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PROPERTIES OF SOLUTIONS IN SEMI-HYPERBOLIC PATCHES FOR UNSTEADY TRANSONIC SMALL DISTURBANCE EQUATIONS

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Abstract. We consider a two-dimensional Riemann problem for the unsteady transonic small disturbance equation resulting in diverging rarefaction waves. We write the problem in self-similar coordinates and we obtain a mixed type (hyperbolic-elliptic) system. Resolving the one-dimensional discontinuities in the far field, where the system is hyperbolic, and using characteristics, we formulate the problem in a semi-hyperbolic patch that is between the hyperbolic and the elliptic regions. A semi-hyperbolic patch is known as a region where one family out of two nonlinear families of characteristics starts on a sonic curve and ends on a transonic shock. We obtain existence of a smooth local solution in this semi-hyperbolic patch and we prove various properties of global smooth solutions based on a characteristic decomposition using directional derivatives.

1. Introduction

Motivation. Two-dimensional Riemann problems for systems of hyperbolic conservation laws have been studied extensively during the last several decades. Various systems have been considered: steady/unsteady transonic small disturbance equation, nonlinear wave system, pressure gradient system, potential flow equations and Euler gas dynamics equations (see [4] by Čanić, Keyfitz and Kim, and [14] by Keyfitz).

Riemann problems for the unsteady transonic small disturbance (UTSD) equation resulting in shock reflection were studied by Brio and Hunter in [1] (Mach reflection), by Čanić, Keyfitz and Kim in [3], and by Jegdić, Keyfitz and Čanić in [13] (transonic or strong regular reflection) and by Čanić, Keyfitz and Kim in [5] (supersonic or weak regular reflection). Solution to a Riemann problem for the steady case with shocks was obtained by Čanić, Keyfitz and Lieberman in [6].

In this article we revisit the Riemann problem for the UTSD equation studied by Chen, Christoforou and Jegdić in [7]. The initial data is chosen to be constant in three sectors, symmetric about x-axis, resulting in diverging rarefaction waves. As in the study of general two-dimensional Riemann problems for systems of hyperbolic conservation laws, we rewrite the problem in self-similar coordinates and we obtain

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a mixed type system. It is well-known that when this system is linearized about a constant state, it changes type across a sonic parabola and it is hyperbolic (elliptic) outside (inside). Since the problem is symmetric with respect to the horizontal axis, we focus our attention on the upper half plane. The one dimensional discontinuity in the hyperbolic region becomes a rarefaction wave that stretches down towards the origin. It interacts with the sonic parabola for one of the constant states and gives rise to a very complicated free boundary problem. Our preliminary local analysis in [7] suggested that there are a degenerate hyperbolic region and a degenerate elliptic region coupled via a sonic curve near the point of intersection of the left-border of the rarefaction wave with the above mentioned sonic parabola. The same Riemann problem for the UTSD equation was studied by Tesdall and Keyfitz [19]. They linearized the problem and solved it exactly, and they presented numerical results for the nonlinear problem indicating that the rarefaction wave reflects off a sonic line and forms a transonic shock. Such regions where one family out of two nonlinear families of characteristics starts at on a sonic curve and ends on a transonic shock are called semi-hyperbolic patches. Existence of solutions in semi-hyperbolic patches was obtained for the pressure gradient system by Song and Zheng in [18], for the Euler equations by Li and Zheng in [16] and by Zheng in [21], and for the nonlinear wave system (NLWS) by Hu and Wang in [10]. They derive characteristics decomposition for the problem and use it to

(1) prove that a hyperbolic state adjacent to a constant state is a simple wave,
(2) formulate a Goursat-type boundary value problem,
(3) prove existence of a local solution to the Goursat problem,
(4) derive various properties of global smooth solutions, and
(5) use those properties and results by Dai and Zhang in [8] for gas dynamics equations, to obtain a global solution to the degenerate Goursat problem.

We follow ideas in [18, 10] to derive characteristic decomposition of the problem using directional derivatives along characteristics. Čanić and Keyfitz proved in [2] that for 2×2 quasi-one-dimensional systems, a hyperbolic state adjacent to a hyperbolic constant state is a simple wave, completing the above mentioned step (1). In this article, we complete steps (2)-(4) and we believe that techniques from [8] could be extended to the UTSD equation to prove existence of a global solution to the Goursat problem. We leave that study for the future work.

Numerical studies of the full gas dynamics equations involving interaction of rarefaction waves also indicate existence of semi-hyperbolic regions (see [9] by Glimm et al. and [17] by Sheng, Wang and Zhang).

Summary of results. This article is organized as follows. In §2 we formulate the Riemann problem of interest for the UTSD equation and we rewrite it using self-similar coordinates. We recall the structure of the solution in the hyperbolic region (see [7, 19]) and, using analysis via characteristics, we state the problem in the semi-hyperbolic patch. To analyze the problem, we define directional derivatives along characteristics.

In §3 we rewrite the problem, again, using parabolic coordinates and we formulate the Goursat-type boundary value problem with two boundary curves given by two characteristics.

Section §4 deals with existence of a local solution to this Goursat problem (Theorem 4.1) and various properties of smooth solutions in the entire semi-hyperbolic patch. We establish the signs of directional derivatives in the domain and their monotonicity along the characteristics curves. We prove that positive/negative characteristics are convex/concave. We also establish that the global minimum of the directional derivatives is achieved along the boundary which consists of the characteristic curves and we obtain existence of the sonic boundary along which the two directional derivatives coincide. Finally, we derive an estimate of the C^1 norm of the solution away from the sonic curve.

In §5 we remark that the solution in the semi-hyperbolic patch, in the region outside of the Goursat problem, is a simple wave using the result by Čanić and Keyfitz in [2]. We conclude by showing that one family of characteristics forms an envelope before they become sonic, indicating existence of a transonic shock.

2. Formulation of the problem

In this section, we pose our Riemann problem and rewrite it using self-similar coordinates. We formulate the problem in the semi-hyperbolic patch.

Formulation of the initial value problem. We consider the unsteady transonic small disturbance equation

$$\begin{aligned} u_t + uu_x + v_y &= 0,\\ -v_x + u_y &= 0, \end{aligned} \tag{2.1}$$

where $(x, y) \in \mathbb{R}^2$, $t \in [0,\infty)$, and $u, v : [0,\infty) \times \mathbb{R}^2 o\mathbb{R}$ denote velocities in x- and y-directions, respectively. Given parameter $a > 0$, we consider Riemann initial data posed in three sectors (see Figure 1)

$$(u, v)(0, x, y) = \begin{cases} (-1, a), & x/a < y < 0,\\ (-1, -a), & 0 < y < -x/a,\\ (0, 0), & \text{otherwise.} \end{cases} \tag{2.2}$$

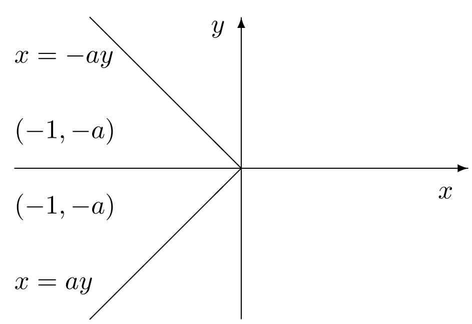

Figure 1. Riemann initial data

Formulation of the problem in self-similar coordinates. We formulate the problem in self-similar coordinates $\xi = x/t$ and $\eta = y/t$ and we obtain the system

$$\begin{aligned} (u - \xi)u_\xi - \eta u_\eta + v_\eta &= 0,\\ -v_\xi + u_\eta &= 0, \end{aligned} \tag{2.3}$$

whose eigenvalues are

$$\Lambda_{\pm} = \frac{\eta \pm \sqrt{\eta^2 + 4(\xi - u)}}{2(\xi - u)}. \tag{2.4}$$

We also recall that by eliminating v in (2.3), we obtain the following second order equation for u

$$(u - \xi)u_{\xi\xi} - \eta u_{\xi\eta} + u_{\eta\eta} + (u_\xi - 1)u_\xi = 0. \tag{2.5}$$

Therefore, when the above system is linearized about a constant state, it changes type across the sonic parabola

$$P_u : \xi + \frac{\eta^2}{4} = u, \tag{2.6}$$

and the system is hyperbolic if $\xi + \eta^2/4 > u$ and it is elliptic if $\xi + \eta^2/4 < u$. Since the problem is symmetric with respect to ξ-axis, we consider the problem in the upper-half plane. The initial condition (2.2) becomes a boundary condition in the far field and results in a rarefaction wave in the upper-half plane given by

$$u = \xi + a\eta - a^2, \quad v = au, \quad a^2 - 1 \le \xi + a\eta \le a^2. \tag{2.7}$$

Hence, in a time-like direction, the left border of the rarefaction wave is given by the line $\xi + a\eta = a^2$ along which $(u, v) = (0, 0)$, and the right border of the rarefaction wave is given by $\xi + a\eta = a^2 - 1$ along which $(u, v) = (-1, -a)$. The rarefaction wave stretches down until it becomes sonic. We compute the sonic line to be $\eta = 2a$ using (2.6) and (2.7). However, it turns that the problem is much more complicated due to the interaction of the rarefaction wave with the sonic parabola

$$P_0 : \xi + \frac{\eta^2}{4} = 0,$$

corresponding to the state $(0, 0)$. More precisely, the left border of the rarefaction wave intersects parabola P_0 at the point $A(-a^2, 2a)$. We find the two characteristics at the point A, by solving the equations

$$\frac{d\eta}{d\xi} = \Lambda_{\pm},$$

and by substituting $\xi - u = a^2 - a\eta$ in (2.4) and noting that $\eta \ge 2a$. We obtain the line

$$\xi + a\eta = a^2 \quad \text{(positive characteristics)}$$

and the parabola

$$\xi = -\frac{(\eta - a)^2}{2} - \frac{a^2}{2} \quad \text{(negative characteristics)}.$$

An easy calculation shows that the negative characteristics through A intersects the right border of the rarefaction wave at the point $B(-a^2 - 1 - a\sqrt{2}, 2a + \sqrt{2})$ which is below the projected parabola P_0 (see Figure 2). Therefore, the information from the point A travels back into the hyperbolic region and, as a consequence the rarefaction wave is curved below the curve AB (which will from now on denote the negative characteristics at the point A). We observe that this negative characteristics continues as a straight line and intersects the sonic parabola $P_{-1} : \xi + \eta^2/4 = -1$, corresponding to the state $(-1, -a)$, tangentially at the point D. The problem we are interested in is formulated as follows:

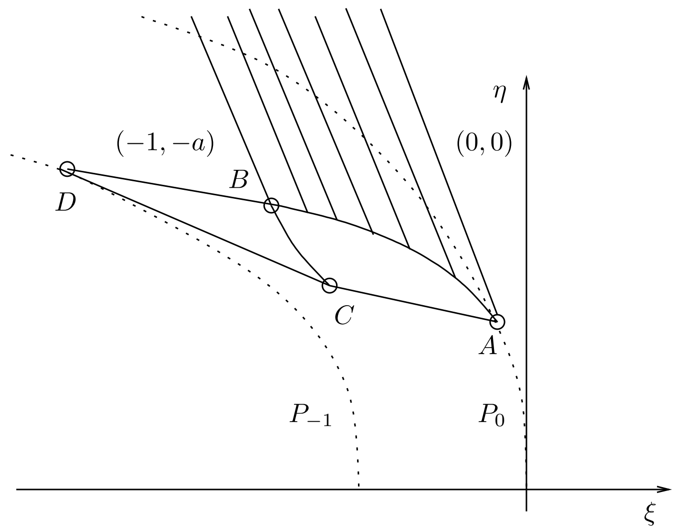

FIGURE 2. Semi-hyperbolic patch in (ξ, η)-plane

> Given a positive characteristics BC, where C is a sonic point, find the solution in the domain $ABDCA$, where the curve ABD is a negative characteristics, AC is a sonic curve and CD is the envelope of negative characteristics.

Directional derivatives along $\Lambda_\pm$ characteristics. We denote the directional derivatives along $\Lambda_\pm$ characteristics by

$$\partial^\pm = \partial_\xi + \Lambda_\pm \partial_\eta. \tag{2.8}$$

Using (2.4) we find

$$u = \xi - \frac{\eta\Lambda_\pm + 1}{\Lambda_\pm^2}.$$

Differentiating this expression, and using $\partial^\pm \xi = 1$ and $\partial^\pm \eta = \Lambda_\pm$, we obtain

$$\begin{aligned} \partial^\pm u &= 1 - \frac{(\Lambda_\pm^2 + \eta\partial^\pm\Lambda_\pm)\Lambda_\pm - (\eta\Lambda_\pm + 1)2\Lambda_\pm\partial^\pm\Lambda_\pm}{\Lambda_\pm^4} \\ &= \frac{\eta\Lambda_\pm + 2}{\Lambda_\pm^3}\partial^\pm\Lambda_\pm, \end{aligned} \tag{2.9}$$

or, equivalently,

$$\partial^\pm\Lambda_\pm = \frac{\Lambda_\pm^3}{\eta\Lambda_\pm + 2}\partial^\pm u. \tag{2.10}$$

3. Formulation of the Goursat problem in parabolic coordinates

It is convenient to formulate the problem in parabolic coordinates (ρ, η), where $\rho = \xi + \eta^2/4$. The system (2.3) becomes

$$(u - \rho)u_\rho - \frac{\eta}{2}u_\eta + v_\eta = 0,$$

$$\frac{\eta}{2}u_\rho - v_\rho + u_\eta = 0,$$

with eigenvalues

$$\lambda_\pm = \pm\frac{1}{\sqrt{\rho - u}}. \tag{3.1}$$

We eliminate variable v in the above system and we obtain the second order equation for u,

$$(u-\rho)u_{\rho\rho} + u_{\eta\eta} - \frac{u_\rho}{2} + u_\rho^2 = 0. \tag{3.2}$$

We find the left and the right borders of the rarefaction wave to be the parabolas

$$\rho = \Big(\frac{\eta}{2} - a\Big)^2 \quad \text{and} \quad \rho = \Big(\frac{\eta}{2} - a\Big)^2 - 1,$$

respectively (see Figure 3). Furthermore, we find points $A(0, 2a)$, $B(-1/2, 2a+\sqrt{2})$ and the negative characteristics at A to be the parabola

$$\rho = -\Big(\frac{\eta}{2} - a\Big)^2. \tag{3.3}$$

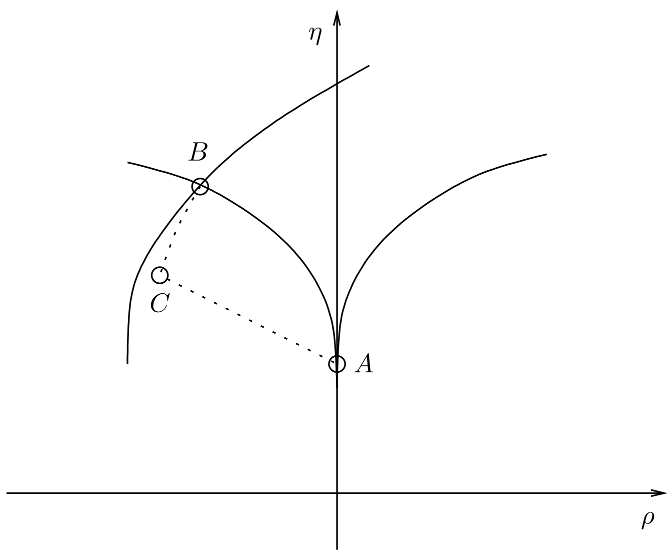

FIGURE 3. Region ABC in the (ρ, η)-plane

Directional derivatives along $\lambda_\pm$ characteristics. We use the notation

$$\lambda^{-1} = \sqrt{\rho - u}, \tag{3.4}$$

and we denote the directional derivatives along $\lambda_\pm$ characteristics by

$$\partial_\pm = \partial_\eta \pm \lambda^{-1}\partial_\rho = \partial_\eta \pm \sqrt{\rho - u}\,\partial_\rho. \tag{3.5}$$

Hence,

$$\begin{aligned}\partial_\pm u &= \partial_\eta u(\xi, \eta) \pm \lambda^{-1}\partial_\rho u(\xi, \eta)\\ &= u_\xi \xi_\eta + u_\eta \pm \lambda^{-1} u_\xi \xi_\rho\end{aligned}$$

$$
\begin{aligned}
&= u_\xi\Big(-\frac{\eta}{2}\pm\lambda^{-1}\Big)+u_\eta \\
&= \Big(-\frac{\eta}{2}\pm\lambda^{-1}\Big)\Big\{u_\xi+\Big(-\frac{\eta}{2}\pm\lambda^{-1}\Big)^{-1}u_\eta\Big\}.
\end{aligned}
$$

An easy calculations shows

$$
\Big(-\frac{\eta}{2}\pm\lambda^{-1}\Big)^{-1}=\Big(-\frac{\eta}{2}\pm\sqrt{\rho-u}\Big)^{-1}=\Lambda_\pm,
$$

implying

$$
\partial_\pm u=\Lambda_\pm^{-1}\partial^\pm u.
$$

Recalling (2.9), we obtain

$$
\partial_\pm u=\frac{\eta\Lambda_\pm+2}{\Lambda_\pm^4}\partial^\pm\Lambda_\pm. \tag{3.6}
$$

We use this relationship between $\partial_\pm u$ and $\partial^\pm\Lambda_\pm$ in §4 to investigate the convexity/concavity of $\Lambda_\pm$ characteristics in the region ABC.

Characteristics decomposition. Using the second order equation (3.2), we compute

$$
\begin{aligned}
\partial_+\partial_- u &= \partial_+\left(u_\eta-\sqrt{\rho-u}\,u_\rho\right)\\
&= \frac{u_\rho}{2\sqrt{\rho-u}}(u_\eta-\sqrt{\rho-u}\,u_\eta)\\
&= \frac{u_\rho}{2\sqrt{\rho-u}}\partial_- u.
\end{aligned}
$$

We obtain, from (3.5), that

$$
u_\rho=\frac{\partial_+u-\partial_-u}{2\sqrt{\rho-u}}. \tag{3.7}
$$

Therefore,

$$
\partial_+\partial_-u=\frac{1}{4(\rho-u)}\left(\partial_+u-\partial_-u\right)\partial_-u.
$$

We define

$$
Q:=\frac{1}{4(\rho-u)}, \tag{3.8}
$$

and note that $Q>0$ in the hyperbolic region. Hence, we obtain

$$
\partial_+\partial_-u=Q(\partial_+u-\partial_-u)\partial_-u. \tag{3.9}
$$

Similarly,

$$
\partial_-\partial_+u=Q(\partial_-u-\partial_+u)\partial_+u. \tag{3.10}
$$

Equations (3.9)-(3.10) are analogous to the characteristic decomposition equations obtained for the pressure-gradient system in [18] and the nonlinear wave system in [10]. We note that

$$
\frac{1}{2}(\partial_+u+\partial_-u)=u_\eta. \tag{3.11}
$$

Formulation of the Goursat problem in the domain ABC. Using (3.9), (3.10) and (3.11), the second order equation (3.2) for u can be written as the system

$$\begin{bmatrix} \partial_+ u \\ \partial_- u \\ u \end{bmatrix}_\eta + \begin{bmatrix} -\lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_+ u \\ \partial_- u \\ u \end{bmatrix}_\rho = \begin{bmatrix} Q(\partial_- u - \partial_+ u)\partial_+ u \\ Q(\partial_+ u - \partial_- u)\partial_- u \\ \frac{1}{2}(\partial_+ u + \partial_- u) \end{bmatrix}. \tag{3.12}$$

We define

$$W = (\partial_+ u, \partial_- u, u)^T,$$

and we specify W along characteristics AB and CB. We denote

$$W_- = W|_{AB} \quad \text{and} \quad W_+ = W|_{CB}. \tag{3.13}$$

First, we recall from (3.3) that

$$AB = \{(\rho, \eta) : \rho = -\Big(\frac{\eta}{2} - a\Big)^2,\ 2a \le \eta \le 2a + \sqrt{2}\}. \tag{3.14}$$

Further, along AB we have

$$u = \xi + a\eta - a^2 = \rho - \frac{\eta^2}{4} + a\eta - a^2 = \rho - \Big(\frac{\eta}{2} - a\Big)^2 = 2\rho. \tag{3.15}$$

Hence, along AB we have

$$\partial_- u = \partial_\eta u - \sqrt{\rho - u}\,\partial_\rho u = -2\sqrt{-\rho}, \tag{3.16}$$

implying $\partial_- u < 0$ on $AB \setminus \{A\}$ and $\partial_- u(A) = 0$.

Secondly, let the positive characteristics CB be given by the curve $\rho = f(\eta)$. Note that this curve is increasing, implying $f' > 0$. Recall that the characteristics AB is given by the parabola (3.3), which has vertex at $(-1, 2a)$ and the second derivative $1/2$. Since the point C is below this parabola, but above the line $\eta = 2a$, it is reasonable to expect that $f'' > 1/2$. For the rest of the paper we assume that $f'' > 1/2$. Since the positive characteristics CB satisfies the equation

$$f' = \frac{d\rho}{d\eta} = \sqrt{\rho - u} = \sqrt{f - u},$$

we obtain

$$u = f - (f')^2. \tag{3.17}$$

Therefore, along CB we have

$$\partial_+ u = \partial_\eta u + \sqrt{\rho - u}\partial_\rho u = f' - 2f'f'' = f'(1 - 2f'') < 0. \tag{3.18}$$

Next, we specify directional derivatives $\partial_+ u$ along AB and $\partial_- u$ along CB. To obtain $\partial_+ u$ along AB, we rewrite (3.10) as

$$-\partial_-\partial_+ u + Q\partial_- u\partial_+ u = Q(\partial_+ u)^2,$$

and, after diving by $(\partial_+ u)^2$, we obtain

$$\partial_-\Big(\frac{1}{\partial_+ u}\Big) + \frac{Q\partial_- u}{\partial_+ u} = Q. \tag{3.19}$$

We recall the expression for ρ along the characteristics AB in (3.14) and expressions for Q, u and $\partial_- u$ from (3.8), (3.15) and (3.16), respectively. Hence, along AB, (3.19) becomes

$$\frac{d}{d\eta}\frac{1}{\partial_+ u} - \frac{1}{\eta - 2a}\frac{1}{\partial_+ u} = \frac{1}{(\eta - 2a)^2}, \tag{3.20}$$

which is a linear ODE for $1/\partial_+ u$. Using (3.18), we impose the condition at the point B to be

$$\frac{1}{\partial_+ u}\Big(2a+\sqrt{2}\Big) = \frac{1}{f'(1-2f'')}\Big(2a+\sqrt{2}\Big). \tag{3.21}$$

The solution of (3.20) is given by

$$\frac{1}{\partial_+ u} = -\frac{1}{2(\eta-2a)} + C(\eta-2a),$$

where constant C is determined from the condition (3.21).

To obtain $\partial_- u$ along CB, similarly as above, we rewrite (3.9) as

$$\partial_+\Big(\frac{1}{\partial_- u}\Big) + \frac{Q\partial_+ u}{\partial_- u} = Q.$$

We recall that $\rho = f(\eta)$ along CB, and using expressions for u, $\partial_+ u$ and Q from (3.17), (3.18) and (3.8), respectively, the above equation along CB becomes

$$\frac{d}{d\eta}\Big(\frac{1}{\partial_- u}\Big) + \frac{1-2f''}{4f'}\frac{1}{\partial_- u} = \frac{1}{4(f')^2}.$$

Using (3.16), we impose a condition for $1/\partial_- u$ at the point B to be

$$\frac{1}{\partial_- u}(2a+\sqrt{2}) = -2\sqrt{-f(2a+\sqrt{2})}.$$

Solution of this linear ODE problem gives $\partial_- u$ along CB.

4. Existence of a local smooth solution to the Goursat problem and properties of smooth solutions

In this section we prove existence of a smooth local solution st the point B to the Goursat problem (3.12) with conditions (3.13) along characteristics AB and CB. In the series of Lemmas that follow, we establish the signs of the directional derivatives $\partial_\pm u$ in the domain ABC, their monotonicity along the characteristic curves, global minimum of $\partial_\pm u$, convexity/concavity of positive/negative characteristics, existence of a sonic curve AC and an estimate of the C^1 norm of W away from the sonic curve.

Theorem 4.1. *The Goursat problem* (3.12)-(3.13) *has a smooth local solution at the point B.*

Proof. The eigenvalues of (3.12) are $\lambda_1 = -\lambda^{-1}$, $\lambda_2 = \lambda^{-1}$ and $\lambda_3 = 0$, and the corresponding eigenvectors are

$$\vec{l}_1 = (1,0,0), \quad \vec{l}_2 = (0,1,0) \quad \text{and} \quad \vec{l}_3 = (0,0,1).$$

To prove the theorem, as in [10, 18], it suffices to check compatibility conditions at the point B (see [20, §4.3])

$$W_+(B) = W_-(B)$$

and

$$\frac{\vec{l}_3(B)W_-'(B) - \frac{\partial_+ u(B)+\partial_- u(B)}{2}}{\lambda_1(B)-\lambda_3(B)} = \frac{\vec{l}_3(B)W_+'(B) - \frac{\partial_+ u(B)+\partial_- u(B)}{2}}{\lambda_2(B)-\lambda_3(B)}.$$

The first condition is clearly satisfied. Noting that $\vec{l}_3(B)W_-'(B) = \partial_- u(B)$ and $\vec{l}_3(B)W_+'(B) = \partial_+ u(B)$, and recalling the definition of λ^{-1} in (3.4), implies that the second condition is also satisfied. □

Lemma 4.2. *Let u be a smooth solution of the Goursat problem for* (3.2) *in the domain ABC with conditions* (3.15) *and* (3.17) *along characteristics AB and CB, respectively. Then*

(a) $\partial_- u < 0$ *and* $\partial_+ u < 0$ *in the interior of ABC,*

(b) $\partial_- u$ *is increasing (decreasing) function along λ_+ characteristics if*

$$\partial_+ u \geq \partial_- u \quad (\partial_+ u \leq \partial_- u),$$

(c) $\partial_+ u$ *is decreasing (increasing) function along λ_- characteristics if*

$$\partial_+ u \geq \partial_- u \quad (\partial_+ u \leq \partial_- u).$$

Proof. Proof of (a) was partially given in [7] where we analyzed the problem only in a neighborhood of the point A, and for completeness we recall it here. Let P be a point in the domain ABC and let P_1 and P_2 be intersections of positive characteristics through P with AB and negative characteristics through P with BC, respectively (Figure 4).

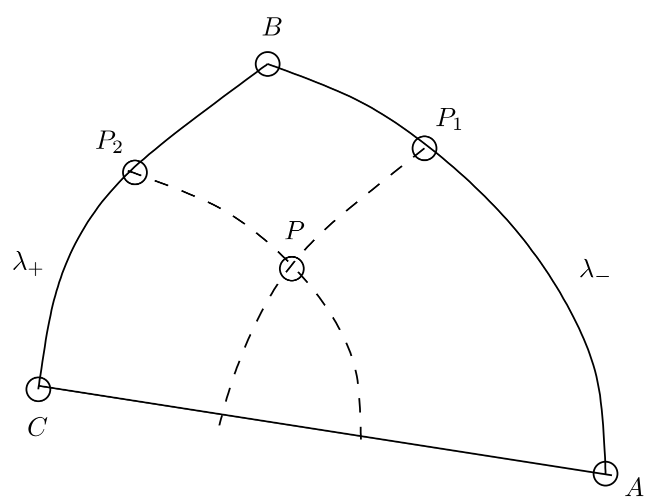

FIGURE 4. Interior point P with its $\lambda_\pm$ characteristics

We recall the characteristic decomposition (3.9)-(3.10) and that $\partial_- u < 0$ and $\partial_+ u < 0$ hold on $AB \backslash \{A\}$ and CB, respectively, from (3.16) and (3.18). Integrating equation (3.9) along positive characteristics PP_1, we obtain

$$\partial_- u(P) = \partial_- u(P_1) e^{\int_P^{P_1} Q(\partial_+ u - \partial_- u) ds}. \tag{4.1}$$

Since $\partial_- u(P_1) < 0$, we obtain $\partial_- u(P) < 0$. Similarly, integrating (3.10) along negative characteristics PP_2, we obtain

$$\partial_+ u(P) = \partial_+ u(P_2) e^{\int_P^{P_2} Q(\partial_- u - \partial_+ u) ds}. \tag{4.2}$$

Using that $\partial_+ u(P_2) < 0$, we obtain $\partial_+ u(P) < 0$.

Parts (b) and (c) follow immediately from (4.1) and (4.2), noting that the sign of the expressions in the exponent depends on the sign of $\partial_+ u - \partial_- u$, since $Q > 0$. $\square$

Lemma 4.3. *Let u be a smooth solution of the Goursat problem for the equation* (3.2) *in the domain ABC with conditions* (3.15) *and* (3.17) *along characteristics AB and CB, respectively. If the parameter a in* (2.2) *is chosen so that $a > 1$, then Λ_- characteristics are concave and Λ_+ characteristics are convex in the region ABC.*

Proof. We remark that the maximum of ξ is at the point A and, that (according to the part (a) of the previous Lemma, the minimum of u is at the point B. Therefore

$$\xi - u \leq -a^2 + 1 < 0,$$

if $a > 1$.

To complete the proof, we recall the relationship (3.6) between $\partial_\pm u$ and $\partial^\pm \Lambda_\pm$. First, we claim that

$$\Lambda_- > -\frac{2}{\eta}. \tag{4.3}$$

Using the expression for Λ_- from (2.4) and rearranging the terms in (4.3), since $\xi - u < 0$, we obtain that proving (4.3) is equivalent to proving

$$-\eta\sqrt{\eta^2 + 4(\xi - u)} < -\eta^2 - 4(\xi - u). \tag{4.4}$$

Note that the right-hand side in the above inequality is $-(\eta^2 + 4(\xi - u))$, which is negative by hyperbolicity. Hence, to show (4.4) we need to show

$$\left(\eta\sqrt{\eta^2 + 4(\xi - u)}\right)^2 > (\eta^2 + 4(\xi - u))^2.$$

After simple calculations, the above inequality is equivalent to

$$\eta^2 + 4(\xi - u) > 0,$$

which is, again, satisfied by hyperbolicity. Since $\partial_- u < 0$ in ABC, using claim the (4.3), from (3.6), we obtain that

$$\partial^- \Lambda_- < 0,$$

implying that Λ_- characteristics are concave in ABC.

Similar calculations show that

$$\Lambda_+ < -\frac{2}{\eta},$$

implying, from (3.6) and the fact that $\partial_+ u < 0$ in ABC, that

$$\partial_+ \Lambda^+ > 0,$$

i.e., Λ^+ characteristics are convex in ABC. □

Lemma 4.4. *If u is a smooth solution of the Goursat problem for* (3.2) *in the domain ABC with conditions* (3.15) *and* (3.17) *along characteristics AB and CB, respectively, then λ_+ characteristics are convex and λ_- characteristics are concave in the domain ABC.*

Proof. Consider a positive characteristics

$$\frac{d\rho}{d\eta} = \sqrt{\rho - u}.$$

Then

$$\frac{d^2\rho}{d\eta^2} = \frac{\sqrt{\rho - u} - \partial_+ u}{2\sqrt{\rho - u}} > \frac{1}{2} > 0,$$

by Lemma 4.2, implying that positive characteristics are convex.

Next, let a negative characteristics be given by $\rho = g(\eta)$. Then

$$g' = \frac{d\rho}{d\eta} = -\sqrt{\rho - u} = -\sqrt{g - u},$$

implying, after solving for u and differentiating along this characteristics, that

$$\frac{\partial_- u}{g'} = 1 - 2g''. \tag{4.5}$$

We claim that

$$\frac{\partial_- u}{g'} > 2. \tag{4.6}$$

Using (3.9) and expression for Q in (3.8), we obtain

$$\begin{aligned}
\partial_+\Big(\frac{\partial_- u}{g'}\Big) &= -\partial_+\Big(\frac{\partial_- u}{\sqrt{\rho - u}}\Big) \\
&= -\frac{Q(\partial_+ u - \partial_- u)\partial_- u\,\sqrt{\rho - u} - \frac{\partial_- u}{2\sqrt{\rho-u}}(\sqrt{\rho - u} - \partial_+ u)}{\rho - u} \\
&= -\frac{1}{4}\frac{\partial_- u}{\sqrt{\rho - u}}\Big(\frac{\partial_+ u}{\rho - u} - \frac{\partial_- u}{\rho - u}\Big) + \frac{\partial_- u}{2(\rho - u)} - \frac{\partial_- u\,\partial_+ u}{2\sqrt{\rho - u}(\rho - u)} \\
&= \frac{\partial_- u}{g'}\Big\{\frac{1}{4}\Big(\frac{\partial_+ u}{\rho - u} - \frac{\partial_- u}{\rho - u}\Big) - \frac{1}{2\sqrt{\rho - u}} + \frac{\partial_+ u}{2(\rho - u)}\Big\}.
\end{aligned} \tag{4.7}$$

We consider two cases.

- If

$$\frac{1}{4}\Big(\frac{\partial_+ u}{\rho - u} - \frac{\partial_- u}{\rho - u}\Big) - \frac{1}{2\sqrt{\rho - u}} + \frac{\partial_+ u}{2(\rho - u)} < 0,$$

then (4.7) implies

$$\partial_+\Big(\frac{\partial_- u}{g'}\Big) < 0,$$

since $\partial_- u < 0$ and $g' < 0$. Hence, $\frac{\partial_- u}{g'}$ decreases along positive characteristics. For an interior point P, let S be the intersection of positive characteristics through P and the curve AB. Then

$$\frac{\partial_- u}{g'}(P) > \frac{\partial_- u}{-\sqrt{\rho - u}}(S) = \frac{2\sqrt{-\rho}}{\sqrt{\rho - u}}(S) = 2,$$

using that along AB we have $u = 2\rho$ and $\partial_- u = -2\sqrt{-\rho}$ from (3.15) and (3.16), respectively. Hence, the claim (4.6) is proved.

- If

$$\frac{1}{4}\Big(\frac{\partial_+ u}{\rho - u} - \frac{\partial_- u}{\rho - u}\Big) - \frac{1}{2\sqrt{\rho - u}} + \frac{\partial_+ u}{2(\rho - u)} \geq 0,$$

then

$$\frac{1}{4}\frac{\partial_- u}{g'} \geq \Big(\frac{1}{2\sqrt{\rho - u}} - \frac{\partial_+ u}{2(\rho - u)} - \frac{1}{4}\frac{\partial_+ u}{\rho - u}\Big)\sqrt{\rho - u} > \frac{1}{2},$$

since $\partial_+ u < 0$ inside ABC. Hence,

$$\frac{\partial_- u}{g'} > 2,$$

and the claim (4.6) is proved.

Therefore, (4.5) implies
$$g'' < -\frac{1}{2} < 0,$$
and, as a consequence, λ_- characteristics are concave in the region ABC. □

Lemma 4.5. *Let u be a smooth solution of the Goursat problem* (3.2) *in the domain ABC with conditions* (3.15) *and* (3.17) *along characteristics AB and CB. Then*

(a) *$\rho - u$ is increasing along positive characteristics,*
(b) *$\rho - u$ is increasing along negative characteristics,*
(c) *level curves of $\rho - u$ are non-characteristic, and*
(d) *AC is a sonic curve and $\partial_+ u = \partial_- u$ on AC.*

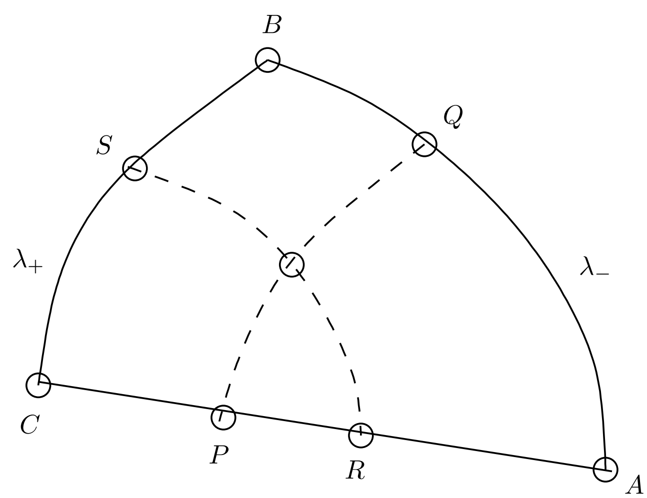

Figure 5. $\lambda_\pm$ characteristics inside the region ABC

Proof. To prove part (a), let PQ be a positive characteristics (Figure 5). Then, along PQ, we have
$$\partial_+(\rho - u) = \partial_+\rho - \partial_+ u > 0,$$
since $\partial_+\rho = \sqrt{\rho - u} > 0$ and $\partial_+ u < 0$, by the previous Lemma.

To prove part (b), let RS be a negative characteristics (Figure 5), given by $\rho = g(\eta)$. Since g is decreasing and concave, we have $g' < 0$ and $g'' < 0$. Then along RS we have
$$g' = \frac{d\rho}{d\eta} = -\sqrt{\rho - u} = -\sqrt{g - u},$$
and, therefore, $u = g - (g')^2$, yielding $\partial_- u = g' - 2g'g''$. Hence
$$\partial_-(\rho - u) = 2g'g'' > 0,$$
implying $\rho - u$ is increasing along RS.

For part (c), clearly, the level curve of the function $\rho - u$, by (a) and (b), must be noncharacteristic.

Finally, to prove part (d), we remark that since $\rho - u$ is increasing along both characteristic families, the level curve AC ($\rho - u = 0$) consists of sonic points only. Further, using the definition of derivatives $\partial_\pm$ from (3.5), since along the sonic curve AC we have $\rho = u$, we obtain $\partial_+ u = \partial_- u$. □

Lemma 4.6. *For a smooth solution u of the Goursat problem* (3.2) *in the domain $\Omega = ABC$, with conditions* (3.15) *and* (3.17) *along characteristics AB and CB, we have*

$$m := \min_{AB\cup CB} \partial_\pm u \le \min_{\Omega} \partial_\pm u \le 0. \tag{4.8}$$

Proof. As in [10, 18], we prove (4.8) in two cases.

• First, assume $\partial_+ u \ge \partial_- u$ in the entire domain Ω. Then, by (b) and (c) of Lemma 4.2, we have that $\partial_- u$ increases along positive characteristics and $\partial_+ u$ decreases along negative characteristics. Let $T \in \Omega$ be an arbitrary point and let S be the intersection of negative characteristics through T with the curve CB (Figure 6). We have

$$\partial_+ u(T) \ge \partial_+ u(S) \ge m.$$

Also, let P be the intersection of positive characteristics through T with the sonic curve AC. Then

$$\partial_- u(T) \ge \partial_- u(P) = \partial_+ u(P),$$

by Lemma 4.5, part (d). Now consider the negative characteristics at P and assume it intersects CB at the point S_1. Since $\partial_+ u$ decreases along negative characteristics we have

$$\partial_+ u(P) \ge \partial_+ u(S_1) \ge m.$$

Hence $\partial_\pm u(T) \ge m$.

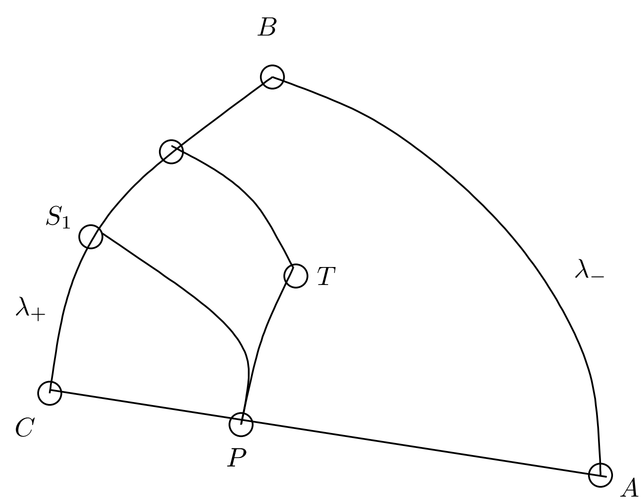

FIGURE 6. Case 1: $\partial_+ u \ge \partial_- u$ in Ω

• Secondly, we assume that there is a point in Ω such that $\partial_+ u < \partial_- u$ at that point. By the continuity, let $\mathcal{N}$ be the largest neighborhood of this point, inside

Ω, such that $\partial_+ u < \partial_- u$ in $\mathcal{N}$. Let $Z \in \overline{\mathcal{N}}$. By Lemma 4.2, $\partial_- u$ decreases along positive characteristics in $\mathcal{N}$. Let $Z_1 \in \partial N$ be the point on positive characteristics through Z, as in Figure 7. Then

$$\partial_- u(Z) \geq \partial_- u(Z_1) = \partial_+ u(Z_1).$$

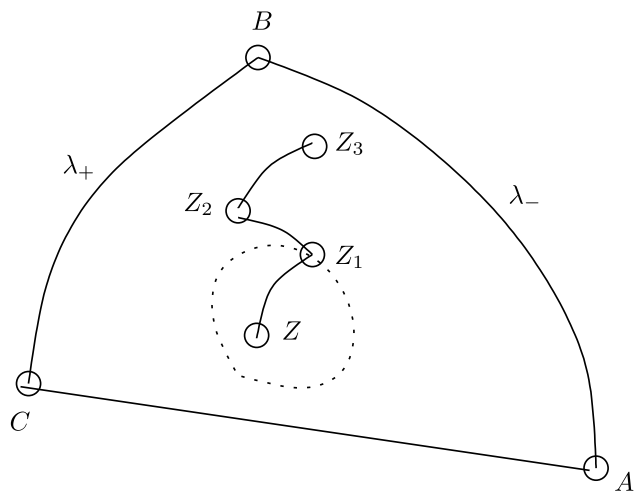

FIGURE 7. Case 2: $\partial_+ u < \partial_- u$ in $\mathcal{N}$

In the neighborhood of Z_1 outside of $\mathcal{N}$ we have $\partial_+ u \geq \partial_- u$ and we draw negative characteristics at Z_1 in the direction of increasing η. For any point Z_2 on this characteristics, we have

$$\partial_+ u(Z_1) \geq \partial_+ u(Z_2),$$

by Lemma 4.2. If this negative characteristics intersects the curve CB, (4.8) is proved for $\partial_- u(Z)$. Otherwise, we draw this characteristics as long as $\partial_+ u \geq \partial_- u$ and assume that Z_2 is the ending point. Next, we draw positive characteristics starting at the point Z_2, in the direction of increasing η, along which $\partial_+ u < \partial_- u$. For any point Z_3 along this curve, we have

$$\partial_+ u(Z_2) = \partial_- u(Z_3) \geq \partial_- u(Z_3).$$

If this positive characteristics intersects the curve AB, (4.8) is proved for $\partial_- u(Z)$. Otherwise, we draw this positive characteristics as long as $\partial_+ u < \partial_- u$ ending at the point Z_3. From the point Z_3, we continue by following negative characteristics. We repeat the procedure, by following positive or negative characteristics, depending on the sign of $\partial_+ u - \partial_- u$, until we reach either AB or CB. Hence, $\partial_- u(Z) \geq m$.

To prove $\partial_+ u(Z) \geq m$, we recall that $\partial_+ u$ increases along negative characteristics that are in $\mathcal{N}$. Let $Z_1 \in \partial \mathcal{N}$ be the point on the negative characteristics through Z as in Figure 8. Then

$$\partial_+ u(Z) \geq \partial_+ u(Z_1) = \partial_- u(Z_1).$$

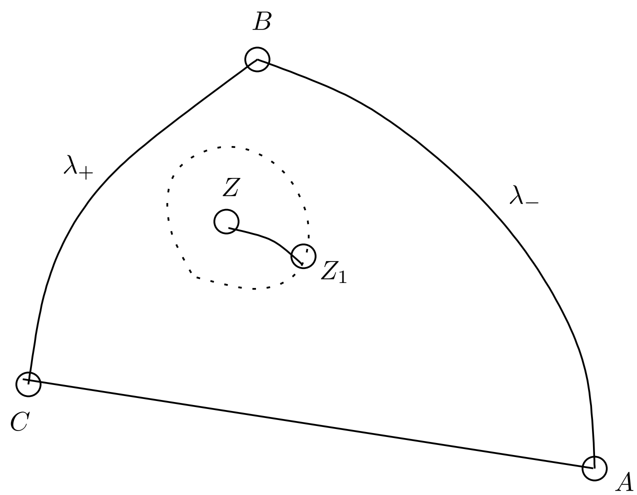

FIGURE 8. Case 2: $\partial_+ u < \partial_- u$ in $\mathcal{N}$

Since $Z_1 \in \overline{\mathcal{N}}$, by the previous part of the proof, we have

$$\partial_- u(Z_1) \geq m.$$

Therefore, $\partial_\pm u(Z) \geq m$. ☐

Lemma 4.7. *Let $\Omega_\epsilon = \{(\rho,\eta) \in ABC : \rho - u > \epsilon\}$ and let $W \in C^1(\Omega_\epsilon)$ be a solution of the Goursat problem* (3.12) *with conditions* (3.13) *along characteristics AB and CB. Then*

$$\|W\|_{C^1(\Omega_\epsilon)} \leq \frac{C}{\epsilon^{5/2}},$$

for a constant C independent of ϵ.

Proof. From equations (3.7) and (3.11) and Lemma 4.5, we have

$$|u_\rho| = \Big|\frac{\partial_+ u - \partial_- u}{2\sqrt{\rho - u}}\Big| \leq \frac{|m|}{\sqrt{\epsilon}}$$

and

$$|u_\eta| = \Big|\frac{\partial_+ u + \partial_- u}{2}\Big| \leq |m|,$$

where m is as in (4.8). Also, (3.9)-(3.10) imply

$$|\partial_\pm \partial_\mp u| = |Q(\partial_\pm u - \partial_\mp u)\partial_\mp u| = \Big|\frac{1}{4(\rho - u)}(\partial_\pm u - \partial_\mp u)\partial_\mp u\Big| \leq \frac{|m|^2}{2\epsilon}. \tag{4.9}$$

Next, we obtain estimates on $\partial_\pm \partial_\pm u$. As in [10, 18], we use the commutator relation from [20, Proposition 2.1]

$$\partial_- \partial_+ I - \partial_+ \partial_- I = \frac{\partial_- \lambda^{-1} - \partial_+ (-\lambda)^{-1}}{(-\lambda)^{-1} - \lambda^{-1}}(\partial_- I - \partial_+ I),$$

where I is any quantity, and $\partial_\pm$ are directional derivatives along characteristics $d\rho/d\eta = (\pm\lambda)^{-1}$. We use this relation for $\partial_- u$ and obtain

$$\partial_-\partial_+\partial_- u - \partial_+\partial_-\partial_- u = \frac{\partial_-\lambda^{-1} - \partial_+(-\lambda)^{-1}}{(-\lambda)^{-1} - \lambda^{-1}}(\partial_-\partial_- u - \partial_+\partial_- u). \tag{4.10}$$

On the other hand, by differentiating expression (3.9) along negative characteristics, we obtain

$$\partial_-\partial_+\partial_- u = \partial_- Q(\partial_+ u - \partial_- u)\partial_- u + Q(\partial_-\partial_+ u - \partial_-\partial_- u)\partial_- u + Q(\partial_+ u - \partial_- u)\partial_-\partial_- u.$$

By substituting this in (4.10), we obtain

$$\begin{aligned}
\partial_+\partial_-\partial_- u = \Big\{ &- \frac{\partial_-\lambda^{-1} - \partial_+(-\lambda)^{-1}}{(-\lambda)^{-1} - \lambda^{-1}} + Q(\partial_+ u - 2\partial_- u)\Big\}\partial_-\partial_- u \\
&+ \frac{\partial_-\lambda^{-1} - \partial_+(-\lambda)^{-1}}{(-\lambda)^{-1} - \lambda^{-1}} Q(\partial_+ u - \partial_- u)\partial_- u \\
&+ \partial_- Q(\partial_+ u - \partial_- u)\partial_- u \\
&+ Q^2(\partial_- u - \partial_+ u)\partial_+ u\,\partial_- u.
\end{aligned} \tag{4.11}$$

We recall that

$$\lambda^{-1} = \sqrt{\rho - u}, \quad (-\lambda)^{-1} = -\sqrt{\rho - u}, \quad (-\lambda)^{-1} - \lambda^{-1} = -2\sqrt{\rho - u},$$

and we compute $\partial_-\rho = -\sqrt{\rho - u}$ and $\partial_+\rho = \sqrt{\rho - u}$. Therefore,

$$\partial_-\lambda^{-1} = \frac{1}{2\sqrt{\rho - u}}(\partial_-\rho - \partial_- u) = -\frac{\sqrt{\rho - u} + \partial_- u}{2\sqrt{\rho - u}},$$

and, similarly,

$$\partial_+(-\lambda)^{-1} = -\frac{\sqrt{\rho - u} - \partial_+ u}{2\sqrt{\rho - u}}.$$

Hence,

$$\frac{\partial_-\lambda^{-1} - \partial_+(-\lambda)^{-1}}{(-\lambda)^{-1} - \lambda^{-1}} = \frac{\partial_- u + \partial_+ u}{4(\rho - u)}.$$

Also, we recall the expression for Q in (3.8), and find

$$\partial_- Q = -\frac{1}{4(\rho - u)^2}(\partial_-\rho - \partial_- u) = \frac{\sqrt{\rho - u} + \partial_- u}{4(\rho - u)^2}.$$

By substituting these expressions in (4.11), we obtain

$$\partial_+\partial_-\partial_- u = -\frac{3}{4(\rho - u)^2}\partial_- u\,\partial_-\partial_- u + \frac{\partial_- u(\partial_+ u - \partial_- u)}{4(\rho - u)^2}\Big\{\frac{5\partial_- u}{4} + \sqrt{\rho - u}\Big\},$$

which is a differential equation for $\partial_-\partial_- u$ along positive characteristics. By integrating this equation along positive characteristics, we obtain

$$|\partial_-\partial_- u| \le \frac{C}{\epsilon^2} \quad \text{on } \Omega_\epsilon.$$

Similarly,

$$|\partial_+\partial_+ u| \le \frac{C}{\epsilon^2} \quad \text{on } \Omega_\epsilon.$$

The last two estimates together with (4.9), (3.11) and (3.7), give

$$|(\partial_\pm u)_\eta| = |\partial_\pm(u_\eta)| = \Big|\partial_\pm\Big(\frac{\partial_+ u - \partial_- u}{2}\Big)\Big| \le \frac{C}{\epsilon^2},$$

$$|(\partial_{\pm}u)_\rho| = |\partial_{\pm}(u_\rho)| = \Big|\partial_{\pm}\Big(\frac{\partial_+ u - \partial_- u}{2\sqrt{\rho - u}}\Big)\Big| \le \frac{C}{\epsilon^{5/2}}.$$

□

Analogous properties for solutions to the pressure-gradient system and the nonlinear wave system to those obtained in Lemmas 4.2–4.7, are used to extend the local smooth solution at the point B to the whole domain ABC using techniques from [8] by Dai and Zhang for gas dynamics equations. We believe that these techniques could be extended for the UTSD equation as well and we leave that study for the future work.

5. The envelope of λ_- characteristics and existence of a shock

As in [10, 18], we conclude by showing that one family of characteristics (in this case, negative characteristics) forms an envelope before they become sonic. We use [2, Theorem 3.2], which states that for 2×2 quasi-one-dimensional systems (such as (2.3)), a hyperbolic state adjacent to a hyperbolic constant state must be a simple wave (also, see the argument using Riemann invariants in §2 in [15]).

Hence, the solution in the region below the curve BD, to the left of CB (see Figure 2), is a simple wave. In our case, the family of negative characteristics are straight lines along which u is constant. Therefore, in this region $\partial_- u = 0$ and

$$\partial_-\Big(\frac{1}{\partial_+ u}\Big) = -\frac{\partial_-\partial_+ u}{(\partial_+ u)^2} = -\frac{Q(\partial_- u - \partial_+ u)\partial_+ u}{(\partial_+ u)^2} = Q = \frac{1}{4(\rho - u)} > 0, \tag{5.1}$$

by (3.10). Hence, $\frac{1}{\partial_+ u}$ is increasing along negative characteristics in the direction from B to D. We recall from (3.18) that $\frac{1}{\partial_+ u} < 0$ along the curve CB. Also note from (5.1) that $\frac{1}{\partial_+ u}$ becomes infinite at the sonic points, since $\rho = u$ there. Hence, $\frac{1}{\partial_+ u}$ must become zero before the negative characteristics reach their sonic points. Consequently, $\partial_+ u$ becomes infinite before negative characteristics become sonic indicating that negative characteristics form an envelope before their sonic points. As in [18] and in §9 in [21], this indicates existence of a shock.

Conclusion. In this article we consider a Riemann problem for the unsteady transonic small disturbance equation resulting in diverging rarefaction waves. We write the problem in self-similar and parabolic coordinates, and obtain a system that changes type from hyperbolic to elliptic. We find the solution in the hyperbolic region and we formulate a problem in a semi-hyperbolic patch where one family out of two families of characteristic curves starts at a sonic curve and ends on a transonic shock. We prove existence of a smooth local solution and we derive various properties of the global solution in the semi-hyperbolic patch adopting the ideas of characteristic decomposition for the nonlinear wave system and the pressure gradient system in [10, 18].

Our future work consists of three main steps: (1) computing the numerical approximations of the global solution to the problem using the numerical method developed in [11], (2) using the approach in [8] to prove existence of the solution in the entire semi-hyperbolic patch, and (3) establishing the iteration scheme to prove existence of a global solution, as outlined for the nonlinear wave system in [12].

References

[1] M. Brio, J. K. Hunter; *Mach reflection for the two dimensional Burgers equation*, Phys. D, **60** (1992), 194–207.

[2] S. Čanić, B. L. Keyfitz; *Quasi-one dimensional Riemann problems and their role in self-similar two-dimensional problems*, Arch. Rational Mech. Anal., **144** (1998), 233–258.

[3] S. Čanić, B. L. Keyfitz, E. H. Kim; *Free boundary problems for the unsteady transonic small disturbance equation: transonic regular reflection*, Method. Appl. Anal, **7** (2000), 313–336.

[4] S. Čanić, B. L. Keyfitz, E. H. Kim; *Mixed hyperbolic-elliptic systems in self-similar flows*, Boletim da Sociedade Brasileira de Mathemática, **32** (2001), 377–399.

[5] S. Čanić, B. L. Keyfitz, E. H. Kim; *A free boundary problem for a quasilinear degenerate elliptic equation: regular reflection of weak shocks*, Comm. Pure Appl. Math., **LV** (2002), 71–92.

[6] S. Čanić, B. L. Keyfitz, G. Lieberman; *A proof of existence of perturbed steady transonic shocks via a free boundary approach* Comm. Pure and Appl. Math., **LIII** (2000), 484–511.

[7] J. Chen, C. Christoforou, K. Jegdić; *Rarefaction wave interaction for the unsteady transonic small disturbance equations*, Proceedings of The 15th American Conference on Applied Mathematics, University of Houston – Downtown (2009), 211–216.

[8] Z. H. Dai, T. Zhang; *Existence of a global smooth solution for a degenerate Goursat problem of gas dynamics*, Arch. Ration. Mech. Anal., **155** (2000), 277–298.

[9] J. Glimm, X. Ji, J. Li, X. Li, P. Zhang, T. Zhang, Y. Zheng; *Transonic shock formation in a rarefaction Riemann problem for the 2D compressible Euler equations*, SIAM J. Appl. Math., **69** (2008), 720–742.

[10] Y. Hu, G. Wang; *Semi-hyperbolic patches of solutions to the two-dimensional nonlinear wave system for Chaplygin gasses*, J. Differential Equations, **257** (2014), 1567–1590.

[11] I. Jegdić; *Large time step and overlapping grids for conservation laws*, Ph.D. Thesis, University of Houston (2014).

[12] I. Jegdić, K. Jegdić; *Remarks on diverging rarefaction waves interactions for the nonlinear wave system*, Journal of Coupled Systems and Multiscale Dynamics, **3** (2015), 87-93.

[13] K. Jegdić, B. L. Keyfitz, S. Čanić; *Transonic regular reflection for the unsteady transonic small disturbance equation - details of the subsonic solution*, Free and Moving Boundaries Analysis, Simulation and Control: R. Glowinski, J. P. Zolesio (Eds.), Lect. Notes Pure Appl. Math., Chapman and Hall/CRC, Boca Raton, FL **252** (2007), 125–165.

[14] B. L. Keyfitz; *Self-similar solutions of two-dimensional conservation laws*, J. Hyp. Diff. Eq., **1** (2004), 445–492.

[15] J. Li, T. Zhang, Y. Zheng; *Simple waves and a characteristic decomposition of the two dimensional compressible Euler equations*, Communications in Mathematical Physics, **267** (2006), 1–12.

[16] M. J. Li, Y. Zheng; *Semi-hyperbolic patches of solutions of the two-dimensional Euler equations*, Arch. Ration. Mech. Anal., **201** (2011), 1069–1096.

[17] W. Sheng, G. Wang, T. Zhang; *Diffraction of a planar rarefaction wave along a compressive corner*, Research Report (2009).

[18] K. Song, Y. Zheng; *Semi-hyperbolic patches of solutions of the pressure gradient system*, Discrete Contin. Dyn. Syst., **24** (2009), 1365–1380.

[19] A. M. Tesdall, B. L. Keyfitz; *A continuous two-way free boundary in the unsteady transonic small disturbance equations*, J. Hyp. Diff. Eq., **7** (2010), 317–338.

[20] Y. Zheng; *The compressible Euler system in two space dimensions*, Lecture notes at Fudan Summer School (2007).

[21] Y. Zheng; *Two-dimensional Riemann problems for the compressible Euler system*, Chinese Annals of Mathematics, Series B, **30 B(6)** (2009), 845–858.

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