Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 248, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILIZATION OF ODE-SCHRÖDINGER CASCADED SYSTEMS SUBJECT TO BOUNDARY CONTROL MATCHED DISTURBANCE

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Abstract. In this article, we consider the state feedback stabilization of ODE-Schrödinger cascaded systems with the external disturbance. We use the backstepping transformation to handle the unstable part of the ODE, then design a feedback control which is used to cope with the disturbance and stabilize the Schrödinger part. By active disturbance rejection control (ADRC) approach, the disturbance is estimated by a constant high gain estimator, then the feedback control law can be designed. Next, we show that the resulting closed-loop system is practical stable, where the peaking value appears in the initial stage and the stabilized result requires that the derivative of disturbance be uniformly bounded. To avoid the peak phenomenon and to relax the restriction on the disturbance, a time varying high gain estimator is presented and asymptotical stabilization of the corresponding closed-loop system is proved. Finally, the effectiveness of the proposed control is verified by numerical simulations.

1. Introduction

Environmental disturbances (e.g., noise, wave, and wind) and modeling uncertainties (e.g., unknown plant parameters) are often encountered problems in practical engineering systems which reduce the system quality, lead to limited productivity and result in premature fatigue failure. As noted in [10], there is an input disturbance in heat PDE-ODE cascade which causes variations in system dynamic characteristics, and makes systems unstable. To suppress vibrations of the systems, many control approaches have been developed to cope with system uncertainty or disturbance. For instance, the result of control design to the systems with unknown plant parameters by adaptive control method is given in [7], with external disturbance by sliding mode control is presented in [13], and with input disturbance by active disturbance rejection control (ADRC) is investigated in [3]. However, it is noticed that stabilized result of [3] requires that the derivative of disturbance is uniformly bounded. Furthermore, the time varying feedback control design has been recently proposed for the unstable wave system in [2], which relaxes the restriction on the disturbance.

2010 *Mathematics Subject Classification.* 93C20.
Key words and phrases. Cascade systems; disturbance; backstepping ; boundary control; active disturbance rejection control.

Submitted August 13, 2015. Published September 23, 2015.

Actuator appears in many control applications such as electromagnetic coupling [8], chemical engineering [12], and industrial oil-drilling plants [1]. Some practical systems with actuator are modelled by ordinary/partial differential equation (ODE/PDE)-PDE cascade, in which the original system is considered as the ODE/PDE part, while the actuator is regarded as a PDE part. Much attention has been dedicated in the past years to the control of unstable systems with infinite-dimensional actuator dynamics. For example, the diffusion PDE-ODE cascaded system is considered in [8], where the compensating actuator dynamics dominated by first-order hyperbolic PDE systems. The results in [8] are extended to the case of actuator dynamics modelled by heat [6], wave [1] or Schrödinger [9] systems. In all these works, without the disturbances and uncertainties, the predictor feedback control law is designed for the actuator and the systems achieve stabilization. More recently, the feedback control law is designed for the cascaded ODE-heat system with the input disturbance in [10] using sliding mode control and backstepping method.

To the best of our knowledge, the predictor feedback control law designing for ODE-Schrödinger cascades with the external disturbances is still open. The Schrödinger equation is challenging due to its complex state and the fact that all of its eigenvalues are on the imaginary axis [9, 11]. When we take into account the input disturbances, the stabilization of the cascaded Schrödinger-ODE systems is difficult .

In this article, we consider the stabilization of the cascaded ODE-Schrödinger systems with input disturbances:

$$
\begin{gathered}
\dot{X}(t) = AX(t) + Bu(0,t), \quad t > 0, \\
u_t(x,t) = -ju_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\
u_x(0,t) = 0, \\
u_x(1,t) = U(t) + d(t),
\end{gathered} \tag{1.1}
$$

where $X \in \mathbb{C}^{n\times 1}$ and u are the states of ODE and PDE respectively, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$; $U(t)$ is the control input; $d(t)$ is the input disturbance at the control end; X_0 and $u_0(x)$ are the initial value of ODE and PDE respectively. The whole system is depicted in Figure 1.

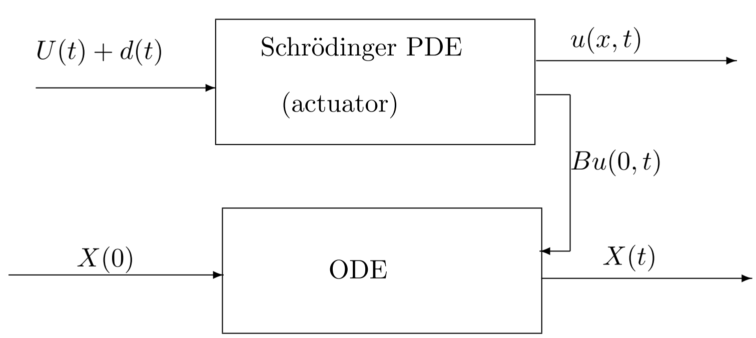


FIGURE 1. Block diagram for the coupled ODE-PDE system

Our aim is to design a feedback control law such that the resulting closed-loop cascaded system is asymptotic stable. First, our design method is based on two-step invertible backstepping transformations that deal with the unstable part of the system, so that the feedback control law is only used to deal with the disturbance and stabilize the PDE part. Second, using ADRC approach, the disturbance is estimated by a constant high gain estimator and time varying high gain estimator respectively; the feedback controllers are designed for the system. Furthermore, we show that the solution of the resulting closed-loop cascaded system is practical stable and asymptotic stable respectively. Finally, the numerical simulation results are provided to illustrate the effectiveness of the proposed design method.

We proceed as follows. In Section 2, the two-step backstepping design is developed using the invertible Volterra integral transformation. In Section 3, we consider the well-posedness of the system obtained from original system (1.1) by designing a feedback control law. In Section 4, we design a constant high gain estimator by the ADRC approach and show the practical stability of the resulting closed-loop system. In Section 5, we design a time varying disturbance estimator and obtain the asymptotic stability of the corresponding closed-loop system. In Section 6, the numerical simulation results are provided to show the effectiveness of the proposed method.

2. Backstepping design

We first introduce the following transformation to stabilize the ODE part [9],

$$\begin{gathered} X(t) = X(t), \\ w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t), \end{gathered} \tag{2.1}$$

where

$$\begin{gathered} q(x,y) = \int_0^{x-y} j\gamma(\sigma)Bd\sigma, \\ \gamma(x) = \begin{bmatrix} K & 0 \end{bmatrix} \exp\Big(\begin{bmatrix} 0 & jA \\ I & 0 \end{bmatrix} x \Big) \begin{bmatrix} I \\ 0 \end{bmatrix}. \end{gathered}$$

The transformation changes system (1.1) into

$$\begin{gathered} \dot X(t) = (A+BK)X(t) + Bw(0,t), \\ w_t(x,t) = -jw_{xx}(x,t), \\ w_x(0,t) = 0, \\ w_x(1,t) = U(t) + d(t) - \int_0^1 q_x(1,y)u(y,t)dy - \gamma'(1)X(t), \end{gathered} \tag{2.2}$$

where K is chosen such that $A+BK$ is Hurwitz. By (2.2), if the PDE part is stable, then the ODE part is also stable.

The transformation (2.1) is invertible, and the inverse transformation $w \mapsto u$ is postulated as follows [9]:

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy + \psi(x)X(t), \tag{2.3}$$

where

$$
\begin{gathered}
l(x,y) = \int_0^{x-y} j\psi(\xi)Bd\xi, \\
\psi(x) = \begin{bmatrix} K & 0 \end{bmatrix} \exp\Big(\begin{bmatrix} 0 & j(A+BK) \\ I & 0 \end{bmatrix} x \Big) \begin{bmatrix} I \\ 0 \end{bmatrix}.
\end{gathered}
$$

For increasing the decay rate, we define the transformation [8]

$$
\begin{gathered}
X(t) = X(t), \\
z(x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)dy,
\end{gathered} \tag{2.4}
$$

where

$$
k(x,y) = -cjx\frac{I_1(\sqrt{cj(x^2-y^2)})}{\sqrt{cj(x^2-y^2)}}, \quad 0 \le y \le x \le 1, \tag{2.5}
$$

and I_1 is the modified Bessel function. Transformation (2.4) changes system (2.2) to the system

$$
\begin{gathered}
\dot{X}(t) = (A+BK)X(t) + Bz(0,t), \\
z_t(x,t) = -jz_{xx}(x,t) - cz(x,t), \\
z_x(0,t) = 0, \\
z_x(1,t) = U(t) + d(t) - \int_0^1 q_x(1,y)u(y,t)dy - \gamma'(1)X(t) \\
- k(1,1)w(1,t) - \int_0^1 k_x(1,y)w(y,t)dy.
\end{gathered} \tag{2.6}
$$

The inverse of transformation (2.4) can be found as follows:

$$
w(x,t) = z(x,t) + \int_0^x p(x,y)z(y,t)dy, \tag{2.7}
$$

where

$$
p(x,y) = -cjx\frac{J_1(\sqrt{cj(x^2-y^2)})}{\sqrt{cj(x^2-y^2)}}, \quad 0 \le y \le x \le 1, \tag{2.8}
$$

where J_1 is a Bessel function.

Therefore, under the two transformations (2.1) and (2.4), systems (1.1) and (2.6) are equivalent. So we only need consider system (2.6). Denote

$$
\begin{aligned}
U_0(t) = U(t) - \int_0^1 q_x(1,y)u(y,t)dy - \gamma'(1)X(t) - k(1,1)w(1,t) \\
- \int_0^1 k_x(1,y)w(y,t)dy.
\end{aligned} \tag{2.9}
$$

By (2.9), the system (2.6) can be rewritten as follows

$$
\begin{gathered}
\dot{X}(t) = (A+BK)X(t) + Bz(0,t), \\
z_t(x,t) = -jz_{xx}(x,t) - cz(x,t), \\
z_x(0,t) = 0, \\
z_x(1,t) = U_0(t) + d(t).
\end{gathered} \tag{2.10}
$$

3. Well-posedness of (2.10)

Let us consider systems (1.1) and (2.6) in the state space $\mathcal{H} = \mathbb{C}^n \times L^2(0,1)$, equipped with the usual inner product:

$$\langle (X,f)^\top, (Y,g)^\top \rangle_{\mathcal{H}} = X^\top \overline{Y} + \int_0^1 f(x)\overline{g(x)}dx, \quad \forall (X,f)^\top, (Y,g)^\top \in \mathcal{H}. \tag{3.1}$$

Define the system operator $\mathcal{A}_0 : D(\mathcal{A}_0)(\subset \mathcal{H}) \to \mathcal{H}$ as

$$D(\mathcal{A}_0) = \big\{(X,f) \in \mathbb{C}^n \times H^2(0,1) | f'(0) = 0, f'(1) = 0\big\}, \tag{3.2}$$

and for any $Z = (X,f)^\top \in D(\mathcal{A}_0)$,

$$\mathcal{A}_0 Z = ((A+BK)X + Bf(0), -jf'' - cf). \tag{3.3}$$

We compute $\mathcal{A}_0^*$, the adjoint of $\mathcal{A}_0$, to obtain

$$\begin{gathered} \mathcal{A}_0^*(Y,g) = (\overline{(A+BK)^\top} Y, jg'' - cg), \quad \forall (Y,g) \in D(\mathcal{A}_0^*), \\ D(\mathcal{A}_0^*) = \{(Y,g) \in \mathbb{C}^n \times H^2(0,1) : g'(0) = j\overline{B^\top} Y,\ g'(1) = 0\}. \end{gathered} \tag{3.4}$$

Define the unbounded operator $\mathcal{B}_0$ by

$$\mathcal{B}_0 = (0, -\delta(x-1))^\top. \tag{3.5}$$

Then system (2.10) can be written as an abstract evolution equation in $\mathcal{H}$,

$$\frac{d}{dt} Z(t) = \mathcal{A}_0 Z(t) + j\mathcal{B}_0 (U_0(t) + d(t)), \tag{3.6}$$

where $Z(t) = (X(t), z(\cdot,t))$.

Lemma 3.1. *Let $\mathcal{A}_0$ be given by (3.2) and (3.3). Then $\mathcal{A}_0^{-1}$ exists and is compact on $\mathcal{H}$ and hence $\sigma(\mathcal{A}_0)$, the spectrum of $\mathcal{A}_0$, consists of isolated eigenvalues of finitely algebraic multiplicity only.*

Proof. For any given $(X_1, z_1) \in \mathcal{H}$, solve

$$\mathcal{A}_0(X,z) = ((A+BK)X + Bz(0), -jz''(x) - cz(x)) = (X_1, z_1). \tag{3.7}$$

We obtain

$$\begin{gathered} (A+BK)X + Bz(0) = X_1. \\ -jz''(x) - cz(x) = z_1(x), \\ z'(0) = 0, z'(1) = 0, \end{gathered} \tag{3.8}$$

with solution

$$\begin{gathered} X = (A+BK)^{-1}(X_1 - Bz(0)), \\ z(x) = c_0\left(e^{c_\lambda x} + e^{-c_\lambda x}\right) - \frac{1}{2c_\lambda}\int_0^x \left(e^{c_\lambda(s-x)} - e^{-c_\lambda(s-x)}\right) jz_1(s)ds, \\ c_0 = -\frac{1}{2c_\lambda\left(e^{c_\lambda} - e^{-c_\lambda}\right)}\int_0^1 \left(e^{c_\lambda(s-1)} + e^{-c_\lambda(s-1)}\right) jz_1(s)ds, \\ c_\lambda = \sqrt{cj}. \end{gathered} \tag{3.9}$$

Hence, we have the unique $(X,z) \in D(\mathcal{A}_0)$. Then, $\mathcal{A}_0^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathcal{A}_0)$ consists of isolated eigenvalues of finite algebraic multiplicity. ∎

Now we consider the eigenvalue problem of $\mathcal{A}_0$. Let $\mathcal{A}_0 Y = \lambda Y$, where $Y = (X, z)$, then we have

$$\begin{gathered} (A+BK)X + Bz(0) = \lambda X, \\ -jz'' - cz = \lambda z(x), \\ z'(0) = z'(1) = 0. \end{gathered} \tag{3.10}$$

Lemma 3.2. *Let $\mathcal{A}_0$ be given by (3.2) and (3.3), let $\lambda_k^0, k = 1, 2, \dots, n$ be the simple eigenvalue of $A + BK$ with the corresponding eigenvector X_k, and assume that*

$$\lambda_k^0 \notin \{\lambda_m^p, m \in \mathbb{N}\}, \tag{3.11}$$

where

$$\lambda_m^p = -c + m^2\pi^2 j. \tag{3.12}$$

Then the eigenvalues of $\mathcal{A}_0$ are

$$\{\lambda_k^0,\ k = 1, 2, \dots, n\} \cup \{\lambda_m^p,\ m = 0, 1, 2, \dots\} \tag{3.13}$$

and the eigenfunctions corresponding to λ_k^0 and λ_m^p are respectively

$$W_k = (X_k, 0), \quad k = 1, 2, \dots, n; \tag{3.14}$$

$$W_m(x) = ([\lambda_m^p I - (A+BK)]^{-1}B, z_m(x)), \quad m \in \mathbb{N}; \tag{3.15}$$

where

$$z_m(x) = \cos m\pi x, \quad m \in \mathbb{N}. \tag{3.16}$$

Proof. Since $A + BK$ is Hurwitz, we have

$$\operatorname{Re} \lambda_k^0 < 0, \quad k = 1, 2, \dots, n. \tag{3.17}$$

A simple computation shows that the eigenvalue problem

$$\begin{gathered} -jz''(x) - cz = \lambda z(x), \\ z'(0) = 0, z'(1) = 0, \end{gathered} \tag{3.18}$$

has the nontrivial solutions

$$(\lambda_m^p, z_m(x)), \quad m \in \mathbb{N}, \tag{3.19}$$

where λ_m^p and $z_m(x)$ are given by (3.12) and (3.16) respectively.

Next, we look for the eigenvalues for (3.10). Let $\lambda = \lambda_k^0,\ k = 1, 2, \dots, n$, since $B \neq 0$ and $(A+BK)X_k + Bz(0) = \lambda_k^0 X_k$, we have $z(0) \equiv 0$. Moreover,

$$\begin{gathered} -jz'' - cz = \lambda_k^0 z, \\ z(0) = z'(0) = z'(1) = 0, \end{gathered} \tag{3.20}$$

only has trivial solutions. So we obtain that $\lambda_k^0, k = 1, 2, \dots, n$ are the eigenvalues of (3.10) and have the corresponding eigenfunctions $(X_k, 0)$, as (3.14).

On the other hand, when $\lambda = \lambda_m^p$, $(\lambda_m^p, z_m(x))$ satisfies the second and third equations of (3.10) and $z_m(0) = (-1)^m \neq 0$. By the first equation of (3.10), we have

$$X_m^p = [\lambda_m^p I - (A+BK)]^{-1}B.$$

So $\lambda_m^p, m \in \mathbb{N}$ is the eigenvalue of (3.10) and has the corresponding eigenfunction

$$([\lambda_m^p I - (A+BK)]^{-1}B, \cos(m\pi x)).$$

The proof is complete. □

Theorem 3.3. *Let $\mathcal{A}_0$ be given by (3.2) and (3.3), let λ_k^0 be the simple eigenvalue of $A+BK$ with the corresponding eigenvector X_k. Then, there is a sequence of eigenfunctions of $\mathcal{A}_0$ which forms a Riesz basis for $\mathcal{H}$. Moreover, the following conclusions hold:*

(1) *$\mathcal{A}_0$ generates a C_0-semigroup $e^{\mathcal{A}_0 t}$ on $\mathcal{H}$.*
(2) *The spectrum-determined growth condition $\omega(\mathcal{A}_0)=s(\mathcal{A}_0)$ holds true for $e^{\mathcal{A}_0}t$, where $\omega(\mathcal{A}_0)=\lim_{t\to\infty}\|e^{\mathcal{A}_0 t}\|/t$ is the growth bound of $e^{\mathcal{A}_0}t$, and $s(\mathcal{A}_0)=\sup\{\operatorname{Re}\lambda|\lambda\in\sigma(\mathcal{A}_0)\}$ is the spectral bound of $\mathcal{A}_0$.*
(3) *The C_0-semigroup $e^{\mathcal{A}_0}t$ is exponentially stable in the sense $\|e^{\mathcal{A}_0 t}\|\le M_1e^{-c_1t}$, where $M_1>0$ and c_1 is an arbitrary pre-designed decay rate.*

Proof. It is noted that $\{X_k, k=1,2,\dots,n\}$ is an orthogonal basis in $\mathbb{C}^n$ and $\{z_m(x), m\in\mathbb{N}\}$ given by (3.16) forms an orthogonal basis in $L^2(0,1)$. We have

$$\{F_k, F_m(x): k=1,2,\dots,n, m\in\mathbb{N}\},$$

which forms an orthogonal basis in $\mathcal{H}$ with $F_k=(X_k,0)$ and $F_m(x)=(0,z_m(x))$. It follows from (3.14), (3.15) that

$$\sum_{k=1}^n\|W_k-F_k\|^2+\sum_{m=0}^\infty\|W_m(x)-F_m(x)\|^2=\sum_{m=0}^\infty\|\left(\lambda_m^pI-(A+BK)\right)^{-1}B\|_{\mathbb{C}^n}^2, \tag{3.21}$$

where $\|\cdot\|_{\mathbb{C}^n}$ denotes the norm in $\mathbb{C}^n$. A simple computation gives

$$\begin{aligned}
&\|[\lambda_m^pI-(A+BK)]^{-1}B\|_{\mathbb{C}^n}^2\\
&=\overline{([\lambda_m^pI-(A+BK)]^{-1}B)^\top}([\lambda_m^pI-(A+BK)]^{-1}B)\\
&=\overline{B^\top([\lambda_m^pI-(A+BK)]^{-1})^\top}([\lambda_m^pI-(A+BK)]^{-1})B\\
&=\frac{1}{|\lambda_m^p|^2}\overline{B^\top}\overline{\Big(\big[I-\frac{1}{\lambda_m^p}(A+BK)\big]^{-1}\Big)^\top}\big[I-\frac{1}{\lambda_m^p}(A+BK)\big]^{-1}B.
\end{aligned}$$

It follows from (3.13) that when $m\to\infty$, $\lambda_m^p\to-\infty$, there is a positive number N such that for $m>N$,

$$\big[I-\frac{1}{\lambda_m^p}(A+BK)\big]^{-1}=I+\mathcal{O}\Big(\frac{1}{|\lambda_m^p|}\Big),\quad m>N.$$

Thus

$$\|[\lambda_m^pI-(A+BK)]^{-1}B\|_{\mathbb{C}^n}^2=\frac{\|B\|_{\mathbb{C}^n}^2}{|\lambda_m^p|^2}\Big(1+\mathcal{O}\Big(\frac{1}{|\lambda_m^p|}\Big)\Big).$$

Hence, it follows from (3.21) that

$$\begin{aligned}
&\sum_{k=1}^n\|W_k-F_k\|^2+\sum_{m=0}^\infty\|W_m(x)-F_m(x)\|^2\\
&=\sum_{m=0}^\infty\|[\lambda_m^pI-(A+BK)]^{-1}B\|_{\mathbb{C}^n}^2\\
&=\sum_{m=0}^N\|[\lambda_m^pI-(A+BK)]^{-1}B\|_{\mathbb{C}^n}^2+\sum_{m=N+1}^\infty\frac{\|B\|_{\mathbb{C}^n}^2}{|\lambda_m^p|^2}\Big(1+\mathcal{O}\Big(\frac{1}{|\lambda_m^p|}\Big)\Big)<\infty.
\end{aligned}$$

Therefore, by Bari's theorem,

$$\{W_k, W_m(x): k=1,2,\dots,n, m=0,1,\dots\}$$

forms a Riesz basis for $\mathcal{H}$. Moreover, by Lemma 3.1, $\mathcal{A}_0$ generates a C_0-semigroup $e^{\mathcal{A}_0 t}$ on $\mathcal{H}$ and the spectrum-determined growth condition $\omega(\mathcal{A}_0) = s(\mathcal{A}_0)$ holds true for $e^{\mathcal{A}_0 t}$. Finally, by the eigenvalues of $\mathcal{A}_0$ given by (3.13), there is a positive constant $M_1 > 0$ and c_1 such that

$$\|e^{\mathcal{A}_0 t}\| \le M_1 e^{-c_1 t}, \qquad \forall t \ge 0. \tag{3.22}$$

The proof is complete. $\square$

Lemma 3.4. *Let $\mathcal{A}_0$, $\mathcal{B}_0$ be defined by* (3.2)-(3.3) *and* (3.5) *respectively. Then $\mathcal{B}_0$ is admissible to the semigroup generated by $\mathcal{A}_0$.*

Proof. Now we show that $\mathcal{B}_0$ is admissible for $e^{\mathcal{A}_0 t}$, or equivalently, $\mathcal{B}_0^*$ is admissible for $e^{\mathcal{A}_0^* t}$. To this end, we consider the dual system of (3.6),

$$\begin{gathered}
\dot{X}^*(t) = \overline{(A+BK)^\top} X^*(t), \\
z_t^*(x,t) = j z_{xx}^*(x,t) - c z^*(x,t), \\
z_x^*(0,t) = \overline{B^\top} j X^*, \\
z_x^*(1,t) = 0, \\
y(t) = \mathcal{B}_0^* \begin{pmatrix} X^* \\ z^*(x,t) \end{pmatrix} = -j z^*(1,t).
\end{gathered} \tag{3.23}$$

From Lemma 3.2, we claim that $\bar{\lambda}_j^0$, $j = 1, 2, \dots, n$ is the simple eigenvalue of $\overline{(A+BK)^\top}$ with the corresponding eigenvector X_j^*. In a similar way as Lemma 3.2, we can find the spectrum $\sigma(\mathcal{A}_0^*)$ of the adjoint operator $\mathcal{A}_0^*$,

$$\sigma(\mathcal{A}_0^*) = \{\bar{\lambda}_j^0 : j = 1, 2, \dots, n\} \cup \{\bar{\lambda}_m^p : m = 0, 1, 2, \dots\}, \tag{3.24}$$

and the eigenvectors corresponding to $\bar{\lambda}_j^0$ and $\bar{\lambda}_m^p$ are respectively

$$Z_j^* = (X_j^*, z_j^*),\ j = 1, 2, \dots, n, \quad Z_m^*(x) = (0, \cos(m\pi x)),\ m \in \mathbb{N}, \tag{3.25}$$

where

$$z_j^* = \frac{\overline{B^\top} i X_j^*}{m(e^{-m} - 1)} \big\{ e^{m(1+x)} + e^{-m(1-x)} - e^{mx} + e^{m(1-x)} \big\}, \tag{3.26}$$

$m = i\sqrt{i(c + \lambda_j^0)}$. Moreover, $\{Z_j, Z_m(x), j = 1, 2, \dots, n, m \in \mathbb{N}\}$ forms a Riesz basis for $C^n \times L^2(0,1)$ and $\mathcal{A}^*$ generates a C_0-semigroup on $\mathcal{H}$. Hence, for any $Z^*(\cdot, 0) \in \mathcal{H}$, we suppose that

$$Z^*(x,0) = \sum_{k=1}^n a_k Z_k + \sum_{m=0}^\infty b_m Z_m(x).$$

Then the solution of (3.23) is

$$[X^*, z^*] = Z^*(x,t) = e^{\mathcal{A}_0^* t} Z^*(x,0) = \sum_{k=1}^n a_k e^{\lambda_k^0 t} Z_k + \sum_{m=0}^\infty b_m e^{\lambda_m^p t} Z_m(x),$$

where

$$X^* = \sum_{k=1}^n a_k e^{\lambda_k^0 t} X_k^*, \quad z^* = \sum_{k=1}^n a_k e^{\lambda_k^0 t} z_k^* + \sum_{m=0}^\infty b_m e^{\lambda_m^p t} Z_m^*(x),$$

hence

$$y(t) = \sum_{k=1}^n a_k e^{\lambda_k^0 t} z_k^* + \sum_{m=0}^\infty b_m e^{\lambda_m^p t}.$$

By Ingham's inequality [5, Theorems 4.3], there exists a $T > 0$, such that

$$\int_0^T |y(t)|^2 dt \leqslant C_{T_1} \sum_{k=1}^{n} |a_k z_k^*|^2 + C_{T_2} \sum_{m=0}^{\infty} |b_m|^2 \leqslant D_T \|Z^*(\cdot,0)\|^2, \tag{3.27}$$

for some constants C_{T_1}, C_{T_2}, D_T that depend on T.

On the other hand, for any given $(X,f)^\top \in \mathcal{H}$, we solve that

$$\mathcal{A}_{\hat{\omega}}^*(Y,g)^\top = (X,f)^\top.$$

Combine the definition of $\mathcal{A}_{\hat{\omega}}^*$ with its boundary condition to obtain

$$\begin{gathered} \overline{(A+BK)^\top} Y = X, \quad jg'' - cg = f, \\ g'(0) = jB^T Y, \quad g'(1) = 0. \end{gathered}$$

A direct computation gives the solution of the above equations

$$\begin{gathered} Y = [(A+BK)^T]^{-1} X, \\ g(x) = c_1 e^{c_\lambda x} - c_2 e^{-c_\lambda x} + \frac{1}{2c_\lambda} \int_0^x \left(e^{-c_\lambda(s-x)} - e^{c_\lambda(s-x)}\right) jf(s) ds, \\ c_1 = \frac{1}{c_\lambda(e^{-c_\lambda} - e^{c_\lambda})} \left\{ \frac{1}{2} \int_0^1 \left(e^{-c_\lambda(s-1)} - e^{c_\lambda(s-1)}\right) jf(s) ds + jB^T Y e^{c_\lambda} \right\}, \\ c_2 = \frac{1}{c_\lambda(e^{-c_\lambda} - e^{c_\lambda})} \left\{ \frac{1}{2} \int_0^1 \left(e^{-c_\lambda(s-1)} - e^{c_\lambda(s-1)}\right) jf(s) ds + jB^T Y e^{-c_\lambda} \right\}, \\ c_\lambda = j\sqrt{cj}. \end{gathered}$$

We obtain

$$\mathcal{B}_0^*(\mathcal{A}_{\hat{\omega}}^*)^{-1}(X,f)^\top = \mathcal{B}_0^*(Y,g)^\top = -g(1),$$

which is bounded from $\mathcal{H}$ to $\mathbb{C}$. This shows that $\mathcal{B}_0^*$ is admissible for $e^{\mathcal{A}_0^* t}$ and so is $\mathcal{B}_0$ for $e^{\mathcal{A}_0 t}$. The proof is complete. □

Proposition 3.5. *The operator $\mathcal{A}_0$ defined by* (3.2) *and* (3.3) *generates an exponential stable C_0-semigroup on $\mathcal{H}$, and the control operator $\mathcal{B}_0$ is admissible to the semigroup $e^{\mathcal{A}_0 t}$. Hence, for any $Z(x,0) \in \mathcal{H}$, there exists a unique (weak) solution to* (3.6)*, which can be written as*

$$Z(\cdot,t) = e^{\mathcal{A}_0 t} Z(\cdot,0) + j \int_0^t e^{\mathcal{A}_0(t-s)} \mathcal{B}_0[U_0(s) + d(s)] ds, \tag{3.28}$$

for all $U_0(s) + d(s) \in L^2_{loc}(0,\infty)$; that is,

$$\frac{d}{dt}\langle Z(\cdot,t), \rho\rangle = \langle Z(\cdot,t), \mathcal{A}_0^*\rho\rangle + j[U_0(t) + d(t)]\mathcal{B}_0^*\rho, \ \forall\, \rho \in D(\mathcal{A}_0^*). \tag{3.29}$$

4. Constant high gain estimator based feedback

In this section, we propose a state disturbance estimator with constant high gain based on the ADRC approach. It is supposed that d and its derivative are uniformly bounded, i.e., $|d(t)| \le M_1$ and $|\dot{d}(t)| \le M_2$ for some $M_1, M_2 > 0$ and all $t \ge 0$. Taking specially $\rho(x) = (0, 2x^3 - 3x^2)^\top$ in (3.29), we obtain

$$\dot{y}_0(t) = U_0(t) + d(t) + y_1(t), \tag{4.1}$$

where

$$y_0(t) = -j\int_0^1 (2x^3 - 3x^2)z(x,t)dx, \tag{4.2}$$

$$y_1(t) = \int_0^1 (-12x + 6 + 2cjx^3 - 3cjx^2)z(x,t)dx. \tag{4.3}$$

Then we are able to design an extended state observer to estimate both $y_0(t)$ and $d(t)$ [4] as follows:

$$\begin{gathered}\dot{\hat{y}}_\epsilon(t) = U_0(t) + \hat{d}_\epsilon(t) + y_1(t) - \frac{1}{\epsilon}(\hat{y}_\epsilon - y_0),\\ \dot{\hat{d}}_\epsilon(t) = -\frac{1}{\varepsilon^2}\{\hat{y}_\epsilon - y_0\},\end{gathered} \tag{4.4}$$

where ϵ is the tuning small parameter and $\hat{d}_\epsilon(t)$ is regarded as approximation of $d(t)$. We have the following result.

Lemma 4.1. *Let $(\hat{y}_\epsilon, \hat{d}_\epsilon)$ be the solution of (4.4) and y_0 be defined as (4.2). The followings hold.*

(1) *For any $\alpha > 0$,*

$$|\hat{y}_\epsilon(t) - y_0(t)| + |\hat{d}_\epsilon(t) - d(t)| \to 0, \quad \text{as } \varepsilon \to 0, \text{ uniformly } t \in [\alpha, \infty). \tag{4.5}$$

(2) *For any $\alpha > 0$,*

$$\int_0^\alpha |\hat{y}_\epsilon(t) - y_0(t)| + |\hat{d}_\epsilon(t) - d(t)|dt \text{ is uniformly bounded as } \epsilon \to 0. \tag{4.6}$$

Proof. Suppose the errors

$$\tilde{y}_\epsilon(t) = \hat{y}_\epsilon(t) - y_0(t), \;\; \tilde{d}_\epsilon(t) = -\hat{d}_\epsilon(t) + d(t), \tag{4.7}$$

satisfy

$$\begin{gathered}\dot{\tilde{y}}_\epsilon(t) = -\tilde{d}_\epsilon(t) - \frac{1}{\varepsilon}\tilde{y}_\epsilon(t),\\ \dot{\tilde{d}}_\epsilon(t) = \frac{1}{\varepsilon^2}\tilde{y}_\epsilon(t) + \dot{d}(t).\end{gathered} \tag{4.8}$$

Then (4.8) can be written as an evolution equation:

$$\frac{d}{ds}\begin{pmatrix}\tilde{y}_\epsilon\\ \tilde{d}_\epsilon\end{pmatrix} = A_1\begin{pmatrix}\tilde{y}_\epsilon\\ \tilde{d}_\epsilon\end{pmatrix} + D_1(s), \tag{4.9}$$

where

$$A_1 = \begin{pmatrix}-\frac{1}{\epsilon} & -1\\ \frac{1}{\epsilon^2} & 0\end{pmatrix}, \quad D_1(s) = \begin{pmatrix}0\\ 1\end{pmatrix}. \tag{4.10}$$

A simple exercise shows that the eigenvalues of the matrix A_1 are

$$\lambda_1 = -\frac{1}{2\epsilon} + \frac{\sqrt{3}}{2}j, \quad \lambda_2 = -\frac{1}{2\epsilon} - \frac{\sqrt{3}}{2}j,$$

which satisfy

$$\begin{gathered}e^{A_1 t} = \begin{pmatrix}\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 t} - \frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 t} & \frac{\lambda_1\lambda_2}{C_\varepsilon(\lambda_2-\lambda_1)}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)\\ \frac{C_\varepsilon}{\lambda_2-\lambda_1}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) & -\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2 t}\end{pmatrix},\\ e^{A_1 t}D_1 = -\begin{pmatrix}\frac{\lambda_1\lambda_2}{C_\varepsilon(\lambda_2-\lambda_1)}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)\\ -\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2 t}\end{pmatrix}, \quad C_\varepsilon = \frac{1}{\varepsilon^2}.\end{gathered} \tag{4.11}$$

The above three equations imply that there exist constants $\hat{L}$ and $\hat{M}$ such that

$$\|e^{A_1 t}\| \leq \frac{\hat{L}}{\epsilon} e^{-\frac{1}{2\epsilon}t}, \quad \|e^{A_1 t} D_1\| \leq \hat{M} e^{-\frac{1}{2\epsilon}t}. \tag{4.12}$$

Since

$$\begin{pmatrix} \tilde{y}_\epsilon(t) \\ \tilde{d}_\epsilon(t) \end{pmatrix} = e^{A_1 t} \begin{pmatrix} \tilde{y}_\epsilon(0) \\ \tilde{d}_\epsilon(0) \end{pmatrix} + \int_0^t e^{A_1(t-s)} D_1 \dot{d}(s) ds, \tag{4.13}$$

the first term above can be arbitrarily small as $t \to \infty$ by the exponential stability of $e^{A_1 t}$, and the second term can also be arbitrarily small as $\epsilon \to 0$ due to boundedness of $\dot{d}$ and the expression of $e^{A_1 t} D_1$. As a result, the solution $(\tilde{y}_\epsilon, \tilde{d}_\epsilon)$ of (4.8) satisfies

$$(\tilde{y}_\epsilon, \tilde{d}_\epsilon) \to 0, \quad \text{as } t \to \infty, \epsilon \to 0. \tag{4.14}$$

By estimating (4.12), we can obtain (4.6). The proof is completed. □

By Lemma 4.1, we deduce that the design of ESO (4.4) is based on the arbitrary decay rate of $\|e^{A_1 t}\|$ and the special structure of D_1. In this way, the ADRC is not well adapted to PDEs because it is hard that a PDE system has the arbitrary decay rate. That also explains why $\dot{d}$ must be uniformly bounded.

In (4.6), it is worth pointing out that $\int_0^\alpha |\hat{d}_\epsilon(t) - d(t)| dt$ is uniformly bounded in ϵ for any fixed $\alpha > 0$, while $\int_0^\alpha |\hat{d}_\epsilon(t) - d(t)|^2 dt$ is unbounded in ϵ. Then we could find that the L^2 unboundedness of $\int_0^\alpha |\hat{d}_\epsilon(t) - d(t)|^2 dt$ brings trouble to PDEs (see (4.23)). To avoid this phenomenon, the feedback controller for system (2.10) is proposed as follows:

$$U_0(t) = -\operatorname{sat}(\hat{d}_\epsilon(t)), \tag{4.15}$$

where

$$\operatorname{sat}(x) = \begin{cases} M_1, & x \geq M_1 + 1, \\ -M_1, & x \leq -M_1 - 1, \\ x, & x \in (-M_1 - 1, M_1 + 1). \end{cases} \tag{4.16}$$

Combining $|d(t)| \leq M_1$ with (4.7), for any given $\alpha > 0$, when ϵ is sufficiently small, we obtain $U_0(t) = -\hat{d}_\epsilon(t)$, for all $t \in [\alpha, \infty)$.

Under the feedback (4.15), system (2.10) becomes

$$\begin{gathered} \dot{X}(t) = (A + BK)X(t) + Bz(0,t), \\ z_t(x,t) = -jz_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -\operatorname{sat}(\hat{d}_\epsilon(t)) + d(t), \\ \dot{\hat{y}}_\epsilon(t) = -\operatorname{sat}(\hat{d}_\epsilon(t)) + \hat{d}_\epsilon(t) + y_1(t) - \frac{1}{\epsilon}(\hat{y}_\epsilon - y_0), \\ \dot{\hat{d}}_\epsilon(t) = -\frac{1}{\varepsilon^2}\{\hat{y}_\epsilon - y_0\}. \end{gathered} \tag{4.17}$$

Using the error dynamics defined in (4.7), we see that (4.17) is equivalent to:

$$\begin{aligned}
\dot{X}(t) &= (A+BK)X(t) + Bz(0,t),\\
z_t(x,t) &= -jz_{xx}(x,t) - cz(x,t),\\
z_x(0,t) &= 0,\\
z_x(1,t) &= \operatorname{sat}(\tilde{d}_\epsilon(t) - d(t)) + d(t),\\
\dot{\tilde{y}}_\epsilon(t) &= -\tilde{d}_\epsilon(t) - \frac{1}{\varepsilon}\tilde{y}_\epsilon(t),\\
\dot{\tilde{d}}_\epsilon(t) &= \frac{1}{\varepsilon^2}\tilde{y}_\epsilon(t) + \dot{d}(t).
\end{aligned} \tag{4.18}$$

By (4.7), it is seen that $(\tilde{y}_\epsilon(t), \tilde{d}_\epsilon(t))$ is independent of the "(X,z)-part", which can be arbitrarily small as $t \to \infty, \varepsilon \to 0$ by Lemma 4.1. Hence, we only need to consider the "(X,z)-part" which is rewritten as

$$\begin{aligned}
\dot{X}(t) &= (A+BK)X(t) + Bz(0,t),\\
z_t(x,t) &= -jz_{xx}(x,t) - cz(x,t),\\
z_x(0,t) &= 0,\\
z_x(1,t) &= \operatorname{sat}(\tilde{d}_\epsilon(t) + d(t)) + d(t) \triangleq \tilde{d}(t).
\end{aligned} \tag{4.19}$$

System (4.19) can be written as the following abstract evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}Z(t) = \mathcal{A}_0 Z(t) + j\mathcal{B}_0\tilde{d}(t), \tag{4.20}$$

where $Z(t) = (X(t), z(\cdot,t))$, $\mathcal{A}_0$ and $\mathcal{B}_0$ are given respectively by (3.3) and (3.5).

Lemma 4.2. *Assume that $|d(t)| \leqslant M_1$ and $\dot{d}(t)$ is measurable, $|\dot{d}(t)| \leq M_2$ for all $t \geq 0$. Then for any initial value $(X(0), z(\cdot,0)) \in \mathcal{H}$, the closed-loop system* (4.19) *admits a unique solution $(X,z)^\top \in C(0,\infty;\mathcal{H})$, and*

$$\lim_{t\to\infty,\,\varepsilon\to 0} \|(X(t), z(\cdot,t), \hat{y}_\varepsilon(t), \hat{d}_\varepsilon(t) - d(t))\|_{\mathcal{H}\times\mathbb{C}^2} = 0.$$

Proof. By Theorem 3.3 and Lemma 3.4, for any initial value $(X(0), z(\cdot,0)) \in \mathcal{H}$, there exists a unique (weak) solution $(X,z) \in C(0,\infty;\mathcal{H})$ which can be written as

$$Z(\cdot,t) = e^{\mathcal{A}_0 t}Z(\cdot,0) + j\int_0^t e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\tilde{d}(t)ds. \tag{4.21}$$

By (4.14), for any given $\varepsilon_0 > 0$, there exist $t_0 > 0$ and $\varepsilon_1 > 0$ such that

$$|\tilde{d}(t)| = |-\operatorname{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t)| < \varepsilon_0,$$

for all $t > t_0$ and $0 < \varepsilon < \varepsilon_1$. We rewrite the solution of (4.21),

$$\begin{aligned}
Z(\cdot,t) = e^{\mathcal{A}_0 t}Z(\cdot,0) &+ je^{\mathcal{A}_0(t-t_0)}\int_0^{t_0} e^{\mathcal{A}_0(t_0-s)}\mathcal{B}_0\tilde{d}(s)ds\\
&+ j\int_{t_0}^t e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\tilde{d}(s)ds.
\end{aligned} \tag{4.22}$$

According to the admissibility of $\mathcal{B}_0$, we obtain

$$\begin{aligned}
\|\int_0^{t_0} e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\tilde{d}(s)ds\|_{\mathcal{H}}^2 &\leqslant C_{t_0}\|\operatorname{sat}(\tilde{d}_\varepsilon + d) + d\|_{L^2(0,t_0)}^2\\
&\leqslant t_0 C_{t_0}(2M_1+1)^2,
\end{aligned} \tag{4.23}$$

where the constant C_{t_0} is independent of $\tilde{d}_\varepsilon$ and d. Since $e^{\mathcal{A}_0 t}$ is exponentially stable and $\mathcal{B}_0$ is admissible to $e^{\mathcal{A}_0 t}$ with L^2_{loc} control, $\mathcal{B}_0$ is admissible to $e^{\mathcal{A}_0 t}$ with L^∞_{loc} control. It follows from [14, Proposition 2.5] that

$$\begin{aligned}\|\int_{t_0}^t e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\tilde{d}(s)ds\| &= \|\int_0^t e^{\mathcal{A}_0(t-s)}\mathcal{B}_0(0 \underset{t_0}{\diamondsuit} \tilde{d}(s)ds\| \\ &\leqslant L\|\tilde{d}(s)\|_{L^\infty(0,\infty)} \\ &\leqslant L\|[\mathrm{sat}(\tilde{d}_\varepsilon + d) + d]\|_{L^\infty(0,\infty)} \leqslant L\varepsilon_0,\end{aligned} \tag{4.24}$$

where L is a constant that is independent of $\tilde{d}_\varepsilon$ and d, and [14]

$$(d_1 \underset{\tau}{\diamondsuit} d_2)(t) = \begin{cases} d_1(t), 0 \leqslant t \leqslant \tau, \\ d_2(t-\tau), t > \tau. \end{cases} \tag{4.25}$$

Assume that $\|e^{\mathcal{A}_0 t}\| \leqslant L_0 e^{-\omega t}$ for some $L_0, \omega > 0$. By (4.22), (4.23), and (4.24),

$$\|Z(\cdot,t)\| \leqslant L_0 e^{-\omega t}\|Z(\cdot,0)\| + L_0 t_0 (2M_1+1)^2 C_{t_0} e^{-\omega(t-t_0)} + L\varepsilon_0. \tag{4.26}$$

This implies that $\|Z(\cdot,t)\|_{L^2(0,1)} \to 0$ as $t \to \infty$. Consequently, by (4.2), $y_0(t) = -j\int_0^1 (2x^3-3x^2)z(x,t)dx \to 0$ as $t \to \infty$. The result then follows with (4.7) and (4.14). The proof is complete. □

Returning to system (1.1) by the inverse transformations (2.3) and (2.7), we have the following theorem.

Theorem 4.3. *Assume that $|d(t)| \leqslant M_1$ and $\dot{d}(t)$ is measurable, $|\dot{d}(t)| \leq M_2$ for all $t \geq 0$. Then for any initial value $(X(0), u(\cdot,0), \hat{y}_\varepsilon(0), \hat{d}_\varepsilon(0)) \in \mathcal{H} \times \mathbb{C}^2$, the closed-loop of system (1.1) as follows:*

$$\begin{gathered}\dot{X}(t) = AX(t) + Bu(0,t), \quad t > 0, \\ u_t(x,t) = -ju_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ u_x(0,t) = 0, \\ u_x(1,t) = U(t) + d(t), \\ \dot{\hat{y}}_\epsilon(t) = -\mathrm{sat}(\hat{d}_\epsilon(t)) + \hat{d}_\epsilon(t) + y_1(t) - \frac{1}{\epsilon}(\hat{y}_\epsilon - y_0), \\ \dot{\hat{d}}_\epsilon(t) = -\frac{1}{\varepsilon^2}\{\hat{y}_\epsilon - y_0\},\end{gathered} \tag{4.27}$$

admits a unique solution $(X, u, \hat{y}_\varepsilon, \hat{d}_\varepsilon)^\top \in C(0,\infty; \mathcal{H} \times \mathbb{C}^2)$, and

$$\lim_{t\to\infty,\, \varepsilon\to 0} \|(X(t), u(\cdot,t), \hat{y}_\varepsilon(t), \hat{d}_\varepsilon(t) - d(t))\|_{\mathcal{H}\times\mathbb{C}^2} = 0,$$

where the feedback control is

$$\begin{aligned}U(t) = \int_0^1 q_x(1,x)u(x,t)dx + \gamma'(1) + k(1,1)w(1,t) \\ + \int_0^1 k_x(1,x)w(x,t)dx - \mathrm{sat}(\hat{d}_\varepsilon(t)), \quad t \geqslant 0,\end{aligned} \tag{4.28}$$

and

$$y_0(t) = -j\int_0^1 (2x^3 - 3x^2)z(x,t)dx,$$

$$
\begin{gathered}
y_1(t)=\int_0^1(-12x+6+2cjx^3-3cjx^2)z(x,t)dx,\\
z(x,t)=w(x,t)-\int_0^x k(x,y)w(y,t)dy,\\
k(x,y)=-cxj\frac{I_1(\sqrt{cj(x^2-y^2)})}{\sqrt{cj(x^2-y^2)}},\\
w(x,t)=u(x,t)-\int_0^x q(x,y)u(y,t)dy-\gamma(x)X(t),\\
q(x,y)=\int_0^{x-y} j\gamma(\sigma)Bd\sigma.
\end{gathered}
$$

5. Time varying high gain estimator based feedback

In this section, we stabilize system (2.10) by ADRC with a time varying high gain state feedback disturbance estimator which is different from that in Section 4. The advantage of using the disturbance estimator by time varying high gain lies in four aspects:

- (a) the stability of system (5.18) we obtain in Theorem 5.3 is irrelevant to gain ϵ, it is different from Theorem 4.3;
- (b) the boundedness of derivative of disturbance is relaxed in some extent by choosing properly the time varying gain;
- (c) the peaking value is reduced significantly;
- (d) the possible non-smooth control (4.15) becomes smooth.

Now, we design the following extended state observer with time varying high gain for $y_0(t)$ and $d(t)$:

$$
\begin{gathered}
\dot{\hat{y}}(t)=U_0(t)+\hat{d}(t)+y_1(t)-g(t)[\hat{y}(t)-y_0(t)],\\
\dot{\hat{d}}(t)=-g^2(t)[\hat{y}(t)-y_0(t)],
\end{gathered}
\tag{5.1}
$$

where $g\in C^1[0,\infty)$ is a time varying gain real value function satisfying

$$
\begin{gathered}
g(t)>0,\quad \dot{g}(t)>0,\quad \forall t\ge 0,\\
g(t)\to\infty\quad\text{as } t\to\infty,\quad \sup_{t\in[0,\infty)}\Big|\frac{\dot{g}(t)}{g(t)}\Big|<\infty.
\end{gathered}
\tag{5.2}
$$

In addition, we assume that the disturbance $d(t)\in H^1_{loc}(0,\infty)$ satisfies

$$
\lim_{t\to\infty}\frac{|\dot{d}(t)|}{g(t)}=0. \tag{5.3}
$$

By (5.3), $\dot{d}(t)$ is allowed to grow exponentially at any rate by choosing properly the gain function $g(t)$. This relaxes the condition in Section 4 where $\dot{d}(t)$ is assumed to be uniformly bounded. We use $\hat{d}(t)$ to estimate $d(t)$, then the convergence is stated in the following lemma.

Lemma 5.1. *Let $(\hat{y},\hat{d})$ be the solution of (5.1). Then*

$$
\lim_{t\to\infty}|\hat{y}(t)-y(t)|=0,\quad \lim_{t\to\infty}|\hat{d}(t)-d(t)|=0. \tag{5.4}
$$

Proof. Set
$$\tilde{y}(t) = g(t)[\hat{y}(t) - y_0(t)], \quad \tilde{d}(t) = \hat{d}(t) - d(t). \tag{5.5}$$
Then the error $(\tilde{y}, \tilde{d})$ is governed by
$$\begin{gathered} \dot{\tilde{y}}(t) = -g(t)[\tilde{y}(t) - \tilde{d}(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t), \\ \dot{\tilde{d}}(t) = -g(t)\tilde{y}(t) - \dot{d}(t). \end{gathered} \tag{5.6}$$
The existence of the local classical solution to (5.6) is guaranteed by the local Lipschitz condition of the right side of (5.6). We consider the stability of this ODE. To this end, we introduce the following Lyapunov function (in addition, the global solution is ensured by the following Lyapunov function argument). Define
$$\begin{gathered} E(t) = |\tilde{y}(t)|^2 + \frac{3}{2}|\tilde{d}(t)|^2, \quad \rho(t) = \tilde{y}(t)\overline{\tilde{d}(t)}, \\ V(t) = E(t) - \operatorname{Re}\rho(t). \end{gathered} \tag{5.7}$$
Then
$$\frac{1}{2}E(t) \le V(t) \le 2E(t). \tag{5.8}$$
Differentiating $E(t)$ and $\rho(t)$ with respect to t, we obtain
$$\begin{aligned} \dot{E}(t) &= 2\operatorname{Re}[\overline{\tilde{y}(t)}\dot{\tilde{y}}(t)] + 3\operatorname{Re}[\overline{\tilde{d}(t)}\dot{\tilde{d}}(t)] \\ &= -2g(t)|\tilde{y}(t)|^2 + 2\frac{\dot{g}(t)}{g(t)}|\tilde{y}(t)|^2 - g(t)\operatorname{Re}[\overline{\tilde{y}(t)}\tilde{d}(t)] - 3\overline{\tilde{d}(t)}\dot{d}(t), \end{aligned}$$
and
$$\begin{aligned} \dot{\rho}(t) &= \dot{\tilde{y}}(t)\overline{\tilde{d}(t)} + \tilde{y}(t)\overline{\dot{\tilde{d}}(t)} \\ &= -g(t)|\tilde{y}(t)|^2 + g(t)|\tilde{d}(t)|^2 - g(t)\tilde{y}(t)\overline{\tilde{d}(t)} + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t)\overline{\tilde{d}(t)} - \tilde{y}(t)\overline{\dot{d}(t)}. \end{aligned}$$
Then
$$\begin{aligned} \dot{V}(t) &= [-g(t) + \frac{2\dot{g}(t)}{g(t)}]|\tilde{y}(t)|^2 - g(t)|\tilde{d}(t)|^2 - \frac{\dot{g}(t)}{g(t)}\operatorname{Re}\rho(t) \\ &\quad + \operatorname{Re}[-3\dot{d}(t)\overline{\tilde{d}(t)} + \tilde{y}(t)\overline{\dot{d}(t)}] \\ &\le -\frac{1}{2}\kappa(t)V(t) + m_0|\dot{d}(t)|\|(\tilde{y}(t), \tilde{d}(t))\| \\ &\le -\frac{1}{2}\kappa(t)V(t) + m_0|\dot{d}(t)|\sqrt{V(t)}, \end{aligned} \tag{5.9}$$
where m_0 is a constant,
$$\kappa(t) = g(t) - \sup_{t\in[0,\infty)}|\frac{3\dot{g}(t)}{g(t)}| \to \infty \quad \text{as } t \to \infty, \tag{5.10}$$
and there exists $t_0 > 0$ such that
$$\kappa(t) > 0, \quad \forall t \ge t_0.$$
This, together with (5.9), yields
$$\frac{d\sqrt{V(t)}}{dt} \le -\frac{1}{4}\kappa(t)\sqrt{V(t)} + \frac{m_0}{2}|\dot{d}(t)|, \quad \forall t \ge t_0. \tag{5.11}$$

We deduce that

$$\sqrt{V(t)} \leq \sqrt{V(t_0)} e^{-\frac{1}{4}\int_{t_0}^t \kappa(s)ds} + e^{-\frac{1}{4}\int_{t_0}^t \kappa(s)ds} \frac{m_0}{2} \int_{t_0}^t |\dot{d}(s)| e^{\frac{1}{4}\int_{t_0}^s \kappa(\tau)d\tau} ds. \tag{5.12}$$

The first term on the right-hand side of (5.12) is obviously convergent to zero as $t \to \infty$ owing to (5.10). Applying the L'Hospital rule to the second term on the right-hand side of (5.12), we obtain

$$\begin{aligned} \lim_{t\to\infty} \frac{m_0 \int_{t_0}^t |\dot{d}(s)| e^{\frac{1}{4}\int_{t_0}^s \kappa(\tau)d\tau} ds}{2e^{\frac{1}{4}\int_{t_0}^t \kappa(s)ds}} &= \lim_{t\to\infty} \frac{m_0}{2} \frac{|\dot{d}(t)| e^{\frac{1}{4}\int_{t_0}^t \kappa(\tau)d\tau}}{\frac{1}{4} e^{\frac{1}{4}\int_{t_0}^t \kappa(s)ds} \kappa(t)} \\ &= \lim_{t\to\infty} 2m_0 \frac{|\dot{d}(s)|}{g(t)} \cdot \frac{g(t)}{\kappa(t)} = 0, \end{aligned} \tag{5.13}$$

which implies $\lim_{t\to\infty} \sqrt{V(t)} = 0$; that is,

$$\lim_{t\to\infty} [|\tilde{y}(t)|^2 + |\tilde{d}(t)|^2] = 0. \tag{5.14}$$

The proof is complete. □

By Lemma 5.1, we design the feedback control

$$U_0(t) = -\hat{d}(t), \tag{5.15}$$

then we can rewrite the closed-loop of system (2.10) as

$$\begin{gathered} \dot{X}(t) = (A + BK)X(t) + Bz(0,t), \\ z_t(x,t) = -jz_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -\hat{d}(t) + d(t), \\ \dot{\hat{y}}(t) = y_1(t) - g(t)[\hat{y}(t) - y_0(t)], \\ \dot{\hat{d}}(t) = -g^2(t)[\hat{y}(t) - y_0(t)]. \end{gathered} \tag{5.16}$$

Proposition 5.2. *Assume that the time varying gain $g(t) \in C^1[0,\infty)$ satisfies (5.2) and the disturbance $d(t) \in H^1_{loc}(0,\infty)$ satisfies (5.3). Then for any initial value $(X(0), z(0), \hat{y}(0), \hat{d}(0)) \in \mathcal{H} \times \mathbb{C}^2$, there exists a unique solution $(X, z, \hat{y}, \hat{d}) \in C(0,\infty; \mathcal{H} \times \mathbb{C}^2)$ to system (5.16) and system (5.16) is asymptotically stable in the sense that*

$$\lim_{t\to\infty} \|(X(t), z(\cdot,t), \hat{y}(t), \hat{d}(t) - d(t))\|_{\mathcal{H}\times\mathbb{C}^2} = 0$$

Proof. By the error variables $(\tilde{y}, \tilde{d})$ defined in (5.5), we have the following equivalent system for system (5.16),

$$\begin{gathered} \dot{X}(t) = (A + BK)X(t) + Bz(0,t), \\ z_t(x,t) = -jz_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -\hat{d}(t) + d(t), \\ \dot{\tilde{y}}(t) = -g(t)[\tilde{y}(t) - \tilde{d}(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t), \\ \dot{\tilde{d}}(t) = -g(t)\tilde{y}(t) - \dot{d}(t). \end{gathered} \tag{5.17}$$

The "ODE part" of (5.17) is just the system (5.6), which is shown to be convergent by Lemma 5.1. The "(X, z) part" of (5.17) is similar to (2.10), hence the proof becomes similar to the proof of Lemma 4.2. The details are omitted. $\square$

Returning to system (1.1) by the inverse transformations (2.3) and (2.7), we prove the following theorem.

Theorem 5.3. *Assume that the time varying gain $g(t) \in C^1[0, \infty)$ satisfies (5.2) and the disturbance $d(t) \in H^1_{loc}(0, \infty)$ satisfies (5.3). Then for any initial value $(X(0), u(\cdot, 0), \hat{y}(0), \hat{d}(0)) \in \mathcal{H} \times \mathbb{C}^2$, the closed-loop of system*

$$
\begin{gathered}
\dot{X}(t) = AX(t) + Bu(0,t), \quad t > 0 \\
u_t(x,t) = -ju_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\
u_x(0,t) = 0, \\
\begin{aligned}
u_x(1,t) = -\hat{d}(t) + \int_0^1 q_x(1,y)u(y,t)dy + \gamma'(1) + k(1,1)w(1,t) \\
+ \int_0^1 k_x(1,y)w(y,t)dy + d(t),
\end{aligned} \\
\dot{\hat{y}}(t) = y_1(t) - g(t)[\hat{y}(t) - y_0(t)], \\
\dot{\hat{d}}(t) = -g^2(t)[\hat{y}(t) - y_0(t)],
\end{gathered}
\tag{5.18}
$$

admits a unique solution $(X, u, \hat{y}, \hat{d})^\top \in C(0, \infty; \mathcal{H} \times \mathbb{C}^2)$, and system (5.18) is asymptotically stable

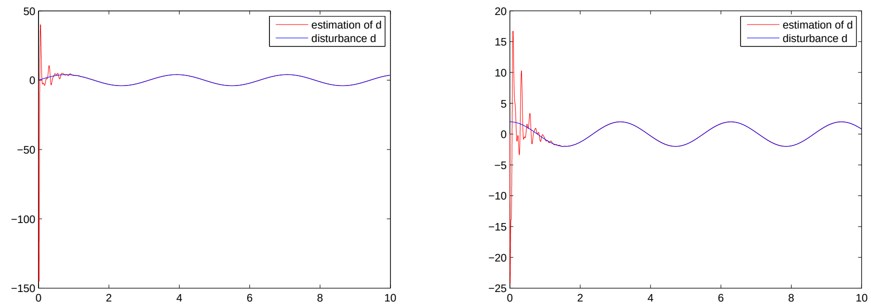
$$
\lim_{t\to\infty} \|(X(t), u(\cdot,t), \hat{y}(t), \hat{d}(t) - d(t))\|_{\mathcal{H}\times\mathbb{C}^2} = 0,
$$

where

$$
\begin{gathered}
y_0(t) = -j\int_0^1 (2x^3 - 3x^2)z(x,t)dx, \\
y_1(t) = \int_0^1 (-12x + 6 + 2cjx^3 - 3cjx^2)z(x,t)dx, \\
z(x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)dy, \\
k(x,y) = -cxj\frac{I_1(\sqrt{cj(x^2-y^2)})}{\sqrt{cj(x^2-y^2)}}, \\
w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t), \\
q(x,y) = \int_0^{x-y} j\gamma(\sigma)Bd\sigma.
\end{gathered}
$$

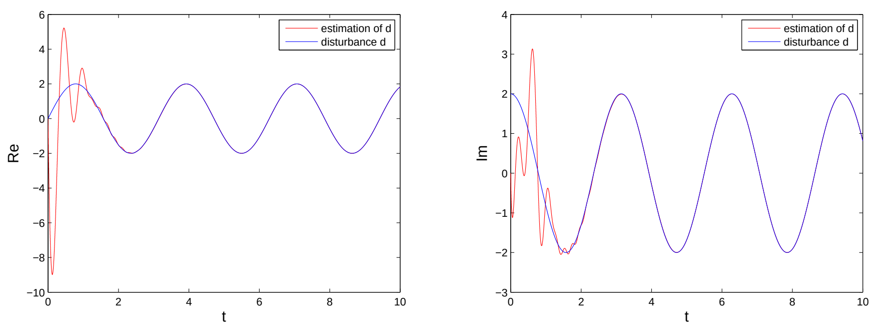
6. Numerical simulation

In this section, we present some numerical simulations to show visually the effectiveness of the proposed controllers for systems (4.27) and (5.18) respectively. We choose the initial values and the parameters as following: $u(x,0) = -x + 3xj$, $A = 2$, $B = -6$, $K = 4$, $\epsilon = 0.01$, $g(t) = 10 + 12t^2$. To estimate the unknown disturbance $d(t)$, we assume that the time varying gain function $g(t)$ satisfies assumption (5.2). However, from the practice standpoint, the increasing $g(t)$ can not



(a) Real part (b) Imaginary part

FIGURE 2. The $\hat{d}(t)$ and disturbance $d(t) = 2\sin(2t) + 2j\cos(2t)$ by constant high gain



(a) Real part (b) Imaginary part

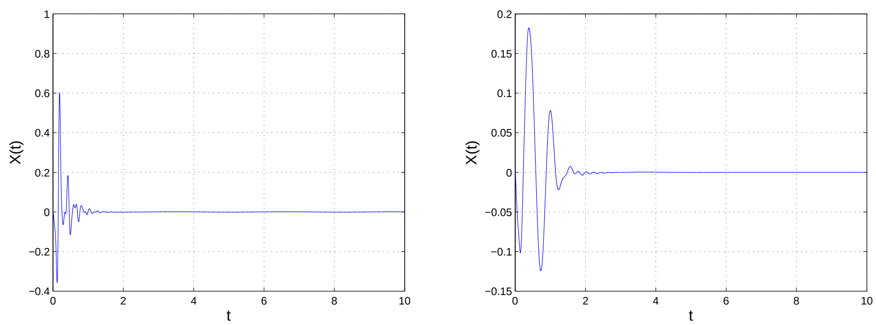
FIGURE 3. The $\hat{d}(t)$ and disturbance $d(t) = 2\sin(2t) + 2j\cos(2t)$ by time varying gain

be applied in extended time interval. A recommended scheme is to use the time varying gain first to reduce the peaking value in the initial stage to a reasonable level and then use the constant high gain. To this end, we take the gain function

$$\hat{g}(t) = \begin{cases} g(t), & t \le t_0, \\ g(t_0), & t > t_0, \end{cases} \tag{6.1}$$

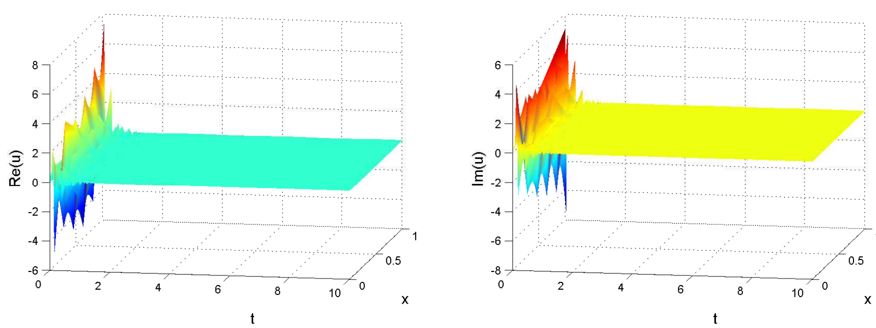
where $t_0 > 0$ and $\epsilon = 1/g(t_0)$. Using combined varying gain (6.1) with $t_0 \simeq 2.8$.

On the one hand, the disturbance is taken as $d(t) = 2\sin(2t) + 2j\cos(2t)$. It is seen that with the constant high gain, the peaking value is observed in the initial stage in Figure 2, whereas with the time varying gain, the peaking value is dramatically reduced as shown in Figure 3. This is an advantage of the time varying gain approach. The price is that the convergence by the time varying gain is slightly slow which is observed from Figure 3. The ODE state $X(t)$ of the systems (4.27) and (5.18) are shown in Figure 4(a) and Figure 4(b), respectively. The PDE



(a) Real part (b) Imaginary part

FIGURE 4. The ODE state $X(t)$ for systems (4.27) and (5.18)



(a) Real part (b) Imaginary part

FIGURE 5. The Schrödinger sate $u(x,t)$ with constant high gain controllers

part of solutions of systems (4.27) and (5.18) are plotted in Figure 5 and Figure 6 respectively.

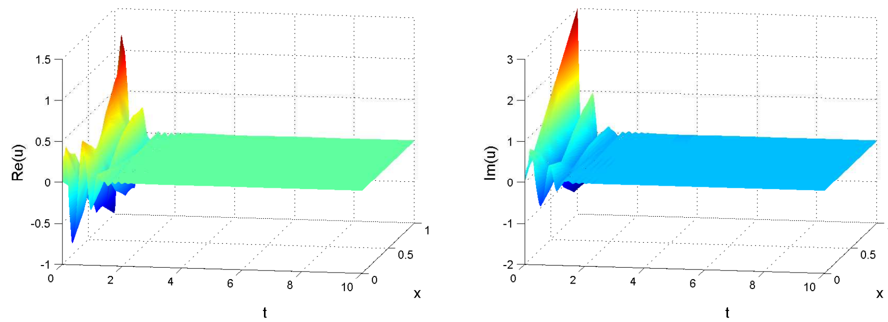
On the other hand, the disturbance is taken as $d(t) = 2\sin(t^{3/2}) + 2j\cos(t^{3/2})$. The solutions of system (5.18) are plotted in Figure 7, 8, 9, respectively. In spite of the derivative of disturbance is unbounded, we see that convergence of the state is satisfactory with the time varying gain approach. This is another advantage of the time varying gain approach.

Acknowledgments. This work was supported by the National Natural Science Foundation of China.

REFERENCES

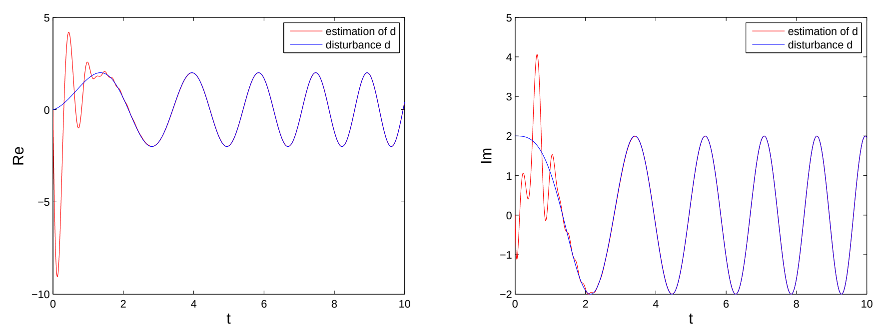
[1] N. Bekiaris-Liberis and M. Krstic, *Compensation of wave actuator dynamics for nonlinear systems*, IEEE Trans. Automat. Control **59** (2014), no. 6, 1555–1570. MR 3225229

[2] H. Feng and B. Z. Guo, *Output feedback stabilization of an unstable wave equation with general corrupted boundary observation*, Automatica J. IFAC **50** (2014), no. 12, 3164–3172. MR 3284151



(a) Real part (b) Imaginary part

FIGURE 6. The Schrödinger sate $u(x,t)$ with time varying high gain controllers



(a) Real part (b) Imaginary part

FIGURE 7. The $\hat{d}(t)$ and the disturbance $d(t) = 2\sin(t^{3/2}) + 2j\cos(t^{3/2})$ by time varying gain

[3] B. Z. Guo and F. F. Jin, *The active disturbance rejection and sliding mode control approach to the stabilization of the Euler-Bernoulli beam equation with boundary input disturbance*, Automatica J. IFAC **49** (2013), no. 9, 2911–2918. MR 3084483

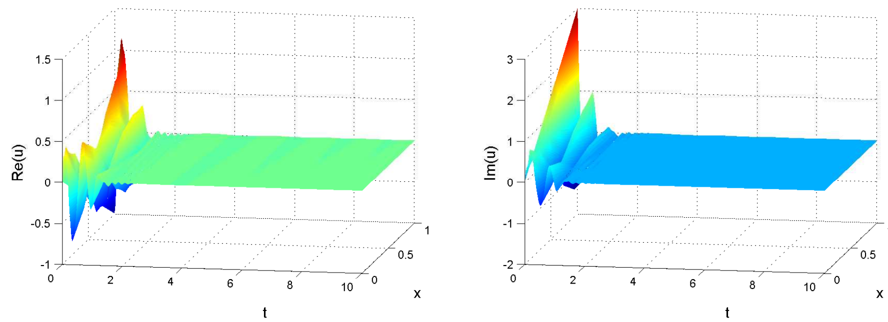
[4] B. Z. Guo and Z. L. Zhao, *On the convergence of an extended state observer for nonlinear systems with uncertainty*, Systems Control Lett. **60** (2011), no. 6, 420–430. MR 2841486 (2012m:93072)

[5] V. Komornik and P. Loreti, *Fourier series in control theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2005. MR 2114325 (2006a:93001)

[6] M. Krstic, *Compensating actuator and sensor dynamics governed by diffusion PDEs*, Systems Control Lett. **58** (2009), no. 5, 372–377. MR 2512493 (2010b:93134)

[7] M. Krstic and A. Smyshlyaev, *Adaptive boundary control for unstable parabolic PDEs. I. Lyapunov design*, IEEE Trans. Automat. Control **53** (2008), no. 7, 1575–1591. MR 2446372 (2009g:93065)

[8] ______, *Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays*, Systems Control Lett. **57** (2008), no. 9, 750–758. MR 2446460 (2010a:93072)



(a) Real part (b) Imaginary part

FIGURE 8. The Schrödinger sate $u(x,t)$ for system (5.18)

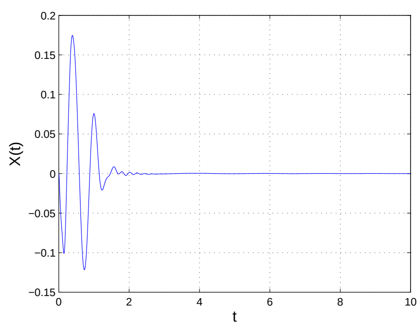


FIGURE 9. The ODE state $X(t)$ for system (5.18)

[9] B. B. Ren, J. M. Wang, and M. Krstic, *Stabilization of an ODE-Schrödinger cascade*, Systems Control Lett. **62** (2013), no. 6, 503–510. MR 3054082

[10] J. M. Wang, J. J. Liu, B. B. Ren, and J. H. Chen, *Sliding mode control to stabilization of cascaded heat PDE-ODE systems subject to boundary control matched disturbance*, Automatica J. IFAC **52** (2015), 23–34. MR 3310810

[11] J. M. Wang, B. B Ren, and M. Krstic, *Stabilization and Gevrey regularity of a Schrödinger equation in boundary feedback with a heat equation*, IEEE Trans. Automat. Control **57** (2012), no. 1, 179–185. MR 2917656

[12] J. M. Wang, L. L. Su, and H. X. Li, *Stabilization of an unstable reaction-diffusion PDE cascaded with a heat equation*, Systems Control Lett. **76** (2015), 8–18. MR 3304624

[13] Z. L Wang and X. R. Shi, *Lag synchronization of two identical Hindmarsh-Rose neuron systems with mismatched parameters and external disturbance via a single sliding mode controller*, Appl. Math. Comput. **218** (2012), no. 22, 10914–10921. MR 2942376

[14] G. Weiss, *Admissibility of unbounded control operators*, SIAM J. Control Optim. **27** (1989), no. 3, 527–545. MR 993285 (90c:93060)

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