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# HOMOGENIZATION OF REACTION-DIFFUSION EQUATIONS IN FRACTURED POROUS MEDIA

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Abstract. The article studies the homogenization of reaction-diffusion equations with large reaction terms in a multi-scale porous medium. We assume that the fractures and pores are equidistributed and that the coefficients of the equations are periodic. Using the multi-scale convergence method, we derive a homogenization result whose limit problem is defined on a fixed domain and is of convection-diffusion-reaction type.

## 1. Introduction

Our aim is to investigate, by means of mathematical homogenization techniques, the diffusion phenomenon in a multi-scale porous medium. The medium consists of a connected network made of pores and fractures which are equidistributed, and the diffusion process is modelled by a semilinear reaction-diffusion equation with a large reaction term. To be more precise, we consider a diffusion process modelled by the boundary-value problem

$$\begin{gathered}
\rho\big(\frac{x}{\varepsilon}\big)\frac{\partial u_\varepsilon}{\partial t} = \operatorname{div}\Big(A\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)\nabla u_\varepsilon\Big) + \frac{1}{\varepsilon}g\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\big) \quad \text{in } Q_T^\varepsilon = \Omega^\varepsilon\times(0,T),\\
A\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)\nabla u_\varepsilon\cdot\nu = 0 \quad \text{on } (\partial\Omega^\varepsilon\setminus\partial\Omega)\times(0,T),\\
u_\varepsilon = 0 \quad \text{on } (\partial\Omega^\varepsilon\cap\partial\Omega)\times(0,T),\\
u_\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega^\varepsilon,
\end{gathered} \tag{1.1}$$

where $T>0$ is a fixed real number representing the final time of the process, $\Omega^\varepsilon$ is a fractured porous domain in which the process occurred and whose structure follows in the lines below (see [23]).

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N\geq 3$) locally located on one side of its Lipschitz continuous boundary $\partial\Omega$. Let $Y=(0,1)^N$ be the unit cell in $\mathbb{R}^N$ and put $Y=\overline{Y}_c\cup Y_m$ where $Y_m$ and $Y_c$ are two disjoint open connected sets representing the local structure of the porous matrix and the cracks (fissures), respectively. We assume that a periodic repetition of $Y_m$ in $\mathbb{R}^N$ is connected and has a Lipschitz continuous boundary. Next, we set $Y_m = Z_s\cup\overline{Z}_p$ where $Z_p$ and $Z_s$ are two disjoint open connected sets representing the local structure of the solid part of the porous

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matrix and the pores, respectively. We assume that $Z_p$ and $Z_s$ have strictly positive Lebesgue measures and that $Z_s$ has a Lipschitz continuous boundary. The fractured porous medium $\Omega^\varepsilon$ is defined as follows. For $\varepsilon > 0$, we set

$$G_m = \cup_{k\in\mathbb{Z}^N}(k + Y_m) \quad \text{and} \quad G_c = \mathbb{R}^N \setminus \overline{G}_m,$$

and

$$G_s = \cup_{k\in\mathbb{Z}^N}(k + Z_s) \quad \text{and} \quad G_p = G_m \setminus \overline{G}_s,$$

and we define the pores space $\Omega_p^\varepsilon = \Omega \cap \varepsilon^2 G_p$ (this include the pores crossing $\partial\Omega$), the cracks space $\Omega_c^\varepsilon = \Omega \cap \varepsilon G_c$ (this includes the cracks crossing $\partial\Omega$) and the fractured porous medium as

$$\Omega^\varepsilon = \Omega \setminus (\Omega_p^\varepsilon \cup \Omega_c^\varepsilon).$$

We assume that both $\Omega^\varepsilon$ and $\Omega_p^\varepsilon \cup \Omega_c^\varepsilon$ are connected.

This being so, the $\varepsilon$-problem (1.1) is constrained as follows:

(A1) (Uniform ellipticity) The matrix $A(y,\tau) = (a_{ij}(y,\tau))_{1\le i,j\le N}$ in the space $(L^\infty(\mathbb{R}^{N+1}))^{N\times N}$ is real, symmetric, positive definite, i.e, there exists $\Lambda > 0$ such that

$$\|a_{ij}\|_{L^\infty(\mathbb{R}^{N+1})} \le \Lambda, \quad 1 \le i,j \le N,$$

$$\sum_{ij=1}^{N} a_{ij}(y,\tau)\zeta_i\zeta_j \ge \Lambda^{-1}|\zeta|^2 \quad \forall (y,\tau) \in \mathbb{R}^{N+1}, \zeta \in \mathbb{R}^N.$$

(A2) (Lipschitz continuity) The function $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses. There exists $C > 0$ such that for any $(y,\tau) \in \mathbb{R}^{N+1}$ and $u \in \mathbb{R}$,

$$|\partial_u g(y,\tau,u)| \le C$$
$$|\partial_u g(y,\tau,u_1) - \partial_u g(y,\tau,u_2)| \le C|u_1 - u_2|(1 + |u_1| + |u_2|)^{-1}.$$

(A3) $g(y,\tau,0) = 0$ for any $(y,\tau) \in \mathbb{R}^{N+1}$.

(A4) (Periodicity) We assume that:
  - (i) $g(\cdot,\cdot,u) \in \mathcal{C}_{\rm per}(Y\times\mathcal{T})$ $(\mathcal{T} = (0,1))$ for any $u \in \mathbb{R}$ with $\int_Y g(y,\tau,u)\,dy = 0$ for all $(\tau,u) \in \mathcal{T} \times \mathbb{R}$;
  - (ii) the functions $a_{ij}$ lie in $L^2_{\rm per}(Y \times \mathcal{T})$ for all $1 \le i,j \le N$;
  - (iii) the density function $\rho$ belongs to $\mathcal{C}_{\rm per}(Y)$ and satisfies $\Lambda^{-1} \le \rho(y) \le \Lambda$ for all $y \in \mathbb{R}^N$.

**Remark 1.1.** As a direct consequence of the periodicity and the zero mean value hypothesis for the function $g$ (see precisely the first item of the hypothesis (A4) above), there exists a unique $R(\cdot,\cdot,u) \in \mathcal{C}_{\rm per}(Y\times T)$ such that $\Delta_y R(\cdot,\cdot,u) = g(\cdot,\cdot,u)$ and $\int_Y R(y,\tau,u)\,dy = 0$ for all $\tau$, $u \in \mathbb{R}$. Moreover $R(\cdot,\cdot,u)$ is at least twice differentiable with respect to $y$. Furthermore, on letting $G = \nabla_y R$ it follows from **A2** and (A3) that

$$|G(y,\tau,u)| \le C|u|, \quad |\partial_u G(y,\tau,u)| \le C, \tag{1.2}$$

$$|\partial_u G(y,\tau,u_1) - \partial_u G(y,\tau,u_2)| \le C|u_1 - u_2|(1 + |u_1| + |u_2|)^{-1}. \tag{1.3}$$

The motivation for problem (1.1) arises from its applicability in the area of modeling of flow and transport in fractured porous media related to environmental and energy problems. In order to overcome difficulties encountered in numerical simulations in multi-scale porous media, we need to upscale such models, that is to find equivalent models by letting $\varepsilon \to 0$. This leads to model problems posed on a fixed domain $Q = \Omega \times (0,T)$ with suitable boundary conditions, hence relatively easy to handle numerically.

Problem (1.1) can also be viewed as modelling the flow of a single phase compressible fluid in a fractured porous medium that obey nonlinear Darcy law. In that case, $u_\varepsilon$ is the density of the fluid, $\rho(y)$ is the porosity of the medium while $A(y,\tau)$ is the permeability of the medium. As the scale (size) of the fractures and that of the pores are separated (ratio of order $\varepsilon$), we use the multi-scale (or reiterated two-scale) convergence method in the framework introduced in [20] to upscale the $\varepsilon$-problem.

The homogenization of parabolic equations in perforated has been widely investigated in the literature. We quote some works similar to ours. In [18] the homogenization of parabolic monotone operator in periodically perforated domain is considered. The problem they consider is degenerate and they use the two-scale convergence method. In [13, 14], the authors study the homogenization of a family of parabolic equations

$$\frac{\partial}{\partial t} b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - \operatorname{div} a(u_\varepsilon, \nabla u_\varepsilon) = f$$

in periodically perforated domain $\Omega_\varepsilon$ with Dirichlet boundary conditions and Neuman boundary conditions, respectively. In [6] is considered the upscaling of a convection-diffusion equation in a perforated domain made of holes periodically distributed. The homogenization limit for the diffusion equation with nonlinear flux condition on the boundary of a periodically perforated domain is studied in [12]. In [7] the homogenization of a semilinear parabolic equation in a periodically perforated domain is considered. In [10] the authors describe some diffusion models for fractured media. We also mention [11] where the author used the $\Gamma$-convergence method associated with multi-scale convergence notions to get a limit law of an incompressible viscous flow in a porous medium with double porosity.

Taking into account the preceding review which is far from being exhaustive, we observe that the study of a problem like (1.1) is relevant due to many reasons. For example, contrarily to the problems studied in [7, 13, 14] where the perforations are on one scale with time independent coefficients, our problem is set up in a fractured porous media (perforation on two scale) and deals with time dependent coefficients with the time variable oscillating at a different speed from the space variable. The perforation on two scale and the oscillations at different speeds drive our problem into a non-standard framework of reiterated homogenization, which, combined with the large nonlinear oscillating reaction term makes the proofs in this paper quite involved. When passing to the limit (as $\varepsilon \to 0$), the large reaction term in the $\varepsilon$-problem generates a convection term and we get a limit problem of convection-diffusion-reaction type. The main result of the paper states as follows.

**Theorem 1.2.** *Assume that the hypotheses* (A1)–(A4) *are satisfied and let $u_\varepsilon$ ($\varepsilon > 0$) be the unique solution to* (1.1). *Then as $\varepsilon \to 0$ we have*

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega_T),$$

*where* $u_0 \in L^2(0,T;H_0^1(\Omega))$ *is the unique solution to*

$$
\begin{aligned}
|Z_s|\Big(\int_{Y_m}\rho(y)\,dy\Big)\frac{\partial u_0}{\partial t} &= \operatorname{div}\big(\hat{A}(x,t)\nabla u_0\big) + \operatorname{div}\, L_1(x,t,u_0)\\
&\quad - L_2(x,t,u_0)\cdot\nabla u_0 - L_3(x,t,u_0) \quad \text{in } \Omega_T\\
u_0 &= 0 \quad \text{on } \partial\Omega\times(0,T)\\
u_0(x,0) &= u^0(x) \quad \text{in } \Omega.
\end{aligned}
$$

The coefficients and operators in the theorem above are defined in Section 4. This article is organized as follows. A priori estimates and compactness results are formulated and proved in Section 2. In Section 3, we recall the concept of multi-scale convergence and prove some preliminary results. Finally, Section 4 deals with the passage to the limit and the derivation of the macroscopic model for problem (1.1).

## 2. A priori estimates and compactness result

Throughout, $C$ denotes a generic constant independent of $\varepsilon$ that can change from one line to the next, the centered dot stands for the Euclidean scalar product in $\mathbb{R}^N$ while the absolute value or modulus is denoted by $|\cdot|$.

With the connectedness of $\Omega^\varepsilon$ in mind, the space

$$
V_\varepsilon = \{u \in H^1(\Omega^\varepsilon) : u = 0 \text{ on } \partial\Omega\cap\partial\Omega^\varepsilon\} \tag{2.1}
$$

is Hilbertian when endowed with the gradient norm,

$$
\|u\|_{V_\varepsilon} = \|\nabla u\|_{L^2(\Omega^\varepsilon)} \quad (u \in V_\varepsilon). \tag{2.2}
$$

Therefore, the Lipschitzity of the function $g(y,\tau,\cdot)$ and the positivity assumption on the density function $\rho$ readily imply (see e.g., [4, 17]) the existence of a unique solution $u_\varepsilon \in L^2(0,T;V_\varepsilon)\cap\mathcal{C}(0,T;L^2(\Omega^\varepsilon))$ to the problem (1.1). Moreover the following uniform estimates hold.

**Lemma 2.1.** *Assume that the hypotheses* (A1)–(A4) *are satisfied. Then the following estimates hold:*

$$
\sup_{0\le t\le T}\|u_\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} \le C, \tag{2.3}
$$

$$
\int_0^T \|\nabla u_\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\,dt \le C, \tag{2.4}
$$

$$
\|\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\|_{L^2(0,T;V_\varepsilon')} \le C, \tag{2.5}
$$

*where $C$ is a positive constant which does not depend on $\varepsilon$.*

*Proof.* Let $t \in (0,T]$. Multiplying the first equation in (1.1) by $u_\varepsilon$ and integrating over $\Omega^\varepsilon\times(0,t)$ yields:

$$
\begin{aligned}
&\|(\rho^\varepsilon)^{1/2}u_\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} - \|(\rho^\varepsilon)^{\frac12}u^0\|^2_{L^2(\Omega^\varepsilon)} + 2\int_0^t\int_{\Omega^\varepsilon}A^\varepsilon|\nabla u_\varepsilon(s)|^2\,dx\,ds\\
&= 2\int_0^t\int_{\Omega^\varepsilon}\frac{1}{\varepsilon}g^\varepsilon(u_\varepsilon(s))u_\varepsilon(s)\,dx\,ds.
\end{aligned} \tag{2.6}
$$

But Remark 1.1 readily implies

$$
\frac{1}{\varepsilon}g\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\big) = \operatorname{div} G\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\big) - \partial_r G\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\big)\cdot\nabla u_\varepsilon,
$$

which combined with (2.6) leads to

$$\begin{aligned}
&\|(\rho^\varepsilon)^{1/2}u_\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + 2\int_0^t\int_{\Omega^\varepsilon} A^\varepsilon|\nabla u_\varepsilon(s)|^2\,dx\,ds\\
&\le \|(\rho^\varepsilon)^{1/2}u^0\|^2_{L^2(\Omega^\varepsilon)} - 2\int_0^t\int_{\Omega^\varepsilon} G^\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon\,dx\,ds\\
&\quad - 2\int_0^t\int_{\Omega^\varepsilon}(\partial_r G^\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon)u_\varepsilon\,dx\,ds,
\end{aligned}$$

where $G^\varepsilon(u_\varepsilon) = G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)$ and $\partial_r G^\varepsilon(u_\varepsilon) = \frac{\partial}{\partial_r}G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)$. Using (1.2), the ellipticity of the matrix $A$ and the boundedness of the function $\rho$, we have

$$\begin{aligned}
&\Lambda^{-1}\|u_\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + 2\Lambda^{-1}\int_0^t\int_{\Omega^\varepsilon}|\nabla u_\varepsilon(s)|^2\,dx\,ds\\
&\le \Lambda\|u^0\|^2_{L^2(\Omega)} + 4C\int_0^t\int_{\Omega^\varepsilon}|u_\varepsilon||\nabla u_\varepsilon|\,dx\,ds.
\end{aligned}\tag{2.7}$$

For any real number $\delta > 0$, we have by Young's inequality,

$$4C\int_0^t\int_{\Omega^\varepsilon}|u_\varepsilon||\nabla u_\varepsilon|\,dx\,ds \le 4C\delta\int_0^t\int_{\Omega^\varepsilon}|u_\varepsilon|^2\,dx\,ds + \frac{C}{\delta}\int_0^t\int_{\Omega^\varepsilon}|\nabla u_\varepsilon|^2\,dx\,ds.$$

Choosing $\delta > 0$ such that $\frac{1}{\Lambda} = \frac{C}{\delta}$, the inequality (2.7) yields:

$$\begin{aligned}
&\Lambda^{-1}\|u_\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \Lambda^{-1}\int_0^t\|\nabla u_\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}\,dx\,ds\\
&\le \Lambda\|u^0\|^2_{L^2(\Omega)} + 4C\delta\int_0^t\|u_\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}\,dx\,ds,
\end{aligned}$$

which by means of the Gronwall's inequality first leads to (2.3), then to (2.4).

As for (2.5), it follows from (1.1) that

$$\|\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\|^2_{L^2(0,T;V'_\varepsilon)} \le C\int_0^T\|\operatorname{div}\, A^\varepsilon\nabla u_\varepsilon\|^2_{V'_\varepsilon}\,dt + C\int_0^T\|\frac{1}{\varepsilon}g^\varepsilon(u_\varepsilon)\|^2_{V'_\varepsilon}\,dt.\tag{2.8}$$

On the one hand, (2.4) and the boundedness of the matrix $A$ imply

$$\int_0^T\|\operatorname{div}\, A^\varepsilon\nabla u_\varepsilon\|_{V'_\varepsilon}\,dt \le C.\tag{2.9}$$

On the other hand, we have

$$\|\frac{1}{\varepsilon}g^\varepsilon(u_\varepsilon)\|_{V'_\varepsilon} = \sup_{\varphi\in V_\varepsilon,\|\varphi\|_{V_\varepsilon}=1}\Big|\int_{\Omega^\varepsilon}G^\varepsilon(u_\varepsilon)\cdot\nabla\varphi\,dx + \int_{\Omega^\varepsilon}(\partial_r G^\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon)\varphi\,dx\Big|,$$

which by means of the Poincaré's inequality and (1.2) yields

$$\begin{aligned}
\|\frac{1}{\varepsilon}g^\varepsilon(u_\varepsilon)\|_{V'_\varepsilon} &\le C\sup_{\varphi\in V_\varepsilon,\|\varphi\|_{V_\varepsilon}=1}\left(\|u_\varepsilon(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla u_\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)}\right)\\
&\le C\left(\|u_\varepsilon(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla u_\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}\right)\quad t\in(0,T).
\end{aligned}$$

Therefore, using (2.3)-(2.4) and the Hölder's inequality we get

$$\int_0^T\|\frac{1}{\varepsilon}g^\varepsilon(u_\varepsilon)\|^2_{V'_\varepsilon}dt \le C.\tag{2.10}$$

We combine (2.8), (2.9) and (2.10) to get (2.5). □

The next result relies on the following classical extension property (see e.g., [1]).

**Proposition 2.2.** *For any $\varepsilon > 0$, there exists a bounded linear operator $P_\varepsilon$ from $V_\varepsilon$ into $H_0^1(\Omega)$ such that for any $u \in V_\varepsilon$ we have*

$$P_\varepsilon u = u \quad \text{in } \Omega^\varepsilon, \tag{2.11}$$

$$\|P_\varepsilon u\|_{H_0^1(\Omega)} \leq C\|u\|_{V_\varepsilon}, \tag{2.12}$$

*where $C$ is a positive constant independent of $\varepsilon$.*

For a function $u \in L^2(0,T;V_\varepsilon)$ we define its extension $P_\varepsilon u$ as follows

$$(P_\varepsilon u)(t) = P_\varepsilon(u(t)) \quad \text{a.e. } t \in (0,T), \tag{2.13}$$

and $P_\varepsilon u \in L^2(0,T;H_0^1(\Omega))$.

Bearing this in mind and owing to Proposition 2.2 (see precisely (2.12)), we have the following corollary.

**Corollary 2.3.** *Under the hypotheses of Lemma 2.1, we have the uniform estimate*

$$\|P_\varepsilon u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{2.14}$$

*where $C > 0$ is a positive constant independent of $\varepsilon$ and where $P_\varepsilon$ is the extension operator defined in Proposition 2.1.*

The next estimate requires some preliminaries. We define $R_\varepsilon : H_0^1(\Omega) \to V_\varepsilon$ by $R_\varepsilon u = u|_{\Omega_\varepsilon}$ for $u \in H_0^1(\Omega)$ (where $u|_{\Omega_\varepsilon}$ denotes the restriction of $u$ to $\Omega_\varepsilon$). Then, $R_\varepsilon$ is continuous since

$$\|R_\varepsilon u\|_{V_\varepsilon} \leq \|u\|_{H_0^1(\Omega)} \quad \text{for } u \in H_0^1(\Omega).$$

We recall that the adjoint $R_\varepsilon^* : V_\varepsilon' \to H^{-1}(\Omega)$ of $R_\varepsilon$ satisfies, for all $v \in V_\varepsilon'$ and $\psi \in H_0^1(\Omega)$,

$$\langle R_\varepsilon^* v, \varphi\rangle = \langle v, R_\varepsilon \varphi\rangle,$$

where the brackets on the left hand side denote the duality pairing between the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ while those on the right hand side denote the duality pairing between $V_\varepsilon'$ and $V_\varepsilon$. It is straightforward that

$$R_\varepsilon^* u = \chi_{\Omega^\varepsilon} u \quad \text{for } u \in L^2(\Omega^\varepsilon \times (0,T)). \tag{2.15}$$

Indeed, for any $\varphi \in L^2(0,T;H_0^1(\Omega))$, we have

$$\begin{aligned}
\langle R_\varepsilon^* u, \varphi\rangle = \langle u, R_\varepsilon \varphi\rangle &= \int_0^T \int_{\Omega^\varepsilon} u\,(\varphi|_{\Omega^\varepsilon}) dx\,dt \\
&= \int_0^T \int_\Omega \chi_{\Omega^\varepsilon}(u\,\varphi) dx\,dt = \int_0^T \int_\Omega (\chi_{\Omega^\varepsilon}\, u)\,\varphi\,dx\,dt.
\end{aligned}$$

It is worth noticing that combining (2.15) and Proposition 2.2 (see precisely (2.11) therein), we have

$$R_\varepsilon^* u = \chi_{\Omega^\varepsilon}(P_\varepsilon u) \quad \text{for } u \in L^2(0,T;V_\varepsilon). \tag{2.16}$$

Likewise, one can easily check that, for any $u \in L^2(0,T;V_\varepsilon)$ with $\frac{\partial u}{\partial t} \in L^2(0,T;V_\varepsilon')$, we have

$$R_\varepsilon^*\big(\frac{\partial u_\varepsilon}{\partial t}\big) = \frac{\partial(R_\varepsilon^* u_\varepsilon)}{\partial t}. \tag{2.17}$$

We are now in a position to formulate another estimate.

**Lemma 2.4.** *There exists a constant $C$ independent of $\varepsilon$ such that*
$$\|(\rho^\varepsilon\chi_{\Omega^\varepsilon})\frac{\partial(P_\varepsilon u_\varepsilon)}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))}\le C. \tag{2.18}$$

*Proof.* We first prove that there exists a constant $C$ independent of $\varepsilon$ such that
$$\|R_\varepsilon^*\big(\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\big)\|_{L^2(0,T;H^{-1}(\Omega))}\le C. \tag{2.19}$$

To do this, let $\varphi$ be arbitrarily fixed in $L^2(0,T;H_0^1(\Omega))$. We have
$$\begin{aligned}
\big|\big\langle R_\varepsilon^*\big(\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\big),\varphi\big\rangle\big|
&=\big|\big\langle\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t},R_\varepsilon\varphi\big\rangle\big|\\
&=\big|\int_0^T\big\langle\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t},R_\varepsilon\varphi\big\rangle_{V_\varepsilon',V_\varepsilon}dt\big|\\
&\le\|\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\|_{L^2(0,T,V_\varepsilon')}\|R_\varepsilon\varphi\|_{L^2(0,T;V_\varepsilon)}\\
&\le C\|R_\varepsilon\varphi\|_{L^2(0,T;V_\varepsilon)}\qquad\text{(see (2.5))}\\
&\le C\|\varphi\|_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}$$

Having done this, it remains to prove that
$$R_\varepsilon^*\big(\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\big)=\rho^\varepsilon\chi_{\Omega^\varepsilon}\frac{\partial(P_\varepsilon u_\varepsilon)}{\partial t}. \tag{2.20}$$

But with (2.16) and (2.17) in mind, it is easy to see that
$$R_\varepsilon^*\big(\rho^\varepsilon\frac{\partial u_\varepsilon}{\partial t}\big)=\rho^\varepsilon R_\varepsilon^*\big(\frac{\partial u_\varepsilon}{\partial t}\big)=\rho^\varepsilon\frac{\partial(R_\varepsilon^*u_\varepsilon)}{\partial t}=\rho^\varepsilon\frac{\partial(\chi_{\Omega^\varepsilon}P_\varepsilon u_\varepsilon)}{\partial t}=\rho^\varepsilon\chi_{\Omega^\varepsilon}\frac{\partial(P_\varepsilon u_\varepsilon)}{\partial t},$$
and the proof is complete. □

The following compactness result will be the starting point of our homogenization process.

**Theorem 2.5.** *Assume that the sequence $(\rho^\varepsilon\chi_{\Omega^\varepsilon})_{\varepsilon>0}$ weakly $*$-converges in $L^\infty(\Omega)$, as $\varepsilon\to 0$, to some real function that is different from zero almost everywhere in $\Omega$. Then the sequence $(P_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^2(0,T;L^2(\Omega))$.*

*Proof.* This is a direct consequence of the convergence hypothesis on the sequence $(\rho^\varepsilon\chi_{\Omega^\varepsilon})_{\varepsilon>0}$, Corollary 2.3 and Lemma 2.4, by using [5, Theorem 2.3 and Remark 2.5]. □

## 3. Multi-scale convergence and preliminary convergence results

We recall the definition and some compactness results of the multi-scale convergence theory [2, 21, 22]. We also introduce our functional setting and adapt some results of the multi-scale convergence method to our framework. We finally prove some preliminary convergence results needed in the homogenization process of the problem under consideration. We introduce the following notations: $\Omega_T=\Omega\times(0,T)$ and $\mathcal{T}=(0,1)$.

3.1. **Multi-scale convergence method.**

**Definition 3.1.** (i) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega_T)$ is said to weakly multi-scale converge towards $u_0 \in L^2(\Omega_T \times Y \times Z \times \mathcal{T})$, and denoted $u_\varepsilon \xrightarrow{w-ms} u_0$ in $L^2(\Omega_T)$, if as $\varepsilon \to 0$,

$$\begin{aligned} &\int_{\Omega_T} u_\varepsilon(x,t)\varphi(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})\,dx\,dt \\ &\to \iiiint_{\Omega_T\times Y\times Z\times\mathcal{T}} u_0(x,t,y,z,\tau)\varphi(x,t,y,z,\tau)\,dx\,dt\,dy\,dz\,d\tau \end{aligned} \tag{3.1}$$

for all $\varphi \in L^2(\Omega_T;\mathcal{C}_{\rm per}(Y\times Z\times\mathcal{T}))$.

(ii) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega_T)$ is said to strongly multi-scale converge towards $u_0 \in L^2(\Omega_T \times Y \times Z \times \mathcal{T})$, and denoted $u_\varepsilon \xrightarrow{s-ms} u_0$ in $L^2(\Omega_T)$, if it multi scale converges weakly to $u_0$ in $L^2(\Omega_T \times Y \times Z \times \mathcal{T})$ and further satisfies

$$\|u_\varepsilon\|_{L^2(\Omega_T)} \to \|u_0\|_{L^2(\Omega_T\times Y\times Z\times\mathcal{T})} \quad \text{as } \varepsilon\to 0. \tag{3.2}$$

**Remark 3.2.** (i) Let $u \in L^2(\Omega_T;\mathcal{C}_{\rm per}(Y\times Z\times\mathcal{T}))$ and define $u^\varepsilon : \Omega_T \to \mathbb{R}$ by $u^\varepsilon(x,t) = u(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})$, for $\varepsilon > 0$ and $(x,t) \in \Omega_T$. Then $u^\varepsilon \xrightarrow{w-ms} u$ and $u^\varepsilon \xrightarrow{s-ms} u$ in $L^2(\Omega_T)$ as $\varepsilon \to 0$. We also have $u^\varepsilon \to \widetilde{u}$ in $L^2(\Omega_T)$ -weak as $\varepsilon \to 0$, with

$$\widetilde{u}(x,t) = \iiint_{Y\times Z\times\mathcal{T}} u(\cdot,\cdot,y,z,\tau)\,dy\,dz\,d\tau.$$

(ii) If a sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega_T)$ multi-scale converges weakly in $L^2(\Omega_T)$ to some $u_0 \in L^2(\Omega_T \times Y \times Z \times \mathcal{T})$, in the sense of Definition 3.1, then (3.1) still holds for $\varphi \in \mathcal{C}(\overline{\Omega}_T; L^\infty_{\rm per}(Y\times Z\times\mathcal{T}))$.

(iii) Let $u \in \mathcal{C}(\overline{\Omega}_T; L^\infty_{\rm per}(Y\times Z\times\mathcal{T}))$ and define $u^\varepsilon$ like in (i) above. Then $u^\varepsilon \xrightarrow{w-ms} u$ in $L^2(\Omega_T)$ as $\varepsilon \to 0$.

The following two compactness results are the cornerstones of the multi-scale convergence theory.

**Theorem 3.3.** *Any bounded sequence in $L^2(\Omega_T)$ admits a weakly multi-scale convergent subsequence.*

Let $E$ be an ordinary sequence of real number converging to zero with $\varepsilon$.

**Theorem 3.4.** *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^2(0,T;H^1_0(\Omega))$. There exist a subsequence $E'$ of $E$ and a triplet $(u_0,u_1,u_2)$ in the space*

$$L^2(0,T;H^1_0(\Omega)) \times L^2(\Omega_T;L^2(\mathcal{T};H^1_{\rm per}(Y))) \times L^2(\Omega_T;L^2(Y\times\mathcal{T};H^1_{\rm per}(Z)))$$

*such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \quad \textit{in } L^2(0,T;H^1_0(\Omega))\textit{-weak} \tag{3.3}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \xrightarrow{w-ms} \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} + \frac{\partial u_2}{\partial z_i} \quad \textit{in } L^2(\Omega_T) \quad (1 \le j \le N). \tag{3.4}$$

We need to tailor Theorem 3.4 according to our needs. The functions $u_1$ and $u_2$ in Theorem 3.4 are unique up to additive function of variables $x,t,\tau$ and $x,t,y,\tau$, respectively. It is crucial to fix the choice of $u_1$. We introduce the space

$$H^1_\rho(Y_m) = \{u \in H^1_{\rm per}(Y) : \int_{Y_m} \rho(y)u(y)\,dy = 0\},$$

which is a closed subspace of $H^1_{\rm per}(Y)$ since it is the kernel of the bounded linear functional $u \mapsto \int_{Y_m} \rho(y)u(y)\,dy$ defined on $H^1_{\rm per}(Y)$. The version of Theorem 3.4 that will be used in the sequel formulates as follows.

**Theorem 3.5.** *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^2(0,T;H^1_0(\Omega))$. There exist a subsequence $E'$ of $E$ and a triplet $(u_0,u_1,u_2)$ in the space*

$$L^2(0,T;H^1_0(\Omega)) \times L^2(\Omega_T;L^2(\mathcal{T};H^1_\rho(Y_m))) \times L^2(\Omega_T;L^2(Y\times\mathcal{T};H^1_{\rm per}(Z)))$$

*such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \quad \text{in } L^2(0,T;H^1_0(\Omega))\text{-weak} \tag{3.5}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \xrightarrow{w-ms} \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} + \frac{\partial u_2}{\partial z_i} \quad \text{in } L^2(\Omega_T) \quad (1\le j\le N). \tag{3.6}$$

The proof of the above theorem is similar to that of [8, Theorem 2.5], and is omitted. The following weak-strong convergence result (see [19, Theorem 6] for its proof) and its corollary are worth recalling since they will be used in the sequel.

**Theorem 3.6.** *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^2(\Omega_T)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^2(\Omega_T)$ be two sequences such that $u_\varepsilon \xrightarrow{w-ms} u_0$ and $v_\varepsilon \xrightarrow{s-ms} v_0$ in $L^2(\Omega_T)$ with $u_0, v_0 \in L^2(\Omega_T\times Y\times Z\times\mathcal{T})$. Then $u_\varepsilon v_\varepsilon \xrightarrow{w-ms} u_0v_0$ in $L^1(\Omega_T)$.*

**Corollary 3.7.** *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^2(\Omega_T)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^2(\Omega_T)\cap L^\infty(\Omega_T)$ be two sequences such that $u_\varepsilon \xrightarrow{w-ms} u_0$ and $v_\varepsilon \xrightarrow{s-ms} v_0$ in $L^2(\Omega_T)$ with $u_0, v_0 \in L^2(\Omega_T\times Y\times Z\times\mathcal{T})$. Assume further that $(v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^\infty(\Omega_T)$. Then $u_\varepsilon v_\varepsilon \xrightarrow{w-ms} u_0v_0$ in $L^2(\Omega_T)$.*

3.2. **Preliminary convergence results.** We start this subsection by studying the limiting behavior of the sequence $(\chi_{\Omega^\varepsilon})_{\varepsilon>0}$ as $\varepsilon\to 0$. To do this, we first express the characteristic function of $\Omega^\varepsilon$ in $\Omega$, in terms of those of $Y_m$ and $Z_s$. Denoting by $\chi^\varepsilon_c$ and $\chi^\varepsilon_p$ the characteristic functions of $\Omega^\varepsilon_c$ and $\Omega^\varepsilon_p$, respectively, it appears that

$$\begin{aligned}
\chi^\varepsilon_c(x) &= \chi_{G_c}\big(\frac{x}{\varepsilon}\big) = \chi_{Y_c}\big(\frac{x}{\varepsilon}\big) \quad \text{(by $Y$-periodicity)}\\
\chi^\varepsilon_p(x) &= \big(1-\chi_{G_c}\big(\frac{x}{\varepsilon}\big)\big)\chi_{G_p}\big(\frac{x}{\varepsilon^2}\big)\\
&= \big(1-\chi_{Y_c}\big(\frac{x}{\varepsilon}\big)\big)\chi_{Z_p}\big(\frac{x}{\varepsilon^2}\big) \quad \text{(by $Y$ and $Z$-periodicity).}
\end{aligned}$$

Hence

$$\begin{aligned}
\chi_{\Omega^\varepsilon}(x) &= 1-(\chi^\varepsilon_c(x)+\chi^\varepsilon_p(x)) \quad (x\in\Omega)\\
&= 1-\big[\chi_{Y_c}\big(\frac{x}{\varepsilon}\big) + \big(1-\chi_{Y_c}\big(\frac{x}{\varepsilon}\big)\big)\chi_{Z_p}\big(\frac{x}{\varepsilon^2}\big)\big]\\
&= \big(1-\chi_{Y_c}\big(\frac{x}{\varepsilon}\big)\big)\big(1-\chi_{Z_p}\big(\frac{x}{\varepsilon^2}\big)\big)\\
&= \chi_{Y_m}\big(\frac{x}{\varepsilon}\big)\chi_{Z_s}\big(\frac{x}{\varepsilon^2}\big).
\end{aligned}$$

But, $\chi_{Y_m}(\cdot)\otimes\chi_{Z_s}(\cdot) \in L^\infty_{\rm per}(Y\times Z)$ so that according to (iii) of Remark 3.2,

$$\chi_{\Omega^\varepsilon} \xrightarrow{w-ms} \chi_{Y_m}\otimes\chi_{Z_s} \quad \text{in} \quad L^2(\Omega_T),$$

where the tensor product $\chi_{Y_m}\otimes\chi_{Z_s}$ is $(\chi_{Y_m}\otimes\chi_{Z_s})(y,z) = \chi_{Y_m}(y)\chi_{Z_s}(z)$, $((y,z)\in\mathbb{R}^N\times\mathbb{R}^N)$. We have proved the following result.

**Proposition 3.8.** *As $\varepsilon \to 0$, the characteristic function $\chi_{\Omega^\varepsilon}$ of $\Omega^\varepsilon$ multi-scale converges weakly in $L^2(\Omega_T)$ to $\chi_{Y_m} \otimes \chi_{Z_p}$.*

We now recall properties of some functional spaces that we will use. The topological dual of $H^1_\rho(Y_m)$ is denoted in the sequel by $(H^1_\rho(Y_m))'$ while $L^2_\rho(Y_m)$ stands for the space of functions $u \in L^2_{\mathrm{per}}(Y)$ satisfying $\int_{Y_m} \rho(y)u(y)\,dy = 0$. We first recall that, since the space $H^1_\rho(Y_m)$ is densely embedded in $L^2_\rho(Y_m)$, the following continuous embeddings hold:

$$H^1_\rho(Y_m) \subset L^2_\rho(Y_m) \subset (H^1_\rho(Y_m))'.$$

We also recall that the topological dual of $L^2(\mathcal{T}; H^1_\rho(Y_m))$ is $L^2(\mathcal{T}; (H^1_\rho(Y_m))')$. This readily follows from the reflexiveness of the space $H^1_\rho(Y_m)$. We denote the duality pairing between $H^1_\rho(Y_m)$ and $(H^1_\rho(Y_m))'$ by $(\cdot,\cdot)$, and that of $L^2(\mathcal{T}; H^1_\rho(Y_m))$ and $L^2(\mathcal{T}; (H^1_\rho(Y_m))')$ by $[\cdot,\cdot]$. Thus, we have

$$(u, v) = \int_Y u(y)v(y)\,dy$$

for $u \in L^2_\rho(Y_m)$ and $v \in H^1_\rho(Y_m)$, and

$$[u, v] = \int_0^1 (u(\tau), v(\tau))\,d\tau = \int_0^1 \int_Y u(y,\tau)v(y,\tau)\,dy\,d\tau$$

for $u \in L^2(\mathcal{T}; H^1_\rho(Y_m))$ and $v \in L^2(\mathcal{T}; (H^1_\rho(Y_m))')$. Furthermore, let $\mathcal{D}_\rho(Y_m)$ stands for the space of functions $u \in \mathcal{D}_{\mathrm{per}}(Y)$ with $\int_{Y_m} \rho(y)u(y)\,dy = 0$. Owing to the fact that the space $\mathcal{D}_\rho(Y_m)$ is dense in $H^1_\rho(Y_m)$ the following result holds (see e.g. [20, Lemma 2 and Lemma 3]).

**Theorem 3.9.** *Let $u \in \mathcal{D}'_{\mathrm{per}}(Y \times \mathcal{T})$ and assume that $u$ is continuous on $\mathcal{D}_\rho(Y_m) \otimes \mathcal{D}_{\mathrm{per}}(\mathcal{T})$ endowed with the $L^2_{\mathrm{per}}(\mathcal{T}; H^1_\rho(Y_m))$-norm. Then $u \in L^2_{\mathrm{per}}(\mathcal{T}; (H^1_\rho(Y_m))')$, and further*

$$\langle u, \varphi\rangle = \int_0^1 (u(\tau), \varphi(\cdot,\tau))\,d\tau$$

*for all $\varphi \in \mathcal{D}_\rho(Y_m) \otimes \mathcal{D}_{\mathrm{per}}(\mathcal{T})$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{D}'_{\mathrm{per}}(Y \times \mathcal{T})$ and $\mathcal{D}_{\mathrm{per}}(Y \times \mathcal{T})$, whereas the right-hand side is the product of $u$ and $\varphi$ in the duality between $L^2_{\mathrm{per}}(\mathcal{T}; (H^1_\rho(Y_m))')$ and $L^2_{\mathrm{per}}(\mathcal{T}; H^1_\rho(Y_m))$.*

We now define an operator

$$\begin{aligned} R : L^2_{\mathrm{per}}(Y) &\to L^2_{\mathrm{per}}(Y) \\ u &\mapsto \chi_{Y_m}\rho u. \end{aligned} \tag{3.7}$$

It is clear that $R$ is a non-negative and linear bounded self-adjoint operator. Using the positivity of the weight $\rho$ we prove that the kernel of $R$ is defined by:

$$\ker(R) = \{u \in L^2_{\mathrm{per}}(Y) : u = 0 \text{ a.e. in } Y_m\}.$$

We denote by $\ker(R)^\perp$ the orthogonal of the kernel of $R$ in $L^2_{\mathrm{per}}(Y)$ while $L^2_0(Y_m)$ stands for the completion of $\ker(R)^\perp$ with respect to the norm

$$\|u\|_+ = \|\chi_{Y_m}\rho^{\frac12}u\|_{L^2_{\mathrm{per}}(Y)} \quad (u \in \ker(R)^\perp).$$

We denote by $P$ the orthogonal projection from $L^2_{\rm per}(Y)$ onto $L^2_0(Y_m)$. We recall that for $u \in L^2_{\rm per}(\mathcal{T}; L^2_{\rm per}(Y))$ we define $Ru$ and $Pu$ by

$$(Ru)(\tau) = R(u(\tau)) \quad \text{and} \quad (Pu)(\tau) = P(u(\tau)) \quad \text{for a.e. } \tau \in (0,1).$$

Considered as an unbounded operator on $L^2_{\rm per}(\mathcal{T}; H^1_\rho(Y_m))$, the domain of $R' \equiv \chi_{Y_m}\rho\frac{\partial}{\partial\tau}$ is

$$\mathcal{W} = \big\{u \in L^2_{\rm per}(\mathcal{T}; H^1_\rho(Y_m)) : \chi_{Y_m}\rho\frac{\partial u}{\partial \tau} \in L^2_{\rm per}(\mathcal{T}; (H^1_\rho(Y_m))')\big\}.$$

We endow $\mathcal{W}$ with its natural norm

$$\|u\|_{\mathcal{W}} = \|u\|_{L^2_{\rm per}(\mathcal{T}; H^1_\rho(Y_m))} + \|\chi_{Y_m}\rho\frac{\partial u}{\partial\tau}\|_{L^2_{\rm per}(\mathcal{T}; (H^1_\rho(Y_m))')} \quad (u \in \mathcal{W}),$$

and recall an important result (see e.g., [16, 17]) we will use in the sequel.

**Proposition 3.10.** *The operator $P$ maps continuously $\mathcal{W}$ into $\mathcal{C}([0,1]; L^2_0(Y_m))$, i.e., there exists a constant $c > 0$ such that*

$$\|Pu\|_{\mathcal{C}([0,1]; L^2_0(Y_m))} = \sup_{0\le\tau\le1} \|\chi_{Y_m}\rho^{1/2}u(\tau)\|_{L^2(Y)} \le c\|u\|_{\mathcal{W}} \quad \textit{for all } u \in \mathcal{W}.$$

*Moreover,*

$$[\chi_{Y_m}\rho\frac{\partial u}{\partial\tau}, v] = -[\chi_{Y_m}\rho\frac{\partial v}{\partial\tau}, u] \quad \textit{for all } u, v \in \mathcal{W}. \tag{3.8}$$

We will also use the following convergence results in the forthcoming homogenization process.

**Proposition 3.11.** *Let $\varphi \in \mathcal{C}^\infty_0(Q_T) \otimes \mathcal{D}_{\rm per}(\mathcal{T}) \otimes \mathcal{D}_\rho(Y_m)$. Let $E'$, $(u_\varepsilon)_{\varepsilon\in E}$ and $(u_0, u_1) \in L^2(0,T; H^1_0(\Omega)) \times L^2(\Omega_T; L^2(\mathcal{T}; H^1_\rho(Y_m)))$ be as in Theorem 3.5. Then*

$$\lim_{E\ni\varepsilon\to0} \frac{1}{\varepsilon}\int_{Q^\varepsilon_T} u_\varepsilon(x,t)\rho(\frac{x}{\varepsilon})\varphi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})dx\,dt = |Z_s|\int_{Q_T} [\chi_{Y_m}\,\rho\, u_1(x,t), \varphi(x,t)]\,dx\,dt,$$

*where $0 < |Z_s| < 1$ denotes the Lebesgue measure of the set $Z_s$.*

*Proof.* We have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{Q^\varepsilon_T} u_\varepsilon(x,t)\rho(\frac{x}{\varepsilon})\varphi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})dx\,dt \\
&= \frac{1}{\varepsilon}\int_{Q_T} u_\varepsilon(x,t)\chi_{\Omega^\varepsilon}(x)\rho(\frac{x}{\varepsilon})\varphi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})dx\,dt \\
&= \frac{1}{\varepsilon}\int_{Q_T} u_\varepsilon(x,t)\chi_{Y_m}(\frac{x}{\varepsilon})\chi_{Z_s}(\frac{x}{\varepsilon^2})\rho(\frac{x}{\varepsilon})\varphi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})dx\,dt
\end{aligned}$$

Bearing in mind that

$$\int_Y \chi_{Y_m}(y)\rho(y)\varphi(x,t,y,\tau)dy = 0 \quad \text{for all } (x,t,\tau) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}),$$

the same line of reasoning as in the proof of [8, Theorem 2.3] yields, as $E \ni \varepsilon \to 0$,

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{Q_T} u_\varepsilon(x,t)\chi_{Y_m}(\frac{x}{\varepsilon})\chi_{Z_s}(\frac{x}{\varepsilon^2})\rho(\frac{x}{\varepsilon})\varphi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\,dx\,dt \\
&\to \int_{Q_T\times Y\times Z\times\mathcal{T}} \chi_{Y_m}\chi_{Z_s}\rho u_1\varphi\,dx\,dt\,dy\,dz\,d\tau,
\end{aligned}$$

which concludes the proof. $\square$

We finally introduce the following notation-definition

$$\mathbb{F}_0^1 = L^2(0,T;H_0^1(\Omega)) \times L^2(\Omega_T;L^2_{\rm per}(\mathcal{T};H^1_\rho(Y_m))) \times L^2(\Omega_T;L^2_{\rm per}(Y\times\mathcal{T};H^1_{\rm per}(Z_s))),$$

where $H^1_{\rm per}(Z_s)$ stands for the space of functions $u\in H^1_{\rm per}(Z)$ with $\int_{Z_s} u(z)dz=0$. We similarly define $\mathcal{D}_{\rm per}(Z_s)$ and remark that the space $\mathbb{F}_0^1$ admits the following dense subspace

$$\begin{aligned}\mathfrak{F}_0^\infty &= \mathcal{C}_0^\infty(\Omega_T)\times\big(\mathcal{C}_0^\infty(\Omega_T)\otimes\mathcal{D}_{\rm per}(\mathcal{T})\otimes\mathcal{D}_\rho(Y_m)\big)\times\Big(\mathcal{C}_0^\infty(\Omega_T)\otimes\mathcal{D}_{\rm per}(Y\times\mathcal{T})\\ &\quad\otimes\mathcal{D}_{\rm per}(Z_s)\Big).\end{aligned}$$

Moreover, $\mathbb{F}_0^1$ is a Banach space under the norm

$$\begin{aligned}&\|(u_0,u_1,u_2)\|_{\mathbb{F}_0^1}\\ &=\|u_0\|_{L^2(0,T;H_0^1(\Omega))}+\|u_1\|_{L^2(\Omega_T;L^2(\mathcal{T};H^1_\rho(Y_m)))}+\|u_2\|_{L^2(\Omega_T;L^2(Y\times\mathcal{T};H^1_{\rm per}(Z)))}.\end{aligned}$$

## 4. Homogenization process and main results

Let $E$ be an ordinary sequence of real numbers $\varepsilon$ converging to zero with $\varepsilon$.

### 4.1. Derivation of the global limit problem.

**4.1. Derivation of the global limit problem.** By Corollary 2.3 and Theorem 3.5, there exist

$$(u_0,u_1,u_2)\in\mathbb{F}_0^1 \tag{4.1}$$

and a subsequence $E'$ of $E$ such that, as $E'\ni\varepsilon\to 0$,

$$\begin{gathered}P_\varepsilon u_\varepsilon\to u_0\quad\text{in } L^2(0,T;H_0^1(\Omega))\text{-weak},\\ \frac{\partial(P_\varepsilon u_\varepsilon)}{\partial x_i}\xrightarrow{w-ms}\frac{\partial u_0}{\partial x_i}+\frac{\partial u_1}{\partial y_i}+\frac{\partial u_2}{\partial z_i}\quad\text{in } L^2(\Omega_T)\ (1\le j\le N).\end{gathered}$$

Moreover, as $E'\ni\varepsilon\to 0$ we have

$$\rho^\varepsilon\chi_{\Omega^\varepsilon}\to\rho\chi_{Y_m}\otimes\chi_{Z_s}\quad\text{in } L^\infty(\Omega)\text{ weak-}*$$

so that Theorem 2.5 yields

$$P_\varepsilon u_\varepsilon\to u_0\quad\text{in } L^2(0,T;L^2(\Omega)). \tag{4.2}$$

We are now in a position to formulate the first homogenization result.

**Theorem 4.1.** *The triple $(u_0,u_1,u_2)\in\mathbb{F}_0^1$ determined above by (4.1) is a solution to the variational problem*

$$
\begin{gathered}
(u_0,u_1,u_2)\in\mathbb{F}_0^1,\\
\begin{aligned}
&\Big(\int_{Y_m}\rho(y)dy\Big)\int_{\Omega_T}\frac{\partial u_0}{\partial t}\psi_0\,dx\,dt-\int_{\Omega_T}\big[\rho\chi_{Y_m}u_1(x,t),\frac{\partial\psi_1}{\partial\tau}(x,t)\big]\,dx\,dt\\
&=-\iiint_{\Omega_T\times Y_m\times\mathcal{T}}G(y,\tau,u_0)\cdot\nabla\psi_0\,dx\,dt\,dy\,d\tau\\
&\quad+\iiint_{\Omega_T\times Y_m\times\mathcal{T}}g(y,\tau,u_0)\psi_1\,dx\,dt\,dy\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}A(y,\tau)(\nabla_xu_0+\nabla_yu_1+\nabla_zu_2)\\
&\quad\cdot(\nabla_x\psi_0+\nabla_y\psi_1+\nabla_z\psi_2)\,dx\,dt\,dy\,dz\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}\Big(\partial_rG(y,\tau,u_0)\\
&\quad\cdot(\nabla_xu_0+\nabla_yu_1+\nabla_zu_2)\Big)\psi_0\,dx\,dt\,dy\,dz\,d\tau
\end{aligned}\\
\textit{for all }(\psi_0,\psi_1,\psi_2)\in\mathfrak{F}_0^\infty.
\end{gathered}
\tag{4.3}
$$

*Proof.* Let $\varepsilon>0$ and let $(\psi_0,\psi_1,\psi_2)\in\mathfrak{F}_0^\infty$. The appropriate oscillating test function for our problem is defined as follows:

$$
\psi_\varepsilon(x,t)=\psi_0(x,t)+\varepsilon\psi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})+\varepsilon^2\psi_2(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2}),\tag{4.4}
$$

for $(x,t)\in\Omega\times(0,T)$. Multiplying all terms in the main equation of (1.1) by $\psi_\varepsilon(x,t)$ and integrating over $\Omega^\varepsilon\times(0,T)$ leads to

$$
\int_{\Omega_T^\varepsilon}\rho(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial t}\psi_\varepsilon\,dx\,dt=-\int_{\Omega_T^\varepsilon}A(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla u_\varepsilon\cdot\nabla\psi_\varepsilon\,dx\,dt+\frac{1}{\varepsilon}\int_{\Omega_T^\varepsilon}g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon)\psi^\varepsilon\,dx\,dt,
$$

or equivalently to

$$
\begin{aligned}
&\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial(P_\varepsilon u_\varepsilon)}{\partial t}\psi_\varepsilon\,dx\,dt\\
&=\int_{\Omega_T}\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})A(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(P_\varepsilon u_\varepsilon)\cdot\nabla\psi_\varepsilon\,dx\,dt\\
&\quad+\frac{1}{\varepsilon}\int_{\Omega_T}\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi^\varepsilon\,dx\,dt.
\end{aligned}\tag{4.5}
$$

We now pass to the limit in (4.5) as $E'\ni\varepsilon\to0$. We start with the term in the left hand side. We have

$$
\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial(P_\varepsilon u_\varepsilon)}{\partial t}\psi_\varepsilon\,dx\,dt=-\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})P_\varepsilon u_\varepsilon\frac{\partial\psi_\varepsilon}{\partial t}\,dx\,dt,\tag{4.6}
$$

and we recall that

$$
\begin{aligned}
\frac{\partial\psi_\varepsilon}{\partial t}(x,t)&=\frac{\partial\psi_0}{\partial t}(x,t)+\varepsilon\frac{\partial\psi_1}{\partial t}(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})+\frac{1}{\varepsilon}\frac{\partial\psi_1}{\partial\tau}(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\\
&\quad+\varepsilon^2\frac{\partial\psi_2}{\partial t}(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})+\frac{\partial\psi_2}{\partial\tau}(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2}).
\end{aligned}
$$

It follows from (4.2) that

$$\begin{aligned}
&\lim_{E'\ni\varepsilon\to 0}\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})P_\varepsilon u_\varepsilon\frac{\partial\psi_0}{\partial t}\,dx\,dt\\
&=\iiint_{\Omega_T\times Y\times Z}\rho(y)\chi_{Y_m}(y)\chi_{Z_s}(z)u_0(x,t)\frac{\partial\psi_0}{\partial t}(x,t)\,dx\,dt\,dy\,dz\\
&=-|Z_s|\Big(\int_{Y_m}\rho(y)dy\Big)\int_{\Omega_T}\frac{\partial u_0}{\partial t}\psi_0\,dx\,dt.
\end{aligned}\tag{4.7}$$

By means of Proposition 3.11 we have

$$\begin{aligned}
&\lim_{E'\ni\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})P_\varepsilon u_\varepsilon\frac{\partial\psi_1}{\partial\tau}(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\\
&=\iiint_{\Omega_T\times Y\times Z\times\mathcal{T}}\rho(y)\chi_{Y_m}(y)\chi_{Z_s}(z)u_1\frac{\partial\psi_1}{\partial\tau}\,dx\,dt\,dy\,dz\,d\tau\\
&=|Z_s|\int_{\Omega_T}\big[\rho\chi_{Y_m}u_1(x,t),\frac{\partial\psi_1}{\partial\tau}(x,t)\big]\,dx\,dt
\end{aligned}\tag{4.8}$$

The following limits hold:

$$\lim_{E'\ni\varepsilon\to 0}\varepsilon\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})P_\varepsilon u_\varepsilon\frac{\partial\psi_1}{\partial t}(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\,dx\,dt=0,\tag{4.9}$$

$$\lim_{E'\ni\varepsilon\to 0}\varepsilon^2\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})P_\varepsilon u_\varepsilon\frac{\partial\psi_2}{\partial t}(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})\,dx\,dt=0,\tag{4.10}$$

$$\lim_{E'\ni\varepsilon\to 0}\int_{\Omega_T}\rho(\frac{x}{\varepsilon})\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})P_\varepsilon u_\varepsilon\frac{\partial\psi_2}{\partial\tau}(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})\,dx\,dt=0.\tag{4.11}$$

After the passage to the limit in the left hand side of (4.11), we used the formula $\int_{\mathcal{T}}\frac{\partial\psi_2}{\partial\tau}d\tau=0$. Similar trivial arguments work for (4.9) and (4.10). Thus as $E'\ni\varepsilon\to 0$, we have

$$\begin{aligned}
\int_{\Omega_T^\varepsilon}\rho(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial t}\psi_\varepsilon\,dx\,dt\to|Z_s|\Big(\int_{Y_m}\rho(y)dy\Big)\int_{\Omega_T}\frac{\partial u_0}{\partial t}\psi_0\,dx\,dt\\
-|Z_s|\int_{\Omega_T}[\rho\chi_{Y_m}u_1(x,t),\frac{\partial\psi_1}{\partial\tau}(x,t)]\,dx\,dt.
\end{aligned}\tag{4.12}$$

As regards the first term in the right hand side of (4.5), it is classical that, as $E'\ni\varepsilon\to 0$, we have

$$\begin{aligned}
&\int_{\Omega_T^\varepsilon}A(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(P_\varepsilon u_\varepsilon)\cdot\nabla\psi_\varepsilon\,dx\,dt\\
&=\int_{\Omega_T}\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})A(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(P_\varepsilon u_\varepsilon)\cdot\nabla\psi_\varepsilon\,dx\,dt\\
&\to\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}A(y,\tau)(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\big(\nabla_x\psi_0\\
&\quad+\nabla_y\psi_1+\nabla_z\psi_2\big)\,dx\,dt\,dy\,dz\,d\tau.
\end{aligned}\tag{4.13}$$

Concerning the second term in the right hand side of (4.5), we first rewrite it as follows:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_\varepsilon\,dx\,dt \\
&= \frac{1}{\varepsilon}\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_0\,dx\,dt + \int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\,dx\,dt \\
&\quad + \varepsilon\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_2(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})\,dx\,dt.
\end{aligned}
\tag{4.14}
$$

It is straightforward from [20, Lemma 5] that

$$
\begin{aligned}
&\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\,dx\,dt \\
&= \int_{\Omega_T} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\chi_{\Omega^\varepsilon}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\psi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\,dx\,dt \\
&\to |Z_s|\iiint_{\Omega_T\times Y_m\times\mathcal{T}} g(y,\tau,u_0)\psi_1(x,t,y,\tau)\,dx\,dt\,dy\,d\tau
\end{aligned}
\tag{4.15}
$$

as $E'\ni\varepsilon\to 0$. Likewise, it holds that

$$
\lim_{E'\ni\varepsilon\to 0}\varepsilon\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_2(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2})\,dx\,dt = 0. \tag{4.16}
$$

It then remains to deal with the first term in the right hand side of (4.14). We have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_0\,dx\,dt \\
&= -\int_{\Omega_T^\varepsilon} G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\cdot\nabla_x\psi_0\,dx\,dt \\
&\quad - \int_{\Omega_T^\varepsilon}\big(\partial_r G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\cdot\nabla_x u_\varepsilon\big)\psi_0\,dx\,dt.
\end{aligned}
\tag{4.17}
$$

It follows from [20, Lemma 5 - Remark 2] that as $E'\ni\varepsilon\to 0$,

$$
\begin{aligned}
&\int_{\Omega_T^\varepsilon}\big(\partial_r G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\cdot\nabla_x u_\varepsilon\big)\psi_0\,dx\,dt \\
&= \iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}\left(\partial_r G(y,\tau,u_0)\cdot(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\right)\psi_0\,dx\,dt\,dy\,dz\,d\tau.
\end{aligned}
\tag{4.18}
$$

Likewise,

$$
\begin{aligned}
&\lim_{E'\ni\varepsilon\to 0}\int_{\Omega_T^\varepsilon} G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\cdot\nabla_x\psi_0\,dx\,dt \\
&= |Z_s|\iiint_{\Omega_T\times Y_m\times\mathcal{T}} G(y,\tau,u_0)\cdot\nabla_x\psi_0\,dx\,dt\,dy\,d\tau.
\end{aligned}
\tag{4.19}
$$

Thus, as $E' \ni \varepsilon \to 0$, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Omega_T^\varepsilon} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},P_\varepsilon u_\varepsilon)\psi_\varepsilon\,dx\,dt \\
&\to -\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}} (\partial_r G(y,\tau,u_0)\cdot(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2))\,\psi_0\,dx\,dt\,dy\,dz\,d\tau \\
&\quad -|Z_s|\iiint_{\Omega_T\times Y_m\times\mathcal{T}} G(y,\tau,u_0)\cdot\nabla_x\psi_0\,dx\,dt\,dy\,d\tau \\
&\quad +|Z_s|\iiint_{\Omega_T\times Y_m\times\mathcal{T}} g(y,\tau,u_0)\psi_1(x,t,y,\tau)\,dx\,dt\,dy\,d\tau,
\end{aligned}
\tag{4.20}
$$

which combined with (4.12)–(4.13) concludes the proof. □

The second term on the left hand side of (4.3) needs further investigations. In fact, for further needs, we would like to rewrite it using formula (3.8) of Proposition 3.10 as follows

$$
-\int_{\Omega_T}\big[\rho\chi_{Y_m}u_1(x,t),\frac{\partial\psi_1}{\partial\tau}(x,t)\big]\,dx\,dt=\int_{\Omega_T}\big[\rho\chi_{Y_m}\frac{\partial u_1}{\partial\tau}(x,t),\psi_1(x,t)\big]\,dx\,dt,
$$

but this requires that $u_1\in\mathcal{W}$.

**Proposition 4.2.** *The function $u_1\in L^2_{\rm per}(\mathcal{T};H^1_\rho(Y_m))$ defined by* (4.1) *and Theorem* 4.1 *belongs to $\mathcal{W}$.*

*Proof.* Let $\psi_0=\psi_2=0$ and $\psi_1=\varphi\otimes\psi$ in ((4.3), where $\varphi\in\mathcal{C}_0^\infty(\Omega_T)$ and $\psi\in\mathcal{D}_{\rm per}(\mathcal{T})\otimes\mathcal{D}_\rho(Y_m)$. Using the arbitrariness of $\varphi$, we are led to

$$
\begin{aligned}
&-\Big[\rho\chi_{Y_m}u_1(x,t),\frac{\partial\psi}{\partial\tau}(x,t)\Big]\,dx\,dt=\iint_{Y\times\mathcal{T}}\chi_{Y_m}\rho\frac{\partial u_1}{\partial\tau}\psi\,dy\,d\tau \\
&=\iint_{Y\times\mathcal{T}} g(y,\tau,u_0)\psi\,dy\,d\tau \\
&\quad-\frac{1}{|Z_s|}\iiint_{Y_m\times Z_s\times\mathcal{T}} A(y,\tau)(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\nabla_y\psi\,dy\,dz\,d\tau.
\end{aligned}
$$

But $g=\operatorname{div}_y G$, and the boundedness of the matrix $A$ implies that the linear functional

$$
\begin{aligned}
\psi\mapsto&\iint_{Y\times\mathcal{T}} g(y,\tau,u_0)\psi\,dy\,d\tau \\
&-\frac{1}{|Z_s|}\iiint_{Y_m\times Z_s\times\mathcal{T}} A(y,\tau)(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\nabla_y\psi\,dy\,dz\,d\tau
\end{aligned}
$$

is continuous on $\mathcal{D}_{\rm per}(\mathcal{T})\otimes\mathcal{D}_\rho(Y_m)$ with the $L^2_{\rm per}(\mathcal{T};H^1_\rho(Y_m))$-norm. Thus can be extended to and element of $L^2_{\rm per}(\mathcal{T};(H^1_\rho(Y_m))')$. In other words, $\chi_{Y_m}\rho\frac{\partial u_1}{\partial\tau}\in L^2_{\rm per}(\mathcal{T};(H^1_\rho(Y_m))')$ and the proof is complete. □

Therefore the global homogenized problem of (1.1) reads

$$
\begin{aligned}
&(u_0,u_1,u_2)\in \mathbb{F}_0^1,\\
&\Big(\int_{Y_m}\rho(y)dy\Big)\int_{\Omega_T}\frac{\partial u_0}{\partial t}\psi_0\,dx\,dt+\int_{\Omega_T}\big[\rho\chi_{Y_m}\frac{\partial u_1}{\partial\tau}(x,t),\psi_1(x,t)\big]\,dx\,dt\\
&=-\iiint_{\Omega_T\times Y_m\times\mathcal{T}}G(y,\tau,u_0)\cdot\nabla\psi_0\,dx\,dt\,dy\,d\tau\\
&\quad+\iiint_{\Omega_T\times Y_m\times\mathcal{T}}g(y,\tau,u_0)\psi_1\,dx\,dt\,dy\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}A(y,\tau)(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\cdot\big(\nabla_x\psi_0\\
&\quad+\nabla_y\psi_1+\nabla_z\psi_2\big)\,dx\,dt\,dy\,dz\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}\big(\partial_r G(y,\tau,u_0)\cdot(\nabla_x u_0+\nabla_y u_1\\
&\quad+\nabla_z u_2)\big)\psi_0\,dx\,dt\,dy\,dz\,d\tau\quad\text{for all }(\psi_0,\psi_1,\psi_2)\in\mathfrak{F}_0^\infty.
\end{aligned}
\tag{4.21}
$$

The variational problem (4.21) is termed global since it contains the macroscopic homogenized problem and the local problem.

4.2. **The macroscopic problem.** We are now in a position to derive the equation describing the macroscopic behavior of the $\varepsilon$-problem (1.1). The variational problem 4.21 is equivalent to the system

$$
\begin{aligned}
&\Big(\int_{Y_m}\rho(y)dy\Big)\int_{\Omega_T}\frac{\partial u_0}{\partial t}\psi_0\,dx\,dt\\
&=-\iiint_{\Omega_T\times Y_m\times\mathcal{T}}G(y,\tau,u_0)\cdot\nabla\psi_0\,dx\,dt\,dy\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}A(y,\tau)(\nabla_x u_0+\nabla_y u_1\\
&\quad+\nabla_z u_2)\cdot(\nabla_x\psi_0)\,dx\,dt\,dy\,dz\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}(\partial_r G(y,\tau,u_0)\cdot(\nabla_x u_0+\nabla_y u_1\\
&\quad+\nabla_z u_2))\psi_0\,dx\,dt\,dy\,dz\,d\tau\quad\text{for all }\psi_0\in\mathcal{C}_0^\infty(\Omega_T),
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&\int_{\Omega_T}\big[\rho\chi_{Y_m}\frac{\partial u_1}{\partial\tau}(x,t),\psi_1(x,t)\big]\,dx\,dt\\
&=\iiint_{\Omega_T\times Y_m\times\mathcal{T}}g(y,\tau,u_0)\psi_1\,dx\,dt\,dy\,d\tau\\
&\quad-\frac{1}{|Z_s|}\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}}A(y,\tau)(\nabla_x u_0+\nabla_y u_1\\
&\quad+\nabla_z u_2)\cdot(\nabla_y\psi_1)\,dx\,dt\,dy\,dz\,d\tau\\
&\qquad\text{for all }\psi_1\in\mathcal{C}_0^\infty(\Omega_T)\otimes\mathcal{D}_{\mathrm{per}}(\mathcal{T})\otimes\mathcal{D}_\rho(Y_m),
\end{aligned}
\tag{4.23}
$$

and

$$\iiiint_{\Omega_T\times Y_m\times Z_s\times\mathcal{T}} A(y,\tau)(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\cdot(\nabla_z\psi_2)\,dx\,dt\,dy\,dz\,d\tau=0$$
$$\text{for all } \psi_2\in\mathcal{C}_0^\infty(\Omega_T)\otimes\mathcal{D}_{\mathrm{per}}(\mathcal{T})\otimes\mathcal{D}_{\mathrm{per}}(Y_m)\otimes\mathcal{D}_{\mathrm{per}}(Z). \tag{4.24}$$

We first study (4.24). We start with a few preliminaries. We define $H^1_\#(Z_s)$ to be the space of functions in $H^1(Z_s)$ assuming same values on the opposites faces of $Z$, and satisfying $\int_{Z_s}u(z)dz=0$. We remark that if $u\in H^1_{\mathrm{per}}(Z)$ with $\int_{Z_s}u(z)dz=0$ then its restriction to $Z_s$ (which is still denoted by $u$ in the sequel) belongs to $H^1_\#(Z_s)$. Vice-versa, the extension of a function $u\in H^1_\#(Z_s)$ belongs to $H^1_{\mathrm{per}}(Z)$ with $\int_{Z_s}u(z)dz=0$. We have the following result whose proof is obvious and therefore omitted.

**Proposition 4.3.** *Let $1\le j\le N$ and let $(y,\tau)\in\mathbb{R}^N\times\mathbb{R}$ be fixed. The following microscopic local problem admits a solution which is uniquely defined almost everywhere in $Z_s$.*

$$\begin{gathered}\chi^j(y,\tau)\in H^1_\#(Z_s):\\ \int_{Z_s}A(y,\tau)\nabla_z\chi^j\cdot\nabla_z\omega\,dz=-\sum_{k=1}^N a_{kj}\int_{Z_s}\frac{\partial\omega}{\partial z_k}\,dz\\ \text{for all } \omega\in H^1_\#(Z_s).\end{gathered}\tag{4.25}$$

Back to (4.24), let $\psi_2=\varphi\otimes\omega$ with $\varphi\in\mathcal{D}(\Omega_T)\otimes\mathcal{D}_{\mathrm{per}}(\mathcal{T})\otimes\mathcal{D}_\rho(Y_m)$ and $\omega\in\mathcal{D}_{\mathrm{per}}(Z_s)$. We get

$$\iiint_{\Omega_T\times Y_m\times\mathcal{T}}\varphi(x,t,y,\tau)\,dx\,dt\Big[\int_{Z_s}A(y,\tau)(\nabla_x u_0+\nabla_y u_1+\nabla_z u_2)\cdot\nabla_z\omega\,dz\Big]=0$$

which by the arbitrariness of $\varphi$ gives, for fixed $(x,t)\in\Omega_T$ and fixed $(y,\tau)\in\mathbb{R}^N\times\mathbb{R}$,

$$\int_{Z_s}A(y,\tau)\nabla_z u_2\cdot\nabla_z\omega\,dz=-\int_{Z_s}A(y,\tau)(\nabla_x u_0+\nabla_y u_1)\cdot\nabla_z\omega\,dz. \tag{4.26}$$

By inspection of the microscopic problems (4.25) and (4.26) it appears by the uniqueness of the solution to (4.25) that for almost all $(x,t,y,\tau)$ fixed in $\Omega_T\times\mathbb{R}^N\times\mathbb{R}$

$$u_2(x,t,y,\tau)=\sum_{j=1}^N\Big(\frac{\partial u_0}{\partial x_j}(x,t)+\frac{\partial u_1}{\partial y_j}(x,t,y,\tau)\Big)\chi^j(y,\tau)\quad\text{a.e., in } Z_s. \tag{4.27}$$

For further needs, we introduce a notation. We define the matrix $\nabla_z\chi$ by

$$(\nabla_z\chi)_{ij}=\frac{\partial\chi^i}{\partial z_j}\quad(1\le i,j\le N).$$

Then we can write in short for almost all $(x,t,y,\tau,z)\in\Omega_T\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$,

$$\nabla_z u_2=\nabla_z\chi\cdot(\nabla_x u_0+\nabla_y u_1). \tag{4.28}$$

We can now look at the mesoscopic scale. Let $(x,t) \in \Omega_T$ and $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$ be freely fixed and let $\pi(x,t,r,\xi)$ be defined by the mesoscopic cell problem:

$$
\begin{aligned}
&\pi_1(x,t,r,\xi) \in \mathcal{W} \\
&[\rho\chi_{Y_m}\frac{\partial \pi_1}{\partial \tau}, \psi_1]\,dx\,dt - \frac{1}{|Z_s|}\iiint_{Y_m\times Z_s\times \mathcal{T}}[A(I+\nabla_z\chi)](\xi+\nabla_y\pi_1)\cdot(\nabla_y\psi_1)\,dy\,dz\,d\tau \\
&= \iint_{Y_m\times\mathcal{T}} g(y,\tau,r)\psi_1\,d\,yd\tau \quad \text{for all } \psi_1 \in \mathcal{W},
\end{aligned}
\tag{4.29}
$$

Where $I$ stands for the identity matrix. For the sake of simplicity, we put $\tilde{A} = \int_{Z_s} A(I+\nabla_z\chi)\,dz$. The following Proposition addresses the question of existence and uniqueness of the solution to the variational problem (4.29).

**Proposition 4.4.** *The following local variational problem admits a solution which is uniquely defined on $Y_m \times \tau$:*

$$
\begin{gathered}
\pi_1(x,t,r,\xi) \in \mathcal{W} \\
[\rho\chi_{Y_m}\frac{\partial \pi_1}{\partial \tau}, \psi_1] - \frac{1}{|Z_s|}\iint_{Y_m\times\mathcal{T}} \tilde{A}(y,\tau)\nabla_y\pi_1\cdot\nabla_y\psi_1\,dy\,d\tau \\
= \iint_{Y_m\times\mathcal{T}} g(y,\tau,r)\psi_1\,dy\,d\tau + \frac{1}{|Z_s|}\iint_{Y_m\times\mathcal{T}} \tilde{A}(y,\tau)\xi\cdot\nabla_y\psi_1\,dy\,d\tau \\
\textit{for all } \psi_1 \in \mathcal{W}.
\end{gathered}
\tag{4.30}
$$

*Proof.* It is clear from the boundedness of the bilinear form on the left hand side, and the boundedness of the linear form on the right hand side of (4.30) that (4.30) admits at least one solution in $\mathcal{W}$. As for the question of uniqueness, let $\pi_1$, $\theta_1$ be two solutions to (4.30). Then $\zeta_1 = \pi_1 - \theta_1$ solves (4.30) with zero right hand side. This yields

$$
[\rho\chi_{Y_m}\frac{\partial \zeta_1}{\partial \tau}, \zeta_1] - \frac{1}{|Z_s|}\iint_{Y_m\times\mathcal{T}} \tilde{A}(y,\tau)\nabla_y\zeta_1\cdot\nabla_y\zeta_1\,dy\,d\tau = 0.
$$

But formula (3.8) of Proposition 3.10 implies

$$
[\rho\chi_{Y_m}\frac{\partial \zeta_1}{\partial \tau}, \zeta_1] = 0.
$$

We are left with

$$
\iint_{Y_m\times\mathcal{T}} \tilde{A}(y,\tau)\nabla_y\zeta_1\cdot\nabla_y\zeta_1\,dy\,d\tau = 0,
$$

which by the uniform ellipticity of the homogenized matrix $\tilde{A}$ yields $\nabla_y\zeta_1 = 0$ a.e. in $Y_m \times \mathcal{T}$. Therefore there exists a function $h$ depending only on $\tau$ such that $\xi_1(y,\tau) = h(\tau)$ for almost every $(y,\tau) \in Y_m \times \mathcal{T}$. But, since $\xi_1 \in L^2_{\rm per}(\mathcal{T}; H^1_\rho(Y_m))$, we have

$$
0 = \int_{Y_m} \rho(y)\xi_1(y,\tau)\,dy = h(\tau)\int_{Y_m}\rho(y)\,dy \quad \text{a.e. in } \mathcal{T}. \tag{4.31}
$$

Thus $h = 0$ since $\int_{Y_m}\rho(y)\,dy \neq 0$. Therefore $\xi_1 = 0$ almost every where in $Y_m \times \mathcal{T}$. □

In particular taking $r = u_0(x,t)$ and $\xi = \nabla u_0(x,t)$ with $(x,t)$ arbitrarily chosen in $\Omega_T$ and then choosing in (4.29) the particular test functions $\psi_1 = \varphi(x,t)v_1$, with $\varphi \in \mathcal{C}_0^\infty(\Omega_T)$ and $v_1 \in (\mathcal{D}_{\rm per}(\mathcal{T}) \otimes \mathcal{D}_\rho(Y_m))$, and finally comparing the resulting equation with (4.23), it follows by means of Proposition 4.2 (bear in mind that $\mathcal{D}_{\rm per}(\mathcal{T})\otimes\mathcal{D}_\rho(Y_m)$ is dense in $\mathcal{W}$), that for almost every $(x,t)\in\Omega_T$ we have

$$u_1(x,t)=\pi_1(x,t,u_0(x,t),\nabla_x u_0(x,t)) \tag{4.32}$$

almost everywhere on $Y_m\times\mathcal{T}$. The linearity of the problem 4.30 suggests a more flexible expression of its solution $\pi_1$. We formulate the following variational problems

$$\begin{gathered}\omega_1(x,t,r)\in\mathcal{W}\\ \big[\rho\chi_{Y_m}\frac{\partial\omega_1}{\partial\tau},\psi\big]-\frac{1}{|Z_s|}\iint_{Y_m\times\mathcal{T}}\tilde{A}(y,\tau)\nabla_y\omega_1\cdot\nabla_y\psi\,dy\,d\tau=\iint_{Y_m\times\mathcal{T}}g(y,\tau,r)\psi\,dy\,d\tau\\ \text{for all }\psi\in\mathcal{W}.\end{gathered} \tag{4.33}$$

and

$$\begin{gathered}\theta(x,t)=(\theta_i(x,t))_{1\le i\le N}\in(\mathcal{W})^N\\ \begin{aligned}&\big[\rho\chi_{Y_m}\frac{\partial\theta_i}{\partial\tau},\psi\big]-\frac{1}{|Z_s|}\iint_{Y_m\times\mathcal{T}}\tilde{A}(y,\tau)\nabla_y\theta_i\cdot\nabla_y\psi\,dy\,d\tau\\ &=\frac{1}{|Z_s|}\sum_{k=1}^N\iint_{Y_m\times\mathcal{T}}\tilde{A}_{ik}(y,\tau)\frac{\partial\psi}{\partial y_k}\,dy\,d\tau\quad\text{for all }\psi\in\mathcal{W},\end{aligned}\end{gathered} \tag{4.34}$$

and leave to the reader to check that they admits solutions that are uniquely defined on $Y_m\times\tau$ and satisfy

$$\pi_1(x,t,r,\xi)(y,\tau)=\theta(x,t,y,\tau)\cdot\xi+\omega_1(x,t,y,\tau,r). \tag{4.35}$$

Hence, the same lines of reasoning as above yields

$$u_1(x,t,y,\tau)=\theta(x,t,y,\tau)\cdot\nabla_x u_0(x,t)+\omega_1(x,t,y,\tau,u_0(x,t)). \tag{4.36}$$

We are now in a position to formulate the strong form of the macroscopic variational problem (4.22). Substituting in (4.22) the expression of $\nabla_z u_2$ obtained in (4.28), we have:

$$\begin{aligned}&\Big(\int_{Y_m}\rho(y)dy\Big)\int_{\Omega_T}\frac{\partial u_0}{\partial t}\psi_0\,dx\,dt\\ &=-\iiint_{\Omega_T\times Y_m\times\mathcal{T}}G(y,\tau,u_0)\cdot\nabla\psi_0\,dx\,dt\,dy\,d\tau\\ &\quad-\frac{1}{|Z_s|}\iiint_{\Omega_T\times Y_m\times\mathcal{T}}\tilde{A}(y,\tau)(\nabla_x u_0+\nabla_y u_1)\cdot(\nabla_x\psi_0)\,dx\,dt\,dy\,d\tau\\ &\quad-\frac{1}{|Z_s|}\iiint_{\Omega_T\times Y_m\times\mathcal{T}}\Big[\partial_r G(y,\tau,u_0)\cdot\Big(\Big(\int_{Z_s}(I+\nabla_z\chi)dz\Big)\cdot(\nabla_x u_0\\ &\quad+\nabla_y u_1)\Big)\Big]\psi_0\,dx\,dt\,dy\,d\tau\quad\text{for all }\psi_0\in\mathcal{C}_0^\infty(\Omega_T).\end{aligned} \tag{4.37}$$

We put $\tilde{B}(y,\tau) = \int_{Z_s} (I + \nabla_z \chi(y,\tau,z))dz$ $((y,\tau) \in \mathbb{R}^N \times \mathbb{R})$ and use (4.36) to get

$$
\begin{aligned}
&\Big(\int_{Y_m} \rho(y)dy\Big) \int_{\Omega_T} \frac{\partial u_0}{\partial t} \psi_0 \, dx\, dt \\
&= - \int_{\Omega_T} \Big(\iint_{Y_m \times \mathcal{T}} \partial_r G(y,\tau,u_0) \cdot \nabla_x u_0 \Big) \psi_0 \, dx\, dt\, dy\, d\tau \\
&\quad - \frac{1}{|Z_s|} \int_{\Omega_T} \hat{A} \nabla_x u_0 \cdot \nabla_x \psi_0 \, dx\, dt \\
&\quad - \frac{1}{|Z_s|} \int_{\Omega_T} \Big(\iint_{Y_m \times \mathcal{T}} \tilde{A}(y,\tau) \nabla_y \omega_1 \, dy\, d\tau \Big) \cdot \nabla_x \psi_0 \, dx\, dt \\
&\quad - \frac{1}{|Z_s|} \int_{\Omega_T} \Big\{\Big(\iint_{Y_m \times \mathcal{T}} \partial_r G(y,\tau,u_0) \big(\tilde{B}(y,\tau)(I + \nabla_y \theta)\big) dy\, d\tau \Big) \\
&\quad \cdot \nabla_x u_0 \Big\} \psi_0 \, dx\, dt \\
&\quad - \frac{1}{|Z_s|} \int_{\Omega_T} \Big(\iint_{Y_m \times \mathcal{T}} \partial_r G(y,\tau,u_0) \Big(\tilde{B}(y,\tau)(I + \nabla_y \theta) \cdot \nabla_y \omega_1 \Big) \, dy\, d\tau \Big) \\
&\quad \cdot \psi_0 \, dx\, dt \quad \text{for all } \psi_0 \in \mathcal{C}_0^\infty(\Omega_T),
\end{aligned} \tag{4.38}
$$

where

$$
\hat{A}(x,t) = \iint_{Y_m \times \mathcal{T}} \tilde{A}(y,\tau)(I + \nabla_y \theta(x,t,y,\tau)) dy\, d\tau \quad ((x,t) \in \Omega_T).
$$

Setting

$$
L_1(x,t,r) = \iint_{Y_m \times \mathcal{T}} \tilde{A}(y,\tau) \nabla_y \omega_1(x,t,y,\tau,r) \, dy\, d\tau,
$$

$$
\begin{aligned}
&L_2(x,t,r) \\
&= \iint_{Y_m \times \mathcal{T}} \Big[|Z_s| \partial_r G(y,\tau,r) + \partial_r G(y,\tau,r) \Big(\tilde{B}(y,\tau)(I + \nabla_y \theta(x,t,y,\tau))\Big)\Big] \, dy\, d\tau, \\
&L_3(x,t,r) \\
&= \iint_{Y_m \times \mathcal{T}} \partial_r G(y,\tau,r) \Big(\tilde{B}(y,\tau)(I + \nabla_y \theta(x,t,y,\tau)) \cdot \nabla_y \omega_1(x,t,y,\tau,r) \Big) \, dy\, d\tau \, .
\end{aligned}
$$

We are led to the following result.

**Theorem 4.5.** *The function $u_0$ determined by* (4.1) *and solution to the variational problem* (4.22)*, is the unique solution to the boundary value problem*

$$
\begin{aligned}
|Z_s| \Big(\int_{Y_m} \rho(y)\, dy\Big) \frac{\partial u_0}{\partial t} &= \operatorname{div}\big(\hat{A}(x,t) \nabla u_0\big) + \operatorname{div} L_1(x,t,u_0) \\
&\quad - L_2(x,t,u_0) \cdot \nabla u_0 - L_3(x,t,u_0) \quad \text{in } \Omega_T \\
u_0 &= 0 \quad \text{on } \partial\Omega \times (0,T) \\
u_0(x,0) &= u^0(x) \quad \text{in } \Omega.
\end{aligned} \tag{4.39}
$$

*Proof.* The claim that $u_0$ solves the problem (4.39) has been proved above and the uniqueness of the solution to (4.39) follows from the fact that the functions $L_i(x,t,\cdot)$ $(1 \le i \le N)$ are Lipschitz. This can be proved by mimicking the reasoning in [3]. $\square$

We can now formulate the homogenization result for problem (1.1).

**Theorem 4.6.** *Assuming that the hypotheses* (A1)–(A4) *are in place and letting* $u_\varepsilon$ *(*$\varepsilon > 0$*) be the unique solution to* (1.1)*, we have, as* $\varepsilon \to 0$*,*

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega_T),$$

*where* $u_0 \in L^2(0,T;H^1_0(\Omega))$ *is the unique solution to* (4.39)*.*

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