\widehat{G}y)(t)\|^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\mathbf{E}\|(Vx)(t) - (Vy)(t)\|^2 \\
&\le \Big(10T^2l_1l_2N_1(1 + T^2C_1) + 15l_1(T + 1)L_1T(1 + 2T^2K_1 + TK_1) \\
&\quad + 5l_1r\left(\sum_{k=1}^r q_k\right) + 10N_1(1 + T^2C_1)\Big) \sup_{s\in[0,T]} \mathbf{E}\|x(s) - y(s)\|^2 \\
&= M_1 \sup_{s\in[0,T]} \mathbf{E}\|x(s) - y(s)\|^2,
\end{aligned}
$$

where

$$\begin{aligned} M_1 &= 5T^2 l_1 l_2 N_1(1+T^2C_1) + 5l_1 r\left(\sum_{k=1}^{r} q_k\right) + 5N_1(1+T^2C_1) \\ &\quad + 5l_1(T+1)L_1T(1+2T^2K_1+TK_1). \end{aligned}$$

□

For a given initial condition and any $u \in U_{ad}$ for $t \in [0,T]$, one can prove the existence and uniqueness of solution $x(t, x_0, u)$ of the of the nonlinear impulsive stochastic integrodifferential state equations (1.1) based on the fixed point technique [26]. The solution of the which can be represented in the following integral form:

$$\begin{aligned} x(t) &= \phi(t)[x_0 - G(0,x_0,0)] + (\widehat{G}x)(t) + \int_0^t A\phi(t-s)(\widehat{G}x)(s)ds \\ &\quad + \int_0^t \phi(t-s)\left(Bu(s) + (\widehat{F}_1x)(s)\right)ds + \int_0^t \phi(t-s)(\widehat{F}_2x)(s)dw(s) \\ &\quad + \sum_{0<t_k<t} \phi(t-t_k)I_k(x(t_k^-)), \\ &= \phi(t)[x_0 - G(0,x_0,0)] + (Vx)(t) + \int_0^t \phi(t-s)Bu(s)ds. \end{aligned} \tag{3.4}$$

The following lemma gives a formula for a control steering the state x_0 to an arbitrary final point x_T.

Lemma 3.4. *Assume Π_0^T is invertible, then for arbitrary $x_T \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ the control*

$$u(t) = B^*\phi^*(T-t)\mathbf{E}\{(\Pi_0^T)^{-1}\left(x_T - \phi(T)[x_0 - G(0,x_0,0)] - (Vx)(T)\right)|\mathcal{F}_t\} \tag{3.5}$$

transfers the system (3.4) *from $x_0 \in \mathbb{R}^n$ to $x_T \in \mathbb{R}^n$ at time T.*

Proof. By substituting (3.5) in (3.4), we obtain

$$\begin{aligned} x(t) &= \phi(t)[x_0 - G(0,x_0,0)] + (Vx)(t) \\ &\quad + \int_0^t \phi(t-s)BB^*\phi^*(t-s)\phi^*(T-t) \\ &\quad \times \mathbf{E}\{(\Pi_0^T)^{-1}\left(x_T - \phi(T)[x_0 - G(0,x_0,0)] - (Vx)(T)\right)|\mathcal{F}_t\} \\ &= \phi(t)[x_0 - G(0,x_0,0)] + (Vx)(t) + \Pi_0^t(\phi^*(T-t)(\Pi_0^T)^{-1} \\ &\quad \times (x_T - \phi(T)[x_0 - G(0,x_0,0)] - (Vx)(T))). \end{aligned} \tag{3.6}$$

Writing $t = T$ in (3.6), we see that the control $u(.)$ transfers the system (3.4) from x_0 to x_T. □

To apply the contraction mapping principle, we define the nonlinear operator $\Upsilon : \mathbf{H}_2 \to \mathbf{H}_2$ by

$$(\Upsilon x)(t) = \phi(t)[x_0 - G(0,x_0,0)] + (Vx)(t) + \int_0^t \phi(t-s)Bu(s)ds,$$

where

$$u(t) = B^*\phi^*(T-t)\mathbf{E}\{(\Pi_0^T)^{-1}\left(x_T - \phi(T)[x_0 - G(0,x_0,0)] - (Vx)(T)\right)|\mathcal{F}_t\}. \tag{3.7}$$

From Lemma (3.4), the control (3.7) transfers the system (3.4) from the initial state x_0 to the final state x_T provided that the operator Υ has a fixed point. So, if the operator Υ has a fixed point then the system (1.1) is completely controllable. As mentioned above, to prove the complete controllability it is enough to show that Υ has a fixed point in $\mathbf{H}_2$. To do this, we use the contraction mapping principle. To apply the contraction principle, first we show that Υ maps $\mathbf{H}_2$ into itself.

Theorem 3.5. *Assume that conditions* (H1)–(H3) *hold. If the inequality*

$$2M_1(1 + Ml_1l_3) < 1 \tag{3.8}$$

holds, then the stochastic control system (1.1) *is completely controllable on* $[0, T]$.

Proof. To prove the complete controllability of the stochastic system (3.4) it is enough to show that Υ has a fixed point in $\mathbf{H}_2$. To apply the contraction principle, first we show that Υ maps $\mathbf{H}_2$ into itself. Let $x \in \mathbf{H}_2$. Now by Lemma (3.4), for $t \in [0, T]$ we have

$$\begin{aligned}
&\mathbf{E}\|(\Upsilon x)(t)\|^2 \\
&= \mathbf{E}\|\phi(t)[x_0 - G(0, x_0, 0)] + (Vx)(t) \\
&\quad + \Pi_0^t\phi^*(T - t)(\Pi_0^T)^{-1}\left(x_T - \phi(T)[x_0 - G(0, x_0, 0)] - (Vx)(T)\right)\|^2, \\
&\le 3\mathbf{E}\|\phi(t)[x_0 - G(0, x_0, 0)]\|^2 + 3\mathbf{E}\|(Vx)(t)\|^2 \\
&\quad + 3\mathbf{E}\|\Pi_0^t\phi^*(T - t)(\Pi_0^T)^{-1}\left(x_T - \phi(T)[x_0 - G(0, x_0, 0)] - (Vx)(T)\right)\|^2.
\end{aligned}$$

From Lemma (3.2) it follows that

$$\begin{aligned}
&\mathbf{E}\|(\Upsilon x)(t)\|^2 \\
&\le 6l_1\left(\|x_0\|^2 + \|G(0, x_0, 0)\|^2\right) + 3\mathbf{E}\|(Vx)(t)\|^2 \\
&\quad + 9Ml_1l_3\left(\mathbf{E}\|x_T\|^2 + 2l_1[\|x_0\|^2 + \|G(0, x_0, 0)\|^2] + \mathbf{E}\|(Vx)(T)\|^2\right), \\
&\le 6l_1\left(\|x_0\|^2 + \|G(0, x_0, 0)\|^2\right) \\
&\quad + 9Ml_1l_3\left(\mathbf{E}\|x_T\|^2 + 2l_1[\|x_0\|^2 + \|G(0, x_0, 0)\|^2]\right) \\
&\quad + 3(1 + 3Ml_1l_3)M_2\Big(1 + T\sup_{s\in[0,T]}\mathbf{E}\|x(s))\|^2\Big),
\end{aligned}$$

therefore, we obtain that $\|(\Upsilon x)(t)\|^2_{\mathbf{H}_2} < \infty$. Since Υ maps $\mathbf{H}_2$ into itself.

Secondly, we show that Υ is a contraction mapping. To see this let $x, y \in \mathbf{H}_2$, so for $t \in [0, T]$ we have

$$\begin{aligned}
&\mathbf{E}\|(\Upsilon x)(t) - (\Upsilon y)(t)\|^2 \\
&= \mathbf{E}\|(Vx)(t) - (Vy)(t) + \Pi_0^t\phi^*(T - t)(\Pi_0^T)^{-1}\left((Vx)(T) - (Vy)(T)\right)\|^2, \\
&\le 2\mathbf{E}\|(Vx)(t) - (Vy)(t)\|^2 + 2Ml_1l_3\mathbf{E}\|(Vx)(T) - (Vy)(T)\|^2, \\
&\le 2(1 + Ml_1l_3)\sup_{s\in[0,T]}\mathbf{E}\|V(x(s)) - V(y(s))\|^2, \\
&\le 2(1 + Ml_1l_3)M_1\Big(\sup_{s\in[0,T]}\mathbf{E}\|x(s) - y(s)\|^2\Big).
\end{aligned}$$

It result from Lemma (3.2) and inequality (3.1) that

$$\sup_{s\in[0,T]}\mathbf{E}\|(\Upsilon x)(s) - (\Upsilon y)(s)\|^2 \le 2M_1(1 + Ml_1l_3)\Big(\sup_{s\in[0,T]}\mathbf{E}\|x(s) - y(s)\|^2\Big).$$

Therefore Υ is contraction mapping if the inequality (3.8) holds. Then the mapping Υ has a unique fixed point $x(.)$ in $\mathbf{H}_2$ which is the solution of the equation (1.1). Thus the system (1.1) is completely controllable. $\square$

4. Applications

Example 4.1. A rocket in vertical motion may be modeled by

$$\begin{aligned} \dot{h} &= v \\ m\dot{v} &= -mg + f\,, \end{aligned} \tag{4.1}$$

where h is altitude, v is velocity; m is mass, f is thrust force. Let $x_1 = h$, $x_2 = v$, $u = \frac{f}{m} - g$, then (4.1) becomes

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= u \end{aligned}$$

The complete controllability is to find a (continuous) control $u(t)$ over the period $[t_0, t_f]$ to move the state of the system from a given initial state $x(t_0) = x_0$ to a desired final state $x(t_f) = x_f$.

Example 4.2. Consider the nonlinear impulsive neutral stochastic systems in the form of (1.1),

$$\begin{aligned} d\{x(t) - (\widehat{G}x)(t)\} &= -5x(t)dt + \{e^{-2t}u(t) + (\widehat{F}_1x)(t)\}dt \\ &\quad + (\widehat{F}_2x)(t)dw(t), \quad t \in [0,T],\ t \neq t_k, \\ \Delta x(t_k) = 0,24e^{0,03}(x(t_k^{-1})), &\quad t = t_k,\ \text{where } t_k = t_{k-1} + 0,5 \text{ for } k = 1,2,\dots r. \\ x(0) &= x_0 \in \mathbb{R}^n. \end{aligned} \tag{4.2}$$

where $w(.)$ is one-dimensional Brownian motion. (BM) is any of various physical phenomena in which some quantity is constantly undergoing small, random fluctuations. If a number of particles subject to (BM) are present in a given medium and there is no preferred direction for the random oscillations, then over a period of time the particles will tend to be spread evenly throughout the medium. (BM) is a Gaussian process with independent increments which are normally distributed. Here

$$A(t) = -5,\ B(t) = e^{-2t}.$$

Moreover,

$$\begin{aligned} (\widehat{F}_1x)(t) &= x(t) + 2t^2e^{-t} + \int_0^t se^{-s}x(s)ds \\ &\quad + \int_0^T \arctan(x(s))ds + \int_0^t \cos(x(s))dw(s), \\ (\widehat{F}_2x)(t) &= e^{-t}\sin(x(t)) + \int_0^t (2s^2+3)x(s)ds \\ &\quad + \int_0^T \frac{1}{\sqrt{1+|x(s)|}}ds + \int_0^t \log(1+|x(s)|)dw(s), \\ (\widehat{G}x)(t) &= \log[e^{2t}\Big|\int_0^t e^{-s}(x(s)+1)ds\Big| + 1], \end{aligned}$$

Note that the above functions satisfy the hypotheses (H1)–(H2). For this system, we have

$$\begin{aligned}\Gamma_0^T &= \int_0^T \phi(T,s)B(s)B^*(s)\phi^*(T,s)ds,\\ &= \int_0^T e^{-4T}ds = Te^{-4T} > 0, \quad \text{for some } T > 0.\end{aligned}$$

Hence, the stochastic system (1.1) is completely controllable on $[0,T]$.

References

[1] K. Balachandran, J. P. Dauer; Controllability of nonlinear systems via fixed-point theorems. *Journal of optimization theory and applications*, 53(3):345–352, 1987.

[2] J. Klamka; Schauder's fixed-point theorem in nonlinear controllability problems. *Control and Cybernetics*, 29(1):153–166, 2000.

[3] Albert T. Bharucha-Reid; *Random integral equations.* Elsevier, 1972.

[4] Xuerong Mao; *Stochastic differential equations and applications.* Elsevier, 2007.

[5] Bernt Øksendal; *Stochastic differential equations.* Springer, 2003.

[6] Rong Situ; Brownian motion, stochastic integral and ito's formula. *Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering*, pages 39–74, 2005.

[7] Szymon Peszat, Jerzy Zabczyk; *Stochastic partial differential equations with Lévy noise: An evolution equation approach*, volume 113. Cambridge University Press, 2007.

[8] J. Klamka L. Socha. Some remarks about stochastic controllability. *Automatic Control, IEEE Transactions on*, 22(5):880–881, 1977.

[9] N. I. Mahmudov, S. Zorlu; Controllability of semilinear stochastic systems. *International Journal of Control*, 78(13):997–1004, 2005.

[10] K. Balachandran, S. Karthikeyan; Controllability of stochastic integrodifferential systems. *International Journal of Control*, 80(3):486–491, 2007.

[11] K. Balachandran, S. Karthikeyan; Controllability of nonlinear itô type stochastic integrodifferential systems. *Journal of the Franklin Institute*, 345(4):382–391, 2008.

[12] Tao Yang; *Impulsive control theory*, volume 272. Springer, 2001.

[13] Drumi Dimitrov Bainov, Pavel S. Simeonov; *Impulsive differential equations: periodic solutions and applications*, volume 66. CRC Press, 1993.

[14] Krishnan Balachandran, F. Paul Samuel; Existence of solutions for quasilinear delay integrodifferential equations with nonlocal conditions. *Electronic Journal of Differential Equations*, 2009(06):1–7, 2009.

[15] Vangipuram Lakshmikantham, Baĭ; *Theory of impulsive differential equations.*

[16] Zhiguo Yang, Daoyi Xu, Li Xiang; Exponential p-stability of impulsive stochastic differential equations with delays. *Physics Letters A*, 359(2):129–137, 2006.

[17] Bin Liu, Xinzhi Liu, Xiaoxin Liao; Existence and uniqueness and stability of solutions for stochastic impulsive systems. *Journal of Systems Science and Complexity*, 20(1):149–158, 2007.

[18] S. Sivasundaram. Josaphat Uvah; Controllability of impulsive hybrid integro-differential systems. *Nonlinear Analysis: Hybrid Systems*, 2(4):1003–1009, 2008.

[19] Daoyi Xu, Zhiguo Yang, Zhichun Yang; Exponential stability of nonlinear impulsive neutral differential equations with delays. *Nonlinear Analysis: Theory, Methods & Applications*, 67(5):1426–1439, 2007.

[20] R. Sakthivel, N. I. Mahmudov, Sang-Gu Lee; Controllability of non-linear impulsive stochastic systems. *International Journal of Control*, 82(5):801–807, 2009.

[21] S. Karthikeyan, K. Balachandran; Controllability of nonlinear stochastic neutral impulsive systems. *Nonlinear Analysis: Hybrid Systems*, 3(3):266–276, 2009.

[22] Jerzy Zabczyk; Controllability of stochastic linear systems. *Systems & Control Letters*, 1(1):25–31, 1981.

[23] M. Ehrhardt, W. Kliemann; Controllability of linear stochastic systems. *Systems & Control Letters*, 2(3):145–153, 1982.

[24] Nazim I. Mahmudov; Controllability of linear stochastic systems. *Automatic Control, IEEE Transactions on*, 46(5):724–731, 2001.

[25] N. I. Mahmudov, S. Zorlu; Controllability of non-linear stochastic systems. *International Journal of Control*, 76(2):95–104, 2003.

[26] M. G. Murge, B. G. Pachpatte; Explosion and asymptotic behavior of nonlinear ito type stochastic integrodifferential equations. *Kodai mathematical journal*, 9(1):1–18, 1986.

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