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TWO-SPECIES COMPETITION MODELS WITH FITNESS-DEPENDENT DISPERSAL ON NON-CONVEX BOUNDED DOMAINS

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Abstract. In this article, we show the existence of global bounded solutions to a two-species competition models with fitness-dependent dispersal posed in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. In particular, we remove the convexity assumption on Ω used by Lou-Tao-Winkler [13].

1. Introduction

In this article, we show the existence and boundedness of global solutions to the two-species competition model with fitness-dependent dispersal

$$
\begin{gathered}
u_t = \nabla \cdot [\mu \nabla u - \alpha u \nabla (m - u - w)] + r_1 u(m - u - w), \quad x \in \Omega,\ t > 0,\\
w_t = \nu \Delta w + r_2 w(m - u - w), \quad x \in \Omega,\ t > 0,\\
[\mu \nabla u - \alpha u \nabla (m - u - w)] \cdot n = \nu \nabla w \cdot n = 0, \quad x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}
$$

where $\mu, \nu, \alpha > 0, r_1 \geq 0, r_2 \geq 0$, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and n denotes the outer unite normal of $\partial\Omega$. The functions $u(x,t)$ and $w(x,t)$ describe the densities of two competing species at time t, at location $x \in \Omega$, and $m(x)$ denotes the distribution of resources. Equation $(1.1)_1$ indicates that the dispersal of organism with density u is dependent on a combination of random motion with random dispersal rate μ and advection upward along its fitness gradient, while equation $(1.1)_2$ indicates that the dispersal of organism with density w is purely random. Moreover, the growth of both species in (1.1) is logistic, with logistic growth rate r_1 and r_2, respectively.

In recent years, equations (1.1) and their variations have been studied by many researchers (see [1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 19] and references therein). To motivate our study, we recall several related ones. Cosner [5] first considered the following fitness-dependent dispersal model for a single species

$$
\begin{gathered}
u_t = \nabla \cdot [\mu \nabla u - \alpha u \nabla (m - u)] + ru(m - u), \quad x \in \Omega,\ t > 0,\\
[\mu \nabla u - \alpha u \nabla (m - u)] \cdot n = 0, \quad x \in \Omega,\ t > 0.
\end{gathered} \tag{1.2}
$$

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Then Cantrell-Cosner-Lou [3] further investigated the global existence of classical solution and the behavior of equilibria to equation (1.2). Recently, Cantrell et al [4] extended the work in [5] to the two-species competition model

$$\begin{gathered} u_t = \nabla\cdot[\mu\nabla u - \alpha u\nabla(m-u-w)] + ru(m-u-w), \quad x\in\Omega,\ t>0, \\ w_t = \nu\Delta w + rw(m-u-w) \quad x\in\Omega,\ t>0, \\ [\mu\nabla u - \alpha u\nabla(m-u-w)]\cdot n = \nu\nabla w\cdot n = 0, \quad x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad w(x,0)=w_0(x), \quad x\in\Omega, \end{gathered} \tag{1.3}$$

and showed that the solutions to equations (1.3) exist globally for $N=1,2$, and for $N\geq 3$ with $\nu>\mu$. They also investigated the nontrivial nonnegative steady states. More recently, Lou et al [13] proved that the corresponding results hold for $N\geq 3$ and $\nu\leq\mu$ under the extra assumption that the domain Ω is convex. The global existence and large time behaviour of the nonnegative weak solution to the limit case (i.e., $\mu=\nu=0$) were also investigated by [13].

The main purpose of this article is to show the global-in-time existence and uniform-in-time boundedness of solutions to (1.1) on a *non-convex* bounded domain. For this purpose, we recall two basic assumptions used in [13]. The first one is related to the parameters and the distribution of resources:

$$m(x)\in C^{2+\gamma}(\overline{\Omega}) \quad \text{for some } \gamma\in(0,1), \text{ and } m(x_0)>0 \text{ for some } x_0\in\Omega. \tag{1.4}$$

The second one relates the initial data:

$$\begin{gathered} (u_0,w_0)\in C^{\gamma}(\overline{\Omega})\times W^{1,\infty}(\Omega) \text{ for some } \gamma\in(0,1), \\ \text{and } u_0(x)\geq 0,\ w_0(x)\geq 0 \text{ in } \overline{\Omega}. \end{gathered} \tag{1.5}$$

To obtain the uniqueness, we also need the following conditions:

$$\begin{gathered} (u_0,w_0)\in W^{s,p}(\Omega)\times W^{s,p}(\Omega) \text{ for some } p>N \text{ and } s>1, \\ \text{and } u_0(x)>0,\ w_0(x)\geq 0 \text{ in } \overline{\Omega}. \end{gathered} \tag{1.6}$$

We now state the main result of this paper as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^N (N\geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$. Then under the assumptions of* (1.4) *and* (1.5)*, equations* (1.1) *have at least one couple of nonnegative classical solutions (u,w) belonging to $C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$, which are bounded in $\Omega\times(0,\infty)$. If in addition (u_0,w_0) also satisfy* (1.6)*, then the solution is unique within the indicated class.*

The rest of this article is organized as follows. We first present the local existence and uniqueness of classical solutions, and some preliminaries in Section 2. Then we establish the global existence of bounded solutions and complete the proof of Theorem 1.1 in Section 3.

2. Preliminaries

In this section, we first present the existence and uniqueness of classical local solutions to (1.1) and then present some basic preliminaries.

Lemma 2.1 (Local existence and uniqueness)**.** *Under assumptions* (1.4) *and* (1.6)*, there exists a maximal existence time T^* and a unique functions pair $(u,w)\in$*

$C^0\big(\overline{\Omega}\times[0,T^*)\big)\cap C^{2,1}\big(\overline{\Omega}\times(0,T^*)\big)$ *such that* (u,w) *are classical solutions of equations* (1.1). *Moreover, if* $T^*<\infty$, *then*

$$\lim_{t\to T^*}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty. \tag{2.1}$$

The proof of the above lemma is standard and we refer to [4, 13] for details. The following boundary derivative estimate plays an important role when we remove the convexity assumption on the domain Ω used by [21].

Lemma 2.2 ([14, Lemma 4.2])**.** *Let* $\Omega\subset\mathbb{R}^N$ *be a bounded domain with smooth boundary* $\partial\Omega$. *If* $f\in C^2(\overline{\Omega})$ *satisfies* $\frac{\partial f}{\partial n}=0$, *then*

$$\frac{\partial|\nabla f|^2}{\partial n}\le C_\Omega|\nabla f|^2, \tag{2.2}$$

where C_Ω *is a positive upper bound for the curvatures of* $\partial\Omega$.

The following embedding theorem comes from [10, Proposition 4.22 (ii) and Theorem 4.24 (i)].

Lemma 2.3. *Let* Ω *be a bounded domain with smooth boundary and let* $r\in(0,\infty)$. *Then*

$$W^{r,2}(\partial\Omega)\hookrightarrow L^2(\partial\Omega)$$

is a compact embedding. Moreover, there exists a linear and bounded map from $W^{r+\frac12,2}(\Omega)$ *onto* $W^{r,2}(\partial\Omega)$.

The proof of global existence will be based on some a priori estimates. To derive these estimates, we will use the following two Gagliardo-Nirenberg inequalities [8, 15, 16].

Lemma 2.4. *Assume that* $u\in W^{1,2}(\Omega)\cap L^r(\Omega)$ *and* $r\in(0,k)$. *Then there exists a positive constant* C_{GN} *such that*

$$\|u\|_{L^k(\Omega)}\le C_{GN}\Big(\|\nabla n\|^{\theta}_{L^2(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}+\|u\|_{L^r(\Omega)}\Big)$$

holds, where $\theta\in(0,1)$ *satisfies*

$$\frac1k=\theta\Big(\frac12-\frac1N\Big)+\frac1r\Big(1-\theta\Big).$$

Lemma 2.5. *Let* $N\in\mathbb{N}$, $s\ge1$ *and* $l\ge1$. *Assume that* $p>0$ *and* $\theta\in(0,1)$ *satisfy*

$$\frac12-\frac pN=(1-\theta)\frac ls+\theta\Big(\frac12-\frac1N\Big)\quad\text{and}\quad p\le\theta. \tag{2.3}$$

Then there exists a positive constant C_0 *such that*

$$\|f\|_{W^{p,2}(\Omega)}\le C_0\|\nabla f\|^{\theta}_{L^2(\Omega)}\|f\|^{1-\theta}_{L^{\frac sl}(\Omega)}+C_0\|f\|_{L^{\frac sl}(\Omega)} \tag{2.4}$$

holds for all $f\in W^{1,2}(\Omega)\cap L^{\frac sl}(\Omega)$.

3. Proof of Theorem 1.1

In this section, we establish the existence of classical global solutions to (1.1). For this purpose, the key is to derive the uniform estimate of L^k norm of the solution. Inspired by an idea in [17, 18] (see also [13]), we establish a combined estimate on $\int_\Omega u^k(x,t)dx+\int_\Omega|\nabla w(x,t)|^{2l}dx$ for appropriately large k and l to obtain the expecting results. We first recall some basic properties of solutions.

Lemma 3.1 ([13, Lemma 2.2]). *Assume that $u_0(x) \in C^2(\overline{\Omega})$ is positive and $w_0(x) \in C^2(\overline{\Omega})$ is nonnegative. Then the classical solution (u, w) to equations (1.1) satisfies the following inequalities:*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le \max\{\|u_0\|_{L^1(\Omega)}, |\Omega|\|m\|_{L^1(\Omega)}\}, \tag{3.1}$$

$$\|w(\cdot,t)\|_{L^1(\Omega)} \le \max\{\|w_0\|_{L^1(\Omega)}, |\Omega|\|m\|_{L^1(\Omega)}\}, \tag{3.2}$$

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \max\{\|u_0\|_{L^1(\Omega)}, |\Omega|\|m\|_{L^\infty(\Omega)}\} \tag{3.3}$$

for all $t \in (0, T^*)$. Moreover, for any $s \in [1, \frac{N}{N-1})$, there exists a positive constant $C(s)$ such that

$$\|w(\cdot,t)\|_{W^{1,s}(\Omega)} \le C(s)\Big(1 + \|u_0\|_{L^1(\Omega)} + \|w_0\|_{W^{1,\infty}(\Omega)}\Big) \quad \text{for all } t \in (0, T^*). \tag{3.4}$$

The following Lemma asserts the L^k-boundedness of solutions, which is the core of the argument concerning the global existence and boundedness. Our proof followed from [13, Lemma 2.5], but we will use the boundary derivative estimates and the Sobolev trace embedding to remove the convexity assumption on the domain Ω used by [13].

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that μ, ν, α, r_1, r_2 and $m(x)$ satisfy (1.4). Then for all $k > 2$ and $l > 2$, there exist two positive constants C_k and C_{2l} depending only on k, l, $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$ such that the solution (u, w) to equations (1.1) emanating from some initial data $(u_0, w_0) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ satisfies*

$$\|u(\cdot,t)\|_{L^k(\Omega)} \le C_k \quad \text{for all } t \in (0, T^*), \tag{3.5}$$

$$\|\nabla w(\cdot,t)\|_{L^{2l}(\Omega)} \le C_{2l} \quad \text{for all } t \in (0, T^*). \tag{3.6}$$

Proof. The L^1-boundedness of $u(\cdot, t)$ has been obtained in the proof of Lemma 3.1. Thus by the interpolation, we may pay our attention to the case that $k > 2$. Multiplying $(1.1)_1$ by ku^{k-1} and integrating the resulted equation over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^k dx \\
&= -\mu k(k-1)\int_\Omega u^{k-2}|\nabla u|^2 dx - \alpha k(k-1)\int_\Omega u^{k-1}|\nabla u|^2 dx \\
&\quad + \alpha k(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla m dx - \alpha k(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla w dx \\
&\quad + kr_1\int_\Omega u^k(m-u-w)dx \\
&\le -\alpha k(k-1)\int_\Omega u^{k-1}|\nabla u|^2 dx + \alpha k(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla m dx \\
&\quad - \alpha k(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla w dx + kr_1\|m\|_{L^\infty(\Omega)}\int_\Omega u^k dx \quad \forall t \in (0, T^*).
\end{aligned} \tag{3.7}$$

Then following the same procedure as Step 1 of [13, Lemma 2.5], we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^k dx + \int_\Omega u^k dx + \frac{\alpha k(k-1)}{(k+1)^2}\int_\Omega \big|\nabla u^{\frac{k+1}{2}}\big|^2 dx \\
&\le \alpha k(k-1)\int_\Omega u^{k-1}|\nabla w|^2 dx + C_1(k, \|u_0\|_{L^\infty(\Omega)}, \|w_0\|_{W^{1,\infty}(\Omega)})
\end{aligned} \tag{3.8}$$

for all $t \in (0, T^*)$. We divide the proof into two cases.

Case (i): $N = 1$. First of all, for any $l \in (2, \infty)$, there exists a positive constant $C_{2l}\big(l, \|u_0\|_{L^\infty(\Omega)}, \|w_0\|_{W^{1,\infty}(\Omega)}\big)$ such that (3.6) holds, i.e. $\int_\Omega |\nabla w|^{2l} \le C_{2l}$ by (3.4). Next, to estimate $\|u(\cdot,t)\|_{L^k(\Omega)}$, we use the Gagliardo-Nirenberg interpolation inequality (Lemma 2.4) and (3.1) to obtain

$$\begin{aligned}
\int_\Omega u^k dx &= \|u^{\frac{k+1}{2}}\|^{\frac{2k}{k+1}}_{L^{\frac{2k}{k+1}}(\Omega)} \\
&\le C(k)\Big(\|\nabla u^{\frac{k+1}{2}}\|^{\theta}_{L^2(\Omega)} \|u^{\frac{k+1}{2}}\|^{1-\theta}_{L^{\frac{2}{k+1}}(\Omega)} + \|u^{\frac{k+1}{2}}\|_{L^{\frac{2}{k+1}}(\Omega)}\Big)^{\frac{2k}{k+1}} \\
&\le C_3\Big(\|\nabla u^{\frac{k+1}{2}}\|^{\frac{2k\theta}{k+1}}_{L^2(\Omega)} + 1\Big),
\end{aligned}$$

where $\theta = \frac{\frac{k+1}{2} - \frac{k+1}{2k}}{1 - \frac{1}{2} + \frac{k+1}{2}} \in (0,1)$, and C_3 is a positive constant depending only on k and $\|u\|_{L^1(\Omega)}$. A simple computation shows that $\frac{2k\theta}{k+1} < 2$. It then follows from Young's inequality and (3.6) that

$$\begin{aligned}
&\int_\Omega u^{k-1}|\nabla w|^2 dx \\
&\le \int_\Omega u^k dx + \int_\Omega |\nabla w|^{2k} dx \le \int_\Omega u^k dx + C_{2k} \\
&\le C_3 \|\nabla u^{\frac{k+1}{2}}\|^{\frac{2k\theta}{k+1}}_{L^2(\Omega)} + C_4\left(l, \|u_0\|_{L^\infty(\Omega)}, \|w_0\|_{W^{1,\infty}(\Omega)}\right) \\
&\le \frac{1}{2(k+1)^2}\int_\Omega \big|\nabla u^{\frac{k+1}{2}}\big|^2 dx + C_5\left(l, \|u_0\|_{L^\infty(\Omega)}, \|w_0\|_{W^{1,\infty}(\Omega)}\right)
\end{aligned} \tag{3.9}$$

for all $t \in (0, T^*)$. Combining (3.9) with (3.8), and using ODE comparison argument, we obtain the desired estimate (3.5).

Case (ii): $N \ge 2$. In this case, the estimate (3.6) can not be derived from (3.4) directly. To overcome this difficulty, we will establish a combined estimate on $\int_\Omega u^k(x,t)dx + \int_\Omega |\nabla w(x,t)|^{2l} dx$. For this purpose, we differentiate equation $(1.1)_2$ to obtain

$$\big(|\nabla w|^2\big)_t = 2\nu \nabla w \cdot \nabla \Delta w + 2r_2 \nabla w \cdot \nabla[w(m-u-w)],$$

which together with the point-wise identity $2\nabla w \cdot \nabla \Delta w = \Delta |\nabla w|^2 - 2|D^2 w|^2$ yields

$$\big(|\nabla w|^2\big)_t = \nu \Delta |\nabla w|^2 - 2\nu |D^2 w|^2 + 2r_2 \nabla w \cdot \nabla[w(m-u-w)]. \tag{3.10}$$

Multiplying both sides of (3.10) by $l|\nabla w|^{2(l-1)}$ and integrating over Ω, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla w|^{2l}dx\\
&= \nu l\int_\Omega |\nabla w|^{2(l-1)}\Delta|\nabla w|^2dx - 2\nu l\int_\Omega |\nabla w|^{2(l-1)}|D^2w|^2dx\\
&\quad + 2r_2 l\int_\Omega |\nabla w|^{2(l-1)}\nabla w\cdot\nabla[w(m-u-w)]dx\\
&= -\nu l(l-1)\int_\Omega |\nabla w|^{2(l-2)}|\nabla|\nabla w|^2|^2dx + l\nu\int_{\partial\Omega}|\nabla w|^{2(l-1)}\frac{\partial|\nabla w|^2}{\partial n}dx\\
&\quad - 2\nu l\int_\Omega |\nabla w|^{2(l-1)}|D^2w|^2dx\\
&\quad + 2r_2 l\int_\Omega |\nabla w|^{2(l-1)}\nabla w\cdot\nabla[w(m-u-w)]dx \quad \text{for all } t\in(0,T^*).
\end{aligned}
\tag{3.11}
$$

The estimate for the second term on the right-hand side of (3.11) is very subtle. We first use Lemma 2.2 to obtain

$$
\int_{\partial\Omega}|\nabla w|^{2(l-1)}\frac{\partial|\nabla w|^2}{\partial n}dx \le C_\Omega\int_{\partial\Omega}|\nabla w|^{2l}dx = C_\Omega\big\||\nabla w|^l\big\|^2_{L^2(\partial\Omega)} \tag{3.12}
$$

for all $t\in(0,T^*)$. Then let us fix a constant $r\in\big(0,\frac12\big)$. Since the embedding $W^{r+\frac12,2}(\Omega)(\hookrightarrow W^{r,2}(\partial\Omega))\hookrightarrow L^2(\partial\Omega)$ is compact by Lemma 2.3, we have

$$
\||\nabla w|^l\|_{L^2(\partial\Omega)} \le C\||\nabla w|^l\|_{W^{r+\frac12,2}(\Omega)}. \tag{3.13}
$$

To estimate the right-hand side of (3.13), we take two constants $s\in\big[1,\frac{N}{N-1}\big)$ and $\theta\in\big(0,1\big)$ such that

$$
\frac12 - \frac{r+\frac12}{N} = (1-\theta)\frac{l}{s} + \theta\Big(\frac12-\frac1N\Big).
$$

Noticing that $l>1$ implies that $r+\frac12\le\theta<1$, we can apply the fractional Gagliardo-Nirenberg inequality (Lemma 2.5) to the right hand side of (3.13) to obtain

$$
\begin{aligned}
\||\nabla w|^l\|_{W^{r+\frac12,2}(\Omega)} &\le C_0\|\nabla|\nabla w|^l\|^\theta_{L^2(\Omega)}\||\nabla w|^l\|^{1-\theta}_{L^{\frac{s}{l}}(\Omega)} + \tilde C_0\||\nabla w|^l\|_{L^{\frac{s}{l}}(\Omega)}\\
&= C_0\|\nabla|\nabla w|^l\|^\theta_{L^2(\Omega)}\|\nabla w\|^{(1-\theta)l}_{L^s(\Omega)} + \tilde C_0\|\nabla w\|^l_{L^s(\Omega)}\\
&\le C\Big(\|\nabla|\nabla w|^l\|^\theta_{L^2(\Omega)}+1\Big) \quad \text{for all } t\in(0,T^*).
\end{aligned}
\tag{3.14}
$$

Here we used the boundedness of $\|w(\cdot,t)\|_{W^{1,s}(\Omega)}$ in the last inequality (see Lemma 3.1). Substituting (3.13) and (3.14) into (3.12), and applying Young's inequality with ϵ, we have

$$
\begin{aligned}
\int_{\partial\Omega}|\nabla w|^{2l-2}\frac{\partial|\nabla w|^2}{\partial n}dx &\le C\Big(\|\nabla|\nabla w|^l\|^{2\theta}_{L^2(\Omega)}+1\Big) = C\Big(\int_\Omega|\nabla|\nabla w|^l|^2dx\Big)^\theta + C\\
&\le \epsilon\int_\Omega|\nabla|\nabla w|^l|^2dx + C(\epsilon)
\end{aligned}
\tag{3.15}
$$

for all $t\in(0,T^*)$, where ϵ is a positive constant to be specified later. For the last term on the right hand of (3.11), we follow the same procedure as [13, (2.26)–(2.29)] and obtain

$$
\begin{aligned}
&2r_2 l\int_\Omega |\nabla w|^{2(l-1)}\nabla w\cdot\nabla[w(m-u-w)]dx\\
&\le 2\nu l\int_\Omega |\nabla w|^{2(l-1)}|D^2w|^2dx+\frac{(l-1)\nu l}{2}\int_\Omega |\nabla w|^{2(l-2)}|\nabla|\nabla w|^2|^2dx \\
&\quad + C_4\int_\Omega |\nabla w|^{2(l-1)}dx + C_5\int_\Omega u^2|\nabla w|^{2(l-1)}dx,
\end{aligned} \tag{3.16}
$$

where C_4 and C_5 are positive constants depending on l, $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$. Since

$$
|\nabla w|^{2(l-2)}|\nabla|\nabla w|^2|^2=\frac{4}{l^2}|\nabla|\nabla w|^l|^2,
$$

we combine (3.15) and (3.16) with (3.11), and utilize Young's inequality to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla w|^{2l}dx+\int_\Omega |\nabla w|^{2l}dx+\Big(\frac{\nu 2(l-1)}{l}-l\nu\epsilon\Big)\int_\Omega |\nabla|\nabla w|^l|^2dx\\
&\le \int_\Omega |\nabla w|^{2l}dx + C_4\int_\Omega |\nabla w|^{2(l-1)}dx + C_5\int_\Omega u^2|\nabla w|^{2(l-1)}dx + C(\epsilon)\\
&\le 2\int_\Omega |\nabla w|^{2l}dx + C_5\int_\Omega u^2|\nabla w|^{2(l-1)}dx + C(\epsilon,|\Omega|,l)\quad \text{for all } t\in(0,T^*).
\end{aligned} \tag{3.17}
$$

For any $s\in\big[1,\min\{\frac{N}{N-1},2l\}\big)$, we take

$$
\tilde\theta=\big(\frac12-\frac{l}{s}\big)\big(\frac12-\frac1N-\frac{l}{s}\big)^{-1}.
$$

Then the Gagliardo-Nirenberg inequality gives

$$
\begin{aligned}
\int_\Omega |\nabla w|^{2l}dx &= \||\nabla w|^l\|^2_{L^2(\Omega)}\\
&\le C(l)\|\nabla|\nabla w|^l\|^{2\tilde\theta}_{L^2(\Omega)}\||\nabla w|^l\|^{2(1-\tilde\theta)}_{L^{\frac{s}{l}}(\Omega)}+C(l)\||\nabla w|^l\|^2_{L^{\frac{s}{l}}(\Omega)}\\
&\le C_6\Big(\|\nabla|\nabla w|^l\|^{2\tilde\theta}_{L^2(\Omega)}+1\Big)
\end{aligned}
$$

for all $t\in(0,T^*)$, where C_6 is a positive constant depending on l, $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$. Here we used the boundedness of $\|w(\cdot,t)\|_{W^{1,s}(\Omega)}$ in the last inequality. Since $s<2l$, a simple computation shows that $\tilde\theta\in(0,1)$, i.e, $2\tilde\theta<2$. Thus by utilizing Young's inequality, we have

$$
2\int_\Omega |\nabla w|^{2l}dx\le l\nu\epsilon\int_\Omega |\nabla|\nabla w|^l|^2dx + C(\epsilon).
$$

Upon substituting into (3.17), and taking $\epsilon=\frac{l-1}{l^2}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla w|^{2l}dx+\int_\Omega |\nabla w|^{2l}dx+\frac{\nu(l-1)}{l}\int_\Omega |\nabla|\nabla w|^l|^2dx\\
&\le C_5\int_\Omega u^2|\nabla w|^{2(l-1)}dx + C_7
\end{aligned} \tag{3.18}
$$

for all $t \in (0, T^*)$, where C_7 is a positive constant depending on l, $|\Omega|$, $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$. Now adding (3.18) to (3.8), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^k dx + \int_\Omega |\nabla w|^{2l} dx\Big) + \int_\Omega u^k dx + \int_\Omega |\nabla w|^{2l} dx \\
&+ \frac{\alpha k(k-1)}{(k+1)^2} \int_\Omega \big|\nabla u^{\frac{k+1}{2}}\big|^2 dx + \frac{\nu(l-1)}{l} \int_\Omega |\nabla |\nabla w|^l|^2 dx \\
&\le \alpha k(k-1) \int_\Omega u^{k-1} |\nabla w|^2 dx + C_5 \int_\Omega u^2 |\nabla w|^{2(l-1)} dx + C_8
\end{aligned} \tag{3.19}
$$

for all $t \in (0, T^*)$, where C_8 is a positive constant depending only on k, l, $|\Omega|$, $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$. Following the same procedure as [13, (2.35)–(2.43)], we can find a positive constant C_9 depending on k, l, $|\Omega|$, $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$ such that

$$
\begin{aligned}
&\alpha k(k-1) \int_\Omega u^{k-1} |\nabla w|^2 dx + C_5 \int_\Omega u^2 |\nabla w|^{2(l-1)} dx \\
&\le \frac{\alpha k(k-1)}{2(k+1)^2} \int_\Omega \big|\nabla u^{\frac{k+1}{2}}\big|^2 dx + \frac{\nu(l-1)}{l} \int_\Omega |\nabla |\nabla w|^l|^2 dx + C_9 \quad \text{for all } t \in (0, T^*).
\end{aligned}
$$

Combing this with (3.19), and setting $y_\delta(t) := \int_\Omega |\nabla w|^{2l} dx + \int_\Omega w^k dx$, we conclude that $y_\delta'(t) + y_\delta(t) \le C_9$ for all $t \in (0, T^*)$. Thus an ODE comparison argument yields the uniform boundedness of $y_\delta(t)$ on $(0, T^*)$, which implies that $\|u(\cdot, t)\|_{L^k(\Omega)}$ and $\|\nabla w(\cdot, t)\|_{L^{2l}(\Omega)}$ are uniformly bounded on $(0, T^*)$. This completes the proof of Lemma 3.2. $\square$

Proof of Theorem 1.1. By Lemma 2.1, there exists a unique local-in-time classical solution (u, w) to equations (1.1) on $[0, T^*)$. By [17, Lemma A.1] and Lemma 3.2, we can establish the uniform boundedness of u and ∇w in $\Omega \times (0, T^*)$. Then we can deduce that $T^* = \infty$ by using the extension criterion in Lemma 2.1. Hence we have completed the proof of Theorem 1.1 under the condition that $(u_0, w_0) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. For the case that u_0 is merely Hölder continuous and nonnegative in $\overline{\Omega}$ and that v_0 belongs to $W^{1,\infty}(\Omega)$ only, we can follow the corresponding proof in [13] to conclude the proof. $\square$

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