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MIXED BOUNDARY-VALUE PROBLEMS FOR MOTION EQUATIONS OF A VISCOELASTIC MEDIUM

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Abstract. We study the mixed boundary-value problem for steady motion equations of an incompressible viscoelastic medium of Jeffreys type in a fixed three-dimensional domain. On one part of the boundary the no-slip condition is provided, while on the other one the impermeability condition and non-homogeneous Dirichlet boundary conditions for tangential component of the surface force is used. The existence of weak solutions of the formulated boundary-value problem is proved. Some estimates for weak solutions are established; it is shown that the set of weak solutions is sequentially weakly closed.

1. Introduction

Mixed boundary problems play significant role in the modeling of fluid flows in domains with a boundary which includes several parts, differing by their physical properties. Mixed boundary conditions arise also when studying boundary flow control problems and at the modeling of flows with free surface.

In this article, we study the nonlinear boundary-value problem for steady motion equations of an incompressible viscoelastic medium of Jeffreys type [13] in a bounded three-dimensional domain with mixed boundary conditions. On a part of the boundary the homogeneous Dirichlet boundary condition is formulated for the field velocity $\mathbf{v}$. This condition has the meaning of non-slip behavior of the viscoelastic medium on this part of the solid wall. On the other part of the boundary we use the impermeability condition ($\mathbf{v}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the outward unit normal vector) and the non-homogeneous Dirichlet boundary condition for the tangential component of the surface force. Obviously, these conditions allow slippage on the corresponding part of the boundary.

This article is organized as follows. In Section 2, the weak formulation of the boundary-value problem is presented. We use a nonstandard approach to definition of weak solutions. The novelty is that the motion equations and the boundary conditions are taken into account in a single integral identity. We use such approach to overcome the difficulties associated with definition of the boundary trace for the low regular extra-stress tensor. We show that a weak solution is well defined. In

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particular, if a weak solution is sufficiently smooth, then it is a classical solution, i.e., the corresponding vector functions satisfy the system of equations and the boundary conditions in the usual sense. In Section 3, we prove the existence of weak solutions and establish some estimates. The proof is based on the Galerkin method, the method of introduction of auxiliary viscosity [10] and topological degree methods [7]. We show also that the set of weak solutions is sequentially weakly closed. All results are obtained without any restriction on the data values.

Note that homogeneous boundary-value problems for liquids described by Jeffreys model and other similar non-Newtonian models were studied by many authors (see e.g. [5, 6, 11, 14, 15] and the references therein). The solvability of the non-homogeneous Dirichlet boundary-value problem for the Jeffreys model was proved in [2]. Some existence results for the equations, describing viscoelastic fluid flows with Navier type slip boundary conditions, were obtained in [3, 9].

2. Problem formulation

As it is well known, the steady motion of any incompressible medium is described by the system of equations in Cauchy form

$$\rho\,\mathbf{v}\cdot\nabla\mathbf{v} = \operatorname{div}\mathbf{T} + \rho\mathbf{f}, \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{2.2}$$

where ρ is the density, $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is the flow velocity at a point $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{T} = \mathbf{T}(\mathbf{x})$ is the Cauchy stress tensor, $\mathbf{f} = \mathbf{f}(\mathbf{x})$ denotes the external force. The Cauchy stress tensor is given by

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{S},$$

where the scalar $p = p(\mathbf{x})$ is the hydrostatic pressure and $\mathbf{S} = \mathbf{S}(\mathbf{x})$ is the extra-stress tensor. The precise form of $\mathbf{S}$ is given by a constitutive law, which depends on the medium. We will use the Jeffreys constitutive law:

$$\mathbf{S} + \lambda_1\mathbf{v}\cdot\nabla\mathbf{S} = 2\eta\big(\mathbf{D} + \lambda_2\mathbf{v}\cdot\nabla\mathbf{D}\big), \tag{2.3}$$

where $\mathbf{D} = \mathbf{D}(\mathbf{v})$ is the strain velocity tensor,

$$\mathbf{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T),$$

η, λ_1, and λ_2 are positive constants. The rheological parameters of the Jeffreys model follow the inequality $\lambda_2/\lambda_1 < 1$, which is explained by thermodynamic limitations (see, for instance [4]).

Equation (2.3) can be rewritten as

$$\mathbf{E} + \lambda_1\mathbf{v}\cdot\nabla\mathbf{E} = 2\epsilon\eta\mathbf{D}(\mathbf{v}), \tag{2.4}$$

where $\mathbf{E}$ is the elastic part of the extra-stress $\mathbf{S}$,

$$\mathbf{E} = \mathbf{S} - 2\eta\lambda_2\lambda_1^{-1}\mathbf{D}(\mathbf{v}), \tag{2.5}$$

and $\epsilon = 1 - \lambda_2\lambda_1^{-1}$.

To write the equations in dimensionless form, choose a characteristic length l and a characteristic speed V and define

$$\mathbf{x}^* = l^{-1}\mathbf{x}, \quad \mathbf{v}^*(\mathbf{x}^*) = V^{-1}\mathbf{v}(\mathbf{x}), \quad \mathbf{E}^*(\mathbf{x}^*) = l(\eta V)^{-1}\mathbf{E}(\mathbf{x}),$$
$$\mathbf{S}^*(\mathbf{x}^*) = l(\eta V)^{-1}\mathbf{S}(\mathbf{x}), \quad p^*(\mathbf{x}^*) = l(\eta V)^{-1}p(\mathbf{x}), \quad \mathbf{f}^*(\mathbf{x}^*) = \rho l^2(\eta V)^{-1}\mathbf{f}(\mathbf{x}).$$

Then, by writing system (2.1), (2.2), (2.4), (2.5) in terms of these dimensionless quantities and omitting the asterisks, we obtain the dimensionless system

$$\mathrm{Re}\,\mathbf{v}\cdot\nabla\mathbf{v}+\nabla p-\operatorname{div}\mathbf{S}=\mathbf{f}, \tag{2.6}$$

$$\operatorname{div}\mathbf{v}=0, \tag{2.7}$$

$$\mathbf{E}+\mathrm{We}\,\mathbf{v}\cdot\nabla\mathbf{E}=2\epsilon\mathbf{D}(\mathbf{v}), \tag{2.8}$$

$$\mathbf{S}=\mathbf{E}+2(1-\epsilon)\mathbf{D}(\mathbf{v}), \tag{2.9}$$

where Re is the Reynolds number, $\mathrm{Re}=\rho lV\eta^{-1}$, and We is the Weissenberg number, $\mathrm{We}=\lambda_1 Vl^{-1}$.

We will investigate the system of equations (2.6)–(2.9). One should of course add suitable conditions at the boundary of the flow domain Ω. We assume that Ω is a bounded domain in $\mathbb{R}^3$ with the boundary $\Gamma\in C^2$, and the boundary is impermeable. Thus

$$\mathbf{v}\cdot\mathbf{n}=0\quad\text{on }\Gamma, \tag{2.10}$$

where $\mathbf{n}=\mathbf{n}(\mathbf{x})$ is the outer unit normal on Γ at the point $\mathbf{x}$, $\mathbf{v}\cdot\mathbf{n}$ is the scalar product of the vectors $\mathbf{v}$ and $\mathbf{n}$ in space $\mathbb{R}^3$.

Moreover, we assume that the flow on the boundary is governed by the following conditions

$$\mathbf{v}=\mathbf{0}\quad\text{on }\Gamma_0, \tag{2.11}$$

$$[\mathbf{Sn}]_\tau=\mathbf{g}\quad\text{on }\Gamma\setminus\Gamma_0, \tag{2.12}$$

where Γ_0 is a part of Γ (the Lebesgue 2-dimensional measure of Γ_0 is positive), $\mathbf{g}$ is a given vector field such that $\mathbf{g}\cdot\mathbf{n}=0$, $[\cdot]_\tau$ denotes the tangential component of the vector, i.e., $\mathbf{u}_\tau=\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}$.

The aim of this article is to prove the existence of weak solutions of problem (2.6)–(2.12). We shall begin by giving the definition of a weak solution. To perform our study, however, we need certain function spaces.

Let $\mathbf{F}$ be a finite-dimensional space. We use the standard notation

$$\mathbf{L}_p(\Omega,\mathbf{F}),\ \mathbf{H}^m(\Omega,\mathbf{F})=\mathbf{W}_2^m(\Omega,\mathbf{F})$$

for the Lebesgue and Sobolev spaces of functions with values in $\mathbf{F}$. The scalar product in $\mathbf{L}_2$ will be denoted $(\cdot,\cdot)$.

By $\mathbf{C}_0^\infty(\Omega,\mathbf{F})$ denote the space of smooth functions with support in Ω and with values in $\mathbf{F}$.

By $\mathbf{H}_0^2(\Omega,\mathbf{F})$ denote the closure $\mathbf{C}_0^\infty(\Omega,\mathbf{F})$ in $\mathbf{H}^2(\Omega,\mathbf{F})$. We will use the following scalar product in $\mathbf{H}_0^2(\Omega,\mathbf{F})$

$$(\mathbf{v},\mathbf{w})_{\mathbf{H}_0^2(\Omega,\mathbf{F})}=(\Delta\mathbf{v},\Delta\mathbf{w})\,.$$

It follows from the properties of the Laplace operator Δ that the norm

$$\|\mathbf{v}\|_{\mathbf{H}_0^2(\Omega,\mathbf{F})}=(\mathbf{v},\mathbf{v})^{1/2}_{\mathbf{H}_0^2(\Omega,\mathbf{F})}$$

is equivalent to the norm induced from $\mathbf{H}^2(\Omega,\mathbf{F})$.

We now introduce the main space

$$\mathbf{X}(\Omega,\mathbb{R}^3)=\{\mathbf{v}\in\mathbf{H}^1(\Omega,\mathbb{R}^3):\operatorname{div}\mathbf{v}=0,\ \mathbf{v}|_\Gamma\cdot\mathbf{n}=0,\ \mathbf{v}|_{\Gamma_0}=\mathbf{0}\}.$$

Here the restriction of $\mathbf{v}\in\mathbf{H}^1(\Omega,\mathbb{R}^3)$ to Γ is given by $\mathbf{v}|_\Gamma=\gamma_0\mathbf{v}$, where $\gamma_0:\mathbf{H}^1(\Omega,\mathbb{R}^3)\to\mathbf{H}^{1/2}(\Gamma,\mathbb{R}^3)$ is the trace operator (see e.g. [1]).

We define the scalar product in $\mathbf{X}(\Omega,\mathbb{R}^3)$ by the formula

$$(\mathbf{v},\mathbf{w})_{\mathbf{X}(\Omega,\mathbb{R}^3)} = \big(\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{w})\big).$$

Let us show that the norm

$$\|\mathbf{v}\|_{\mathbf{X}(\Omega,\mathbb{R}^3)} = (\mathbf{v},\mathbf{v})^{1/2}_{\mathbf{X}(\Omega,\mathbb{R}^3)}$$

is equivalent to the norm induced from the Sobolev space $\mathbf{H}^1(\Omega,\mathbb{R}^3)$.

First we recall an inequality of Korn's type.

Lemma 2.1. *Let $a : \mathbf{H}^1(\Omega,\mathbb{R}^3)\times\mathbf{H}^1(\Omega,\mathbb{R}^3)\to\mathbb{R}$ be a continuous symmetric bilinear form such that*

$$a(\mathbf{v},\mathbf{v})\ge 0 \quad \textit{for all } \mathbf{v}\in\mathbf{H}^1(\Omega,\mathbb{R}^3)$$

and it follows from the conditions

$$\big(\mathbf{D}(\mathbf{w}),\mathbf{D}(\mathbf{w})\big)=0,\ a(\mathbf{w},\mathbf{w})=0,\ \mathbf{w}\in\mathbf{H}^1(\Omega,\mathbb{R}^3)$$

that $\mathbf{w}=0$. Then there exists a positive constant C such that

$$\big(\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{v})\big)+a(\mathbf{v},\mathbf{v})\ge C\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega,\mathbb{R}^3)}$$

for all $\mathbf{v}\in\mathbf{H}^1(\Omega,\mathbb{R}^3)$.

A proof of the above lemma is found in [12]. Define the bilinear form a as

$$a(\mathbf{v},\mathbf{w})=\int_{\Gamma_0}\mathbf{v}\cdot\mathbf{w}\,d\sigma,\quad \mathbf{v},\mathbf{w}\in\mathbf{H}^1(\Omega,\mathbb{R}^3),$$

where σ denotes the Lebesgue 2-dimensional measure. The application of Lemma 2.1 yields

$$\|\mathbf{D}(\mathbf{v})\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3})}+\int_{\Gamma_0}\|\mathbf{v}(\mathbf{x})\|^2_{\mathbb{R}^3}\,d\sigma\ge C\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega,\mathbb{R}^3)},\quad \mathbf{v}\in\mathbf{H}^1(\Omega,\mathbb{R}^3).$$

Thus we have

$$\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega,\mathbb{R}^3)}\ge\|\mathbf{D}(\mathbf{v})\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3})}\ge C\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega,\mathbb{R}^3)}$$

for all $\mathbf{v}\in\mathbf{X}(\Omega,\mathbb{R}^3)$.

We now describe the concept of a weak solution. Assume that

$$\mathbf{f}\in\mathbf{L}_2(\Omega,\mathbb{R}^3),\quad \mathbf{g}\in\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3).$$

Denote by $\mathbb{R}^{3\times3}_s$ the space of 3×3 symmetric matrices.

Definition. We shall say that a triplet

$$(\mathbf{v},\mathbf{E},\mathbf{S})\in\mathbf{X}(\Omega,\mathbb{R}^3)\times\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)\times\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)$$

is a *weak solution* of problem (2.6)–(2.12) if it satisfies equation (2.9) and if the equalities

$$-\operatorname{Re}\sum_{i=1}^3\big(v_i\mathbf{v},\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\big)+\big(\mathbf{S},\mathbf{D}(\boldsymbol{\varphi})\big)=\int_{\Gamma\setminus\Gamma_0}\mathbf{g}\cdot\boldsymbol{\varphi}\,d\sigma+(\mathbf{f},\boldsymbol{\varphi}), \tag{2.13}$$

$$(\mathbf{E},\mathbf{\Phi})-\operatorname{We}\sum_{i=1}^3\Big(\mathbf{E},v_i\frac{\partial\mathbf{\Phi}}{\partial x_i}\Big)=2\epsilon\big(\mathbf{D}(\mathbf{v}),\mathbf{\Phi}\big) \tag{2.14}$$

hold for all $\boldsymbol{\varphi}\in\mathbf{X}(\Omega,\mathbb{R}^3)$ and $\mathbf{\Phi}\in\mathbf{C}_0^\infty(\Omega,\mathbb{R}^{3\times3}_s)$.

Remark 2.2. Equalities (2.13) and (2.14) appear from the following reasoning. Let us assume that $(\mathbf{v}, \mathbf{E}, \mathbf{S}, p)$ is a classical solution of problem (2.6)–(2.12). Taking the scalar product of equality (2.6) with $\boldsymbol{\varphi} \in \mathbf{X}(\Omega, \mathbb{R}^3)$ and integrating over the domain Ω, we obtain

$$\mathrm{Re}\Big(\sum_{i=1}^{3} v_i \frac{\partial \mathbf{v}}{\partial x_i}, \boldsymbol{\varphi}\Big) + (\nabla p, \boldsymbol{\varphi}) - (\operatorname{div} \mathbf{S}, \boldsymbol{\varphi}) = (\mathbf{f}, \boldsymbol{\varphi}). \tag{2.15}$$

Integrating by parts,

$$\begin{aligned}\Big(\sum_{i=1}^{3} v_i \frac{\partial \mathbf{v}}{\partial x_i}, \boldsymbol{\varphi}\Big) &= -(\mathbf{v} \operatorname{div} \mathbf{v}, \boldsymbol{\varphi}) - \sum_{i=1}^{3} \big(v_i \mathbf{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_i}\big) + \int_{\Gamma\setminus\Gamma_0} (\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\boldsymbol{\varphi})\, d\sigma \\ &= -\sum_{i=1}^{3} \big(v_i \mathbf{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_i}\big),\end{aligned} \tag{2.16}$$

$$(\nabla p, \boldsymbol{\varphi}) = -(p, \operatorname{div} \boldsymbol{\varphi}) + \int_{\Gamma\setminus\Gamma_0} p\, (\boldsymbol{\varphi}\cdot\mathbf{n})\, d\sigma = 0, \tag{2.17}$$

$$(\operatorname{div} \mathbf{S}, \boldsymbol{\varphi}) = -\left(\mathbf{S}, \mathbf{D}(\boldsymbol{\varphi})\right) + \int_{\Gamma\setminus\Gamma_0} \mathbf{S}\mathbf{n}\cdot\boldsymbol{\varphi}\, d\sigma. \tag{2.18}$$

Combining (2.15), (2.16), (2.17), (2.18) and (2.12), we obtain equality (2.13). Likewise, taking the $\mathbf{L}_2$-scalar product of (2.8) with a function $\boldsymbol{\Phi} \in \mathbf{C}_0^\infty(\Omega, \mathbb{R}_s^{3\times 3})$ and integrating by parts, we obtain equality (2.14).

Remark 2.3. Let us check that if the weak solution $(\mathbf{v}, \mathbf{E}, \mathbf{S})$ of problem (2.6)–(2.12) is sufficiently smooth, then there exists a function p such that $(\mathbf{v}, \mathbf{E}, \mathbf{S}, p)$ is a classical solution. In fact, multiplying (2.13) by -1 and integrating by parts, we can rewrite (2.13) as follows:

$$\Big(-\mathrm{Re}\sum_{i=1}^{3} v_i \frac{\partial \mathbf{v}}{\partial x_i} + \operatorname{div}\mathbf{S} + \mathbf{f}, \boldsymbol{\varphi}\Big) = \int_{\Gamma\setminus\Gamma_0} ([\mathbf{S}\mathbf{n}]_\tau - \mathbf{g})\cdot\boldsymbol{\varphi}\, d\sigma \tag{2.19}$$

for all $\boldsymbol{\varphi} \in \mathbf{X}(\Omega, \mathbb{R}^3)$. Thus

$$\Big(-\mathrm{Re}\sum_{i=1}^{3} v_i \frac{\partial \mathbf{v}}{\partial x_i} + \operatorname{div}\mathbf{S} + \mathbf{f}, \boldsymbol{\psi}\Big) = 0$$

for all $\psi \in \mathbf{H}^1(\Omega, \mathbb{R}^3)$ such that $\operatorname{div}\psi = 0$ and $\psi|_\Gamma = \mathbf{0}$. Hence (see e.g. [8]), there exists a function p such that

$$-\mathrm{Re}\sum_{i=1}^{3} v_i \frac{\partial \mathbf{v}}{\partial x_i} + \operatorname{div}\mathbf{S} + \mathbf{f} = \nabla p. \tag{2.20}$$

This means that equation (2.6) holds. Also, it can be shown in the standard way that the pair $(\mathbf{v}, \mathbf{E})$ satisfies equation (2.8). Moreover, by definition, equalities (2.7), (2.9), (2.10), and (2.11) are valid.

It remains to check that boundary condition (2.12) holds. Substituting (2.20) in (2.19), we obtain

$$(\nabla p, \boldsymbol{\varphi}) = \int_{\Gamma\setminus\Gamma_0} ([\mathbf{S}\mathbf{n}]_\tau - \mathbf{g})\cdot\boldsymbol{\varphi}\, d\sigma, \quad \boldsymbol{\varphi} \in \mathbf{X}(\Omega, \mathbb{R}^3). \tag{2.21}$$

Integrating by parts, we see that the left-hand side of (2.21) is equal to zero. Thus

$$\int_{\Gamma\setminus\Gamma_0} ([\mathbf{S}\mathbf{n}]_\tau - \mathbf{g})\cdot\boldsymbol{\varphi}\,d\sigma = 0, \quad \boldsymbol{\varphi}\in\mathbf{X}(\Omega,\mathbb{R}^3). \tag{2.22}$$

Since the set $\{\boldsymbol{\varphi}|_{\Gamma\setminus\Gamma_0} : \boldsymbol{\varphi}\in\mathbf{X}(\Omega,\mathbb{R}^3)\}$ is dense in the space

$$\{\mathbf{w}\in\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3) : \mathbf{w}\cdot\mathbf{n}=0\},$$

it follows that equality (2.22) still holds by continuity for any vector function $\boldsymbol{\varphi}\in\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)$ such that $\boldsymbol{\varphi}\cdot\mathbf{n}=0$. This implies that $[\mathbf{S}\mathbf{n}]_\tau - \mathbf{g} = \mathbf{0}$, i.e., condition (2.12) holds.

3. Existence of a weak solution

We formulate our main result as follows.

Theorem 3.1. *Assume that $\mathbf{f}\in\mathbf{L}_2(\Omega,\mathbb{R}^3)$, $\mathbf{g}\in\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)$, and $\mathbf{g}\cdot\mathbf{n}=0$ on $\Gamma\setminus\Gamma_0$. Then*

(a) *problem (2.6)–(2.12) has at least one weak solution such that*

$$\begin{aligned}&\|\mathbf{E}\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)} + 4\epsilon(1-\epsilon)\|\mathbf{D}(\mathbf{v})\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)}\\ &\le C\frac{\epsilon(\|\mathbf{g}\|_{\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)} + \|\mathbf{f}\|_{\mathbf{L}_2(\Omega,\mathbb{R}^3)})^2}{1-\epsilon},\end{aligned}$$

where C is a constant,

(b) *the set of weak solutions of problem (2.6)–(2.12) is sequentially weakly closed in the space $\mathbf{X}(\Omega,\mathbb{R}^3)\times\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)\times\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)$.*

To prove the above Theorem, we need the following lemma.

Lemma 3.2. *Let $\mathbf{B}_R = \{\mathbf{x}\in\mathbb{R}^n : \|\mathbf{x}\|_{\mathbb{R}^n}\le R\}$ be a closed ball and let $\mathbf{F} : \mathbf{B}_R\times[0,1]\to\mathbb{R}^n$ be a continuous map such that*

(i) *$\mathbf{F}(\mathbf{x},\xi)\neq\mathbf{0}$ for all $(\mathbf{x},\xi)\in\partial\mathbf{B}_R\times[0,1]$,*

(ii) *$\mathbf{F}(\mathbf{x},0)=\mathbf{A}\mathbf{x}$ for all $\mathbf{x}\in\mathbf{B}_R$,*

where $\mathbf{A}:\mathbb{R}^n\to\mathbb{R}^n$ is an isomorphism. Then for each $\xi\in[0,1]$ the equation $\mathbf{F}(\mathbf{x},\xi)=\mathbf{0}$ has at least one solution $\mathbf{x}_\xi\in\mathbf{B}_R$.

This lemma can be proved by standard methods of topological degree theory (see [7]).

Proof of Theorem 3.1. Suppose that $\{\boldsymbol{\varphi}^j\}_{j=1}^\infty$ is an orthonormal basis for the space $\mathbf{X}(\Omega,\mathbb{R}^3)$, and $\{\mathbf{Y}^j\}_{j=1}^\infty$ is an orthonormal basis for $\mathbf{H}^2_0(\Omega,\mathbb{R}^{3\times3}_s)$ such that $\mathbf{Y}^j\in\mathbf{C}^\infty_0(\Omega,\mathbb{R}^{3\times3}_s)$ for all $j\in\mathbb{N}$. Let us fix $n\in\mathbb{N}$.

Consider the auxiliary problem: Find a triplet $(\mathbf{v}^n,\mathbf{E}^n,\mathbf{S}^n)$ such that

$$\begin{aligned}&-\xi\,\mathrm{Re}\sum_{i=1}^3\Big(v_i^n\mathbf{v}^n,\frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\Big) + \xi\big(\mathbf{E}^n,\mathbf{D}(\boldsymbol{\varphi}^j)\big) + 2(1-\epsilon)\big(\mathbf{D}(\mathbf{v}^n),\mathbf{D}(\boldsymbol{\varphi}^j)\big)\\ &= \xi\int_{\Gamma\setminus\Gamma_0}\mathbf{g}\cdot\boldsymbol{\varphi}^j\,d\sigma + \xi\left(\mathbf{f},\boldsymbol{\varphi}^j\right), \quad j=1,\dots,n,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&(\mathbf{E}^n,\mathbf{Y}^j) + \xi\,\mathrm{We}\sum_{i=1}^3\big(\frac{\partial\mathbf{E}^n}{\partial x_i},v_i^n\mathbf{Y}^j\big) + \frac{1}{n}(\Delta\mathbf{E}^n,\Delta\mathbf{Y}^j)\\ &= 2\xi\epsilon\big(\mathbf{D}(\mathbf{v}^n),\mathbf{Y}^j\big), \quad j=1,\dots,n,\end{aligned} \tag{3.2}$$

$$\mathbf{v}^n = \sum_{j=1}^{n} \alpha_{nj} \boldsymbol{\varphi}^j, \tag{3.3}$$

$$\mathbf{E}^n = \sum_{j=1}^{n} \beta_{nj} \mathbf{Y}^j, \tag{3.4}$$

$$\mathbf{S}^n = \mathbf{E}^n + 2(1-\epsilon)\mathbf{D}(\mathbf{v}^n), \tag{3.5}$$

where α_{nj} and β_{nj} are unknown real numbers, ξ is a parameter, and $\xi \in [0,1]$.

First we prove some a priori estimates of solutions of (3.1)–(3.5). Let a triplet $(\mathbf{v}^n, \mathbf{E}^n, \mathbf{S}^n)$ satisfies (3.1)–(3.5). We multiply (3.1) by α_{nj} and add these equalities for $j = 1, \dots, n$. Taking into account

$$\Big(\sum_{i=1}^{3} v_i^n \frac{\partial \mathbf{v}^n}{\partial x_i}, \mathbf{v}^n\Big) = 0,$$

we obtain

$$\xi\left(\mathbf{E}^n, \mathbf{D}(\mathbf{v}^n)\right) + 2(1-\epsilon)\big(\mathbf{D}(\mathbf{v}^n), \mathbf{D}(\mathbf{v}^n)\big) = \xi \int_{\Gamma\setminus\Gamma_0} \mathbf{g}\cdot\mathbf{v}^n \, d\sigma + \xi\left(\mathbf{f}, \mathbf{v}^n\right). \tag{3.6}$$

Furthermore, we multiply (3.2) by β_{nj} and add these equalities for $j = 1, \dots, n$. Taking into account the equality

$$\Big(\sum_{i=1}^{3} v_i^n \frac{\partial \mathbf{E}^n}{\partial x_i}, \mathbf{E}^n\Big) = 0,$$

we obtain

$$(\mathbf{E}^m, \mathbf{E}^m) + \frac{1}{n}(\Delta\mathbf{E}^n, \Delta\mathbf{E}^n) = 2\xi\epsilon\big(\mathbf{D}(\mathbf{v}^n), \mathbf{E}^n\big). \tag{3.7}$$

We multiply (3.6) by 2ϵ and add it to (3.7); this gives

$$\begin{aligned}&(\mathbf{E}^m, \mathbf{E}^m) + \frac{1}{n}(\Delta\mathbf{E}^n, \Delta\mathbf{E}^n) + 4\epsilon(1-\epsilon)\big(\mathbf{D}(\mathbf{v}^n), \mathbf{D}(\mathbf{v}^n)\big)\\ &= 2\xi\epsilon \int_{\Gamma\setminus\Gamma_0} \mathbf{g}\cdot\mathbf{v}^n \, d\sigma + 2\xi\epsilon\left(\mathbf{f}, \mathbf{v}^n\right).\end{aligned}$$

Thus we have

$$\begin{aligned}&\|\mathbf{E}^n\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)} + \frac{1}{n}\|\mathbf{E}^n\|^2_{\mathbf{H}^2_0(\Omega,\mathbb{R}^{3\times3}_s)} + 4\epsilon(1-\epsilon)\|\mathbf{v}^n\|^2_{\mathbf{X}(\Omega,\mathbb{R}^3)}\\ &= 2\xi\epsilon \int_{\Gamma\setminus\Gamma_0} \mathbf{g}\cdot\mathbf{v}^n \, d\sigma + 2\xi\epsilon\left(\mathbf{f}, \mathbf{v}^n\right).\end{aligned}$$

Hence

$$\begin{aligned}&\|\mathbf{E}^n\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times3}_s)} + \frac{1}{n}\|\mathbf{E}^n\|^2_{\mathbf{H}^2_0(\Omega,\mathbb{R}^{3\times3}_s)} + 4\epsilon(1-\epsilon)\|\mathbf{v}^n\|^2_{\mathbf{X}(\Omega,\mathbb{R}^3)}\\ &\le 2\epsilon C(\|\mathbf{g}\|_{\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)} + \|\mathbf{f}\|_{\mathbf{L}_2(\Omega,\mathbb{R}^3)})\|\mathbf{v}^n\|_{\mathbf{X}(\Omega,\mathbb{R}^3)},\end{aligned} \tag{3.8}$$

where C is a constant. This yields

$$\|\mathbf{v}^n\|_{\mathbf{X}(\Omega,\mathbb{R}^3)} \le \frac{C(\|\mathbf{g}\|_{\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)} + \|\mathbf{f}\|_{\mathbf{L}_2(\Omega,\mathbb{R}^3)})}{2(1-\epsilon)}. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain the estimate

$$\begin{aligned} &\|\mathbf{E}^n\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times 3}_s)} + \frac{1}{n}\|\mathbf{E}^n\|^2_{\mathbf{H}^2_0(\Omega,\mathbb{R}^{3\times 3}_s)} + 4\epsilon(1-\epsilon)\|\mathbf{v}^n\|^2_{\mathbf{X}(\Omega,\mathbb{R}^3)} \\ &\le \frac{\epsilon C^2(\|\mathbf{g}\|_{\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)} + \|\mathbf{f}\|_{\mathbf{L}_2(\Omega,\mathbb{R}^3)})^2}{1-\epsilon}. \end{aligned} \tag{3.10}$$

An application of Lemma 3.2 yields that problem (3.1)–(3.5) is solvable for each $n \in \mathbb{N}$ and $\xi \in [0,1]$.

Let $(\mathbf{v}^n, \mathbf{E}^n, \mathbf{S}^n)$, $n = 1, 2, \dots$, be a sequence of solutions of problem (3.1)–(3.5) with $\xi = 1$. It follows from estimate (3.10) that the norms $\|\mathbf{v}^n\|_{\mathbf{X}(\Omega,\mathbb{R}^3)}$ and $\|\mathbf{E}^n\|_{\mathbf{L}_2(\Omega,\mathbb{R}^{3\times 3}_s)}$ are uniformly bounded with respect to n. Since the closed balls of Hilbert space are weakly compact, there exists a pair $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{E}}) \in \mathbf{X}(\Omega,\mathbb{R}^3) \times \mathbf{L}_2(\Omega,\mathbb{R}^{3\times 3}_s)$ and a subsequence $\{n_k\}_{k=1}^\infty$ such that $\mathbf{v}^{n_k} \to \widetilde{\mathbf{v}}$ weakly in $\mathbf{X}(\Omega,\mathbb{R}^3)$ and $\mathbf{E}^{n_k} \to \widetilde{\mathbf{E}}$ weakly in $\mathbf{L}_2(\Omega,\mathbb{R}^{3\times 3}_s)$ as $k \to \infty$. Without loss of generality it can be assumed that

$$\mathbf{v}^n \to \widetilde{\mathbf{v}} \text{ weakly in } \mathbf{X}(\Omega,\mathbb{R}^3), \quad \mathbf{E}^n \to \widetilde{\mathbf{E}} \text{ weakly in } \mathbf{L}_2(\Omega,\mathbb{R}^{3\times 3}_s) \tag{3.11}$$

as $n \to \infty$. Due to (3.11) and the compactness theorem (see [1]), we also have

$$\mathbf{v}^n \to \widetilde{\mathbf{v}} \text{ strongly in } \mathbf{L}_4(\Omega,\mathbb{R}^3) \tag{3.12}$$

as $n \to \infty$.

Now define

$$\widetilde{\mathbf{S}} = \widetilde{\mathbf{E}} + 2(1-\epsilon)\mathbf{D}(\widetilde{\mathbf{v}}).$$

Let us show that the triplet $\big(\widetilde{\mathbf{v}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}\big)$ is a weak solution of problem (2.6)–(2.12). Using (3.11) and (3.12), we can pass to the limit $n \to \infty$ in equality (3.1) (with $\xi = 1$) and obtain

$$-\operatorname{Re}\sum_{i=1}^3 \Big(\widetilde{v}_i\widetilde{\mathbf{v}}, \frac{\partial \boldsymbol{\varphi}^j}{\partial x_i}\Big) + \big(\widetilde{\mathbf{S}}, \mathbf{D}(\boldsymbol{\varphi}^j)\big) = \int_{\Gamma\setminus\Gamma_0} \mathbf{g}\cdot\boldsymbol{\varphi}^j\, d\sigma + \big(\mathbf{f}, \boldsymbol{\varphi}^j\big) \tag{3.13}$$

for any $j \in \mathbb{N}$. Recall that $\{\boldsymbol{\varphi}^j\}_{j=1}^\infty$ is a basis of $\mathbf{X}(\Omega,\mathbb{R}^3)$ and thus equality (3.13) remains valid if we replace $\boldsymbol{\varphi}^j$ with an arbitrary vector function $\boldsymbol{\varphi} \in \mathbf{X}(\Omega,\mathbb{R}^3)$.

Further, integrating by parts, we rewrite (3.2) (with $\xi = 1$) as

$$\begin{aligned} &\big(\mathbf{E}^n, \mathbf{Y}^j\big) - \mathrm{We}\sum_{i=1}^3 \Big(\mathbf{E}^n, v_i^n \frac{\partial \mathbf{Y}^j}{\partial x_i}\Big) + \frac{1}{n}\big(\mathbf{E}^n, \Delta(\Delta \mathbf{Y}^j)\big) \\ &= 2\epsilon\big(\mathbf{D}(\mathbf{v}^n), \mathbf{Y}^j\big),\ j = 1, \dots, n. \end{aligned} \tag{3.14}$$

Using (3.11) and (3.12), we can pass to the limit $n \to \infty$ in equality (3.14). We obtain

$$\begin{aligned} &(\widetilde{\mathbf{E}}, \mathbf{Y}^j) - \mathrm{We}\sum_{i=1}^3 \Big(\widetilde{\mathbf{E}}, \widetilde{v}_i \frac{\partial \mathbf{Y}^j}{\partial x_i}\Big) \\ &= 2\epsilon\big(\mathbf{D}(\widetilde{\mathbf{v}}), \mathbf{Y}^j\big) \end{aligned} \tag{3.15}$$

for any $j \in \mathbb{N}$. Since $\{\mathbf{Y}^j\}_{j=1}^\infty$ is a basis of the space $\mathbf{H}^2_0(\Omega,\mathbb{R}^{3\times 3}_s)$, equality (3.14) remains valid if we replace $\mathbf{Y}^j$ with an arbitrary vector function $\boldsymbol{\Phi} \in \mathbf{C}^\infty_0(\Omega, M^{3\times 3}_s)$.

Thus, we have proved that the triplet $\big(\widetilde{\mathbf{v}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}\big)$ is a weak solution of problem (2.6)–(2.12).

From estimate (3.10) it follows that

$$\begin{aligned}&\|\widetilde{\mathbf{E}}\|^2_{\mathbf{L}_2(\Omega,M_s^{3\times 3})}+4\epsilon(1-\epsilon)\|\mathbf{D}(\widetilde{\mathbf{v}})\|^2_{\mathbf{L}_2(\Omega,\mathbb{R}_s^{3\times 3})}\\&\le\frac{\epsilon C^2(\|\mathbf{g}\|_{\mathbf{L}_2(\Gamma\setminus\Gamma_0,\mathbb{R}^3)}+\|\mathbf{f}\|_{\mathbf{L}_2(\Omega,\mathbb{R}^3)})^2}{1-\epsilon}.\end{aligned}$$

Arguing as above, we establish that the weak solution set is sequentially weakly closed in the space $\mathbf{X}(\Omega,\mathbb{R}^3)\times\mathbf{L}_2(\Omega,\mathbb{R}_s^{3\times 3})\times\mathbf{L}_2(\Omega,\mathbb{R}_s^{3\times 3})$. □

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