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LONG-TERM BEHAVIOR OF A CYCLIC MAX-TYPE SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. We study the long-term behavior of positive solutions of the cyclic system of difference equations

$$x_{n+1}^{(i)}=\max\Big\{\alpha,\frac{(x_n^{(i+1)})^p}{(x_{n-1}^{(i+2)})^q}\Big\},\quad i=1,\dots,k,\ n\in\mathbb{N}_0,$$

where $k\in\mathbb{N}$, $\min\{\alpha,p,q\}>0$ and where we regard that $x_n^{(i_1)}=x_n^{(i_2)}$ when $i_1\equiv i_2\pmod k$. We determine the set of parameters α, p and q in $(0,\infty)^3$ for which all such solutions are bounded. In the other cases we show that the system has unbounded solutions. For the case $p=q$ we give some sufficient conditions which guaranty the convergence of all positive solutions. The main results in this paper generalize and complement some recent ones.

1. Introduction

Unlike the linear difference equations and systems, there is no unified theory for nonlinear ones. The lack of the theory, among other reasons, motivated numerous experts to study various concrete nonlinear equations and systems, which seem or look like good prototypes. Since the beginning of 1990's there have been published a lot of papers on such equations and systems (see, e.g., [1]-[9], [11], [12], [14]-[30], [32]-[66]). Many of these papers study only positive solutions of the equations and systems. This is not so unexpected since many of the equations and systems appear in some applications (see, e.g., [10, 13, 31, 34, 39] and the references cited therein).

After studying numerous special cases of rational equations of the form

$$x_{n+1}=\alpha+\frac{x_{n-k}}{x_{n-l}},\quad n\in\mathbb{N}_0,\tag{1.1}$$

where $\alpha>0$, $k,l\in\mathbb{N}_0$, $k\neq l$ (see, e.g., [1, 2, 3, 4, 5, 11, 15, 36, 37, 40, 42, 65]), the topic began developing in a natural direction, that is, into the study of special cases of the non-rational difference equation

$$x_{n+1}=\alpha+\frac{x_{n-k}^p}{x_{n-l}^q},\quad n\in\mathbb{N}_0,\tag{1.2}$$

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where $k, l \in \mathbb{N}_0$, $k \neq l$, and $\min\{\alpha, p, q\} > 0$. The critical moment for investigating positive solutions of equation (1.2) seems the publication of paper [38], where the case $p = q > 0$, $k = l + 1 = 1$ was studied, which was the initial motivation for further investigations. Since that time equation (1.2) and its extensions have been extensively studied by several authors (see, e.g., [6, 16, 17, 20, 30, 40, 41, 44, 45])

The first problem studied was the boudedness character of positive solutions of equation (1.2). One of the first steps in the study was [36, Theorem 2] where it was given an elegant proof for the boundedness of positive solutions of the following equation with a variable coefficient

$$x_{n+1} = \alpha_n + \frac{x_n}{x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $k \in \mathbb{N}$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ is a sequence of real numbers such that $0 < M_1 \leq \alpha_n \leq M_2 < +\infty$, $n \in \mathbb{N}_0$. Another important source related to nonlinear difference equations with variable coefficients is note [35], because it gives some conditions which guarantee the monotonicity of the subsequences (x_{2n}) and (x_{2n+1}) for all solutions of a related difference equation, which along with the boundedness of the solutions produce the eventual periodicity.

On the other hand, Papaschinopoulos and Schinas initiated the study of symmetric systems of difference equations in the second half of 1990's, and since that time there have been published a lot of papers in the topic (see, e.g., [7, 8, 12, 23, 24, 25, 26, 27, 28, 29, 32, 33, 50, 52, 53, 54, 55, 56, 58, 60, 61, 62, 63]).

In 2006 appeared paper [18] which could be the first paper which suggested investigation of cyclic systems of difference equations. There are just a few papers on the topic. Recently, in [57] was studied the boundedness character of the following cyclic system of difference equations, which is a natural extension of the equation in [41] and the system in [62]

$$x_{n+1}^{(i)} = \alpha + \frac{(x_n^{(i+1)})^p}{(x_{n-1}^{(i+2)})^q}, \quad i = 1, \dots, k,\ n \in \mathbb{N}_0, \tag{1.4}$$

where $k \in \mathbb{N}$, $\min\{\alpha, p, q\} > 0$, and where we regard that

$$x_n^{(i_1)} = x_n^{(i_2)}, \quad \text{when} \quad i_1 \equiv i_2 \pmod{k}. \tag{1.5}$$

Note that $i + 2 > k$, for $i \in \{k-1, k\}$, but we will also have some situations in the paper where the superscript is bigger than $k + 2$.

Another topic of recent interest is the investigation of, so called, max-type difference equations and systems, which appeared for the first time in the mid of 1990's. For some results in the area up to 2004, see monograph [14]. A systematic study of non-rational max-type difference equations started in the mid of 2000's (see, e.g., [41, 43, 46, 47, 48, 49, 51, 53, 55, 56, 63, 66]).

The corresponding max-type system of difference equations to (1.4) is

$$x_{n+1}^{(i)} = \max\Big\{\alpha, \frac{(x_n^{(i+1)})^p}{(x_{n-1}^{(i+2)})^q}\Big\}, \quad i = 1, \dots, k,\ n \in \mathbb{N}_0, \tag{1.6}$$

where $k \in \mathbb{N}$, $\min\{\alpha, p, q\} > 0$, and where we also accept the convention in (1.5).

Motivated by above mentioned line of investigations and especially by papers [56, 61, 62] here we study the long-term behavior of positive solutions of system (1.6). Our results generalize and complement some results in these papers.

For a solution $(\vec{x}_n)_{n\ge -1} = (x_n^{(1)},\dots,x_n^{(k)})_{n\ge -1}$ of system (1.6) is said that is *bounded* if there is $L \ge 0$ such that

$$\sup_{n\ge -1}\|\vec{x}_n\|_2 = \sup_{n\ge -1}\Big(\sum_{i=1}^{k}(x_n^{(i)})^2\Big)^{1/2} \le L. \tag{1.7}$$

Of course, in the definition, instead of the Euclidean norm in $\mathbb{R}^k$ we could use any equivalent one (for example, maximum norm). If we say that a solution $(\vec{x}_n)_{n\ge -1}$ of system (1.6) is positive, it will mean that $x_n^{(i)} > 0$ for every $1 \le i \le k$ and $n \ge -1$.

2. Boundedness character of system (1.6)

Prior to stating and proving our theorems we want to mention an obvious estimate which will be frequently used from now on. Namely, note that for any positive solution of system (1.6) the following estimate holds

$$x_n^{(i)} \ge \alpha, \tag{2.1}$$

for every $1 \le i \le k$ and $n \in \mathbb{N}$.

Theorem 2.1. *If $2\sqrt{q} \le p < 1+q$ and $q \in (0,1)$, then all positive solutions of* (1.6) *are bounded.*

Proof. It is easy to see that the conditions $2\sqrt{q} \le p < 1+q$ and $q \in (0,1)$, imply that $\lambda^2 - p\lambda + q = (\lambda-\lambda_1)(\lambda-\lambda_2)$ for λ_1 and λ_2 such that $0 < \lambda_2 \le \lambda_1 < 1$. Hence

$$x_{n+1}^{(i)} = \max\Big\{\alpha, \frac{(x_n^{(i+1)})^{\lambda_1+\lambda_2}}{(x_{n-1}^{(i+2)})^{\lambda_1\lambda_2}}\Big\}.$$

From this and (2.1), for every positive solution of system (1.6) we have

$$\begin{aligned} \frac{x_{n+1}^{(i)}}{(x_n^{(i+1)})^{\lambda_1}} &= \max\Big\{\frac{\alpha}{(x_n^{(i+1)})^{\lambda_1}}, \Big(\frac{x_n^{(i+1)}}{(x_{n-1}^{(i+2)})^{\lambda_1}}\Big)^{\lambda_2}\Big\} \\ &\le \max\Big\{\alpha^{1-\lambda_1}, \Big(\frac{x_n^{(i+1)}}{(x_{n-1}^{(i+2)})^{\lambda_1}}\Big)^{\lambda_2}\Big\} \end{aligned} \tag{2.2}$$

for every $1 \le i \le k$ and $n \in \mathbb{N}$. Let

$$y_n^{(i)} = \frac{x_{n+1}^{(i)}}{(x_n^{(i+1)})^{\lambda_1}}, \quad i = 1,\dots,k, \quad n \in \mathbb{N}_0,$$

and

$$z_n = k\alpha^{1-\lambda_1} + k z_{n-1}^{\lambda_2}, \quad n \in \mathbb{N}, \tag{2.3}$$

where

$$z_0 = \sum_{i=1}^{k} y_0^{(i)}. \tag{2.4}$$

From (2.2) it follows that

$$\begin{aligned}\sum_{i=1}^{k} y_n^{(i)} &\le \sum_{i=1}^{k} \max\Big\{\alpha^{1-\lambda_1}, (y_{n-1}^{(i)})^{\lambda_2}\Big\} \le k\alpha^{1-\lambda_1} + \sum_{i=1}^{k}(y_{n-1}^{(i)})^{\lambda_2} \\ &\le k\alpha^{1-\lambda_1} + k\Big(\sum_{i=1}^{k} y_{n-1}^{(i)}\Big)^{\lambda_2}, \quad \text{for } n\in\mathbb{N}.\end{aligned} \tag{2.5}$$

Since the function $f(x) = k\alpha^{1-\lambda_1} + kx^{\lambda_2}$ is nondecreasing on the interval $(0,\infty)$ and by using (2.4) in (2.5) with $n = 1$, we obtain

$$\sum_{i=1}^{k} y_1^{(i)} \le z_1.$$

From this, by using (2.5), a simple inductive argument yields

$$\sum_{i=1}^{k} y_n^{(i)} \le z_n, \quad n\in\mathbb{N}_0. \tag{2.6}$$

On the other hand, since $\lambda_2 \in (0,1)$, function f is also concave on $(0,\infty)$. Hence, there is a unique solution x^* of the equation $f(x) = x$, and

$$(f(x) - x)(x - x^*) < 0, \quad x\in(0,\infty)\setminus\{x^*\}. \tag{2.7}$$

Moreover, from $f(1) = k\alpha^{1-\lambda_1} + k > 1$, it follows that $x^* > 1$. So, if $z_0 \in (0, x^*]$, then $(z_n)_{n\in\mathbb{N}_0}$ is nondecreasing and $z_n \le x^*$, $n\in\mathbb{N}_0$, while if $z_0 \ge x^*$, it is nonincreasing and $z_n \ge x^*$, $n\in\mathbb{N}_0$. Hence, $(z_n)_{n\in\mathbb{N}_0}$ is bounded. This and (2.6) imply the existence of $M_1 \ge x^* > 1$ such that

$$\sum_{i=1}^{k} y_n^{(i)} \le M_1, \quad n\in\mathbb{N}_0,$$

and consequently

$$x_{n+1}^{(i)} \le M_1 (x_n^{(i+1)})^{\lambda_1},$$

for every $1 \le i \le k$ and $n\in\mathbb{N}_0$. Hence

$$\sum_{i=1}^{k} x_n^{(i)} \le kM_1\Big(\sum_{i=1}^{k} x_{n-1}^{(i)}\Big)^{\lambda_1}, \quad n\in\mathbb{N},$$

from which we have

$$\sum_{i=1}^{k} x_n^{(i)} \le (kM_1)^{\frac{1-{\lambda_1}^n}{1-\lambda_1}}\Big(\sum_{i=1}^{k} x_0^{(i)}\Big)^{{\lambda_1}^n} \le (kM_1)^{\frac{1}{1-\lambda_1}} \max\Big\{1, \sum_{i=1}^{k} x_0^{(i)}\Big\},$$

from which the boundedness of any positive solution follows. □

Before we state the next result we introduce a notation in order to save some space in writing some long formulas. Namely, if a_j, $j = 1,\dots,l$, are nonnegative numbers and r is a positive real number then the notation

$$\max\{a_1, a_2, \dots, a_l\}^r$$

has the same meaning as $(\max\{a_1, a_2, \dots, a_l\})^r$, that is, as $\max\{a_1^r, a_2^r, \dots, a_l^r\}$.

Theorem 2.2. *If $p^2 < 4q$, then all positive solutions of system* (1.6) *are bounded.*

Proof. Since (1.6) is cyclic it is sufficient to prove the boundedness of $(x_n^{(1)})_{n\geq -1}$. Let $p_0 = 0$ and

$$p_{k+1} = \frac{q}{p-p_k}, \quad k \in \mathbb{N}_0. \tag{2.8}$$

We have

$$\begin{aligned}
x_{n+1}^{(1)} &= \max\Big\{\alpha, \frac{(x_n^{(2)})^p}{(x_{n-1}^{(3)})^q}\Big\} = \max\Big\{\alpha, \Big(\frac{x_n^{(2)}}{(x_{n-1}^{(3)})^{\frac{q}{p}}}\Big)^p\Big\} \\
&= \max\Big\{\alpha, \Big(\frac{x_n^{(2)}}{(x_{n-1}^{(3)})^{p_1}}\Big)^p\Big\} \\
&= \max\Big\{\alpha, \max\Big\{\frac{\alpha}{(x_{n-1}^{(3)})^{p_1}}, \frac{(x_{n-1}^{(3)})^{p-p_1}}{(x_{n-2}^{(4)})^q}\Big\}^p\Big\} \\
&= \max\Big\{\alpha, \max\Big\{\frac{\alpha}{(x_{n-1}^{(3)})^{p_1}}, \Big(\frac{x_{n-1}^{(3)}}{(x_{n-2}^{(4)})^{\frac{q}{p-p_1}}}\Big)^{p-p_1}\Big\}^p\Big\} \\
&= \max\Big\{\alpha, \max\Big\{\frac{\alpha}{(x_{n-1}^{(3)})^{p_1}}, \Big(\frac{x_{n-1}^{(3)}}{(x_{n-2}^{(4)})^{p_2}}\Big)^{p-p_1}\Big\}^p\Big\} \\
&= \max\Big\{\alpha, \max\Big\{\frac{\alpha}{(x_{n-1}^{(3)})^{p_1}}, \max\Big\{\frac{\alpha}{(x_{n-2}^{(4)})^{p_2}}, \frac{(x_{n-2}^{(4)})^{p-p_2}}{(x_{n-3}^{(5)})^q}\Big\}^{p-p_1}\Big\}^p\Big\}.
\end{aligned} \tag{2.9}$$

Now assume that for some m such that $1 \leq m \leq n+1$ we have proved the following

$$x_{n+1}^{(1)} = \max\Big\{\alpha, \dots, \max\Big\{\frac{\alpha}{(x_{n-m+2}^{(m)})^{p_{m-2}}}, \frac{(x_{n-m+2}^{(m)})^{p-p_{m-2}}}{(x_{n-m+1}^{(m+1)})^q}\Big\}^{p-p_{m-3}} \dots\Big\}. \tag{2.10}$$

Then by using (1.6) in (2.10) we have

$$\begin{aligned}
&x_{n+1}^{(1)} \\
&= \max\Big\{\alpha, \dots, \max\Big\{\frac{\alpha}{(x_{n-m+2}^{(m)})^{p_{m-2}}}, \Big(\frac{x_{n-m+2}^{(m)}}{(x_{n-m+1}^{(m+1)})^{\frac{q}{p-p_{m-2}}}}\Big)^{p-p_{m-2}}\Big\}^{p-p_{m-3}} \dots\Big\} \\
&= \max\Big\{\alpha, \dots, \max\Big\{\frac{\alpha}{(x_{n-m+2}^{(m)})^{p_{m-2}}}, \Big(\frac{x_{n-m+2}^{(m)}}{(x_{n-m+1}^{(m+1)})^{p_{m-1}}}\Big)^{p-p_{m-2}}\Big\}^{p-p_{m-3}} \dots\Big\} \\
&= \max\Big\{\alpha, \dots, \max\Big\{\frac{\alpha}{(x_{n-m+1}^{(m+1)})^{p_{m-1}}}, \frac{(x_{n-m+1}^{(m+1)})^{p-p_{m-1}}}{(x_{n-m}^{(m+2)})^q}\Big\}^{p-p_{m-2}} \dots\Big\}.
\end{aligned}$$

From this, (2.9) and the method of induction we see that (2.10) holds for every $1 \leq m \leq n+2$. We have to say that if $p_m = p$ for some $m \in \mathbb{N}$, then the above (iterating) procedure is stopped.

If $p^2 \leq q$, which is equivalent to $p \leq p_1$, by using (2.1) and (2.9) for $n \geq 3$, we obtain

$$x_{n+1}^{(1)} = \max\Big\{\alpha, \max\Big\{\frac{\alpha}{(x_{n-1}^{(3)})^{\frac{q}{p}}}, \frac{(x_{n-1}^{(3)})^{p-\frac{q}{p}}}{(x_{n-2}^{(4)})^q}\Big\}^p\Big\} \leq \max\Big\{\alpha, \frac{1}{\alpha^{q-p}}, \frac{1}{\alpha^{q-p^2+pq}}\Big\},$$

which proves the boundedness of $x_n^{(1)}$ in this case.

Since $0 = p_0 < p_1 = q/p$ and the function $g(x) = q/(p-x)$ is increasing for $x < p$, we have that p_k is increasing as far as $p_k < p$. Assume that $p_k < p$, $k \in \mathbb{N}_0$. Then as a monotone and bounded sequence it would have a finite limit p^* such that

$$(p^*)^2 - pp^* + q = 0. \tag{2.11}$$

On the other hand, since $p^2 < 4q$ then equation (2.11) does not have real zeros. This implies that there is a $k_0 \in \mathbb{N}$ such that $p_{k_0-1} < p$ and $p_{k_0} \geq p$.

Using (2.10) for $m = k_0 + 2$ and (2.1), we obtain

$$\begin{aligned} x_{n+1}^{(1)} &= \max\Big\{\alpha, \dots, \max\Big\{\frac{\alpha}{(x_{n-k_0}^{(k_0+2)})^{p_{k_0}}}, \frac{(x_{n-k_0}^{(k_0+2)})^{p-p_{k_0}}}{(x_{n-k_0-1}^{(k_0+3)})^{q}}\Big\}^{p-p_{k_0-1}} \dots\Big\} \\ &\leq \max\Big\{\alpha, \dots, \max\Big\{\frac{1}{\alpha^{p_{k_0}-1}}, \frac{1}{\alpha^{q-p+p_{k_0}}}\Big\}^{p-p_{k_0-1}} \dots\Big\}, \end{aligned}$$

for $n \geq k_0 + 2$, as desired. □

Theorem 2.3. *If $\alpha > 0$, $p = q + 1$ and $q \in (0, 1)$, then all positive solutions of (1.6) are bounded.*

Proof. We may suppose $\alpha = 1$, since the change of variables

$$x_n^{(i)} = \alpha \hat{x}_n^{(i)}, \quad i = 1, \dots, k, \quad n \geq -1,$$

transforms system (1.6) into the same with $\alpha = 1$.

Assume that sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are defined by

$$\begin{gathered} a_0 = q, \quad b_0 = q + 1, \\ a_{2n+1} = (q+1)b_{2n} - a_{2n}, \quad b_{2n+1} = qb_{2n}, \quad n \in \mathbb{N}_0, \\ b_{2n+2} = (q+1)a_{2n+1} - b_{2n+1}, \quad a_{2n+2} = qa_{2n+1}, \quad n \in \mathbb{N}_0. \end{gathered} \tag{2.12}$$

Using (1.6) and (2.12) we have

$$\begin{aligned} x_{n+1}^{(1)} &= \max\Big\{1, \frac{(x_n^{(2)})^{q+1}}{(x_{n-1}^{(3)})^{q}}\Big\} = \max\Big\{1, \frac{(x_n^{(2)})^{b_0}}{(x_{n-1}^{(3)})^{a_0}}\Big\} \\ &= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}} \frac{(x_{n-1}^{(3)})^{(q+1)b_0-a_0}}{(x_{n-2}^{(4)})^{qb_0}}\Big\} \\ &= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}} \frac{(x_{n-1}^{(3)})^{a_1}}{(x_{n-2}^{(4)})^{b_1}}\Big\} \qquad (2.13) \\ &= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}} \frac{1}{(x_{n-2}^{(4)})^{b_1}}, \frac{(x_{n-2}^{(4)})^{(q+1)a_1-b_1}}{(x_{n-3}^{(5)})^{qa_1}}\Big\} \\ &= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}} \frac{1}{(x_{n-2}^{(4)})^{b_1}}, \frac{(x_{n-2}^{(4)})^{b_2}}{(x_{n-3}^{(5)})^{a_2}}\Big\} \qquad (2.14) \end{aligned}$$

Now assume that for some m, $4 \leq 2m \leq n$ we have proved the following two equalities

$$\begin{aligned} &x_{n+1}^{(1)} \\ &= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}} \frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m+3}^{(2m-1)})^{a_{2m-4}}}, \frac{(x_{n-2m+3}^{(2m-1)})^{a_{2m-3}}}{(x_{n-2m+2}^{(2m)})^{b_{2m-3}}}\Big\} \end{aligned} \tag{2.15}$$

$$= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m+2}^{(2m)})^{b_{2m-3}}}, \frac{(x_{n-2m+2}^{(2m)})^{b_{2m-2}}}{(x_{n-2m+1}^{(2m+1)})^{a_{2m-2}}}\Big\}. \tag{2.16}$$

By using (1.6) in (2.16) we obtain

$$\begin{aligned}
&x_{n+1}^{(1)}\\
&= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m+2}^{(2m)})^{b_{2m-3}}}, \frac{(x_{n-2m+2}^{(2m)})^{b_{2m-2}}}{(x_{n-2m+1}^{(2m+1)})^{a_{2m-2}}}\Big\}\\
&= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m+1}^{(2m+1)})^{a_{2m-2}}}, \frac{(x_{n-2m+1}^{(2m+1)})^{(q+1)b_{2m-2}-a_{2m-2}}}{(x_{n-2m}^{(2m+2)})^{qb_{2m-2}}}\Big\}\\
&= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m+1}^{(2m+1)})^{a_{2m-2}}}, \frac{(x_{n-2m+1}^{(2m+1)})^{a_{2m-1}}}{(x_{n-2m}^{(2m+2)})^{b_{2m-1}}}\Big\}\\
&= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m}^{(2m+2)})^{b_{2m-1}}}, \frac{(x_{n-2m}^{(2m+2)})^{(q+1)a_{2m-1}-b_{2m-1}}}{(x_{n-2m-1}^{(2m+3)})^{qa_{2m-1}}}\Big\}\\
&= \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_{n-2m}^{(2m+2)})^{b_{2m-1}}}, \frac{(x_{n-2m}^{(2m+2)})^{b_{2m}}}{(x_{n-2m-1}^{(2m+3)})^{a_{2m}}}.\Big\}
\end{aligned}$$

From this, (2.13), (2.14) and by the method of induction we see that (2.15) and (2.16) hold for $4 \le 2m \le n+2$.

From the relations in (2.12) it is easy to see that

$$\begin{gathered}
b_{2n} = \frac{a_{2n+1} + a_{2n}}{q+1}, \quad n \in \mathbb{N}_0,\\
a_{2n+3} - (q^2+1)a_{2n+1} + q^2 a_{2n-1} = 0, \quad n \in \mathbb{N},
\end{gathered}$$

from which we obtain

$$a_{2n+1} = \frac{1-q^{2n+3}}{1-q}, \quad n \in \mathbb{N}_0. \tag{2.17}$$

Letting $n \to +\infty$ in (2.17) and (2.12) we also obtain

$$\lim_{n\to+\infty} a_{2n} = \lim_{n\to+\infty} b_{2n+1} = \frac{q}{1-q}, \tag{2.18}$$

$$\lim_{n\to+\infty} b_{2n} = \lim_{n\to+\infty} a_{2n+1} = \frac{1}{1-q}, \tag{2.19}$$

(see [62] for a detailed explanation).

Now note that from (2.15) and (2.16) we have that

$$x_{2n+1}^{(1)} = \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_0^{(2n+2)})^{b_{2n-1}}}, \frac{(x_0^{(2n+2)})^{b_{2n}}}{(x_{-1}^{(2n+3)})^{a_{2n}}}\Big\} \tag{2.20}$$

$$x_{2n}^{(1)} = \max\Big\{1, \frac{1}{(x_{n-1}^{(3)})^{a_0}}\frac{1}{(x_{n-2}^{(4)})^{b_1}}, \dots, \frac{1}{(x_0^{(2n+1)})^{a_{2n-2}}}, \frac{(x_0^{(2n+1)})^{a_{2n-1}}}{(x_{-1}^{(2n+2)})^{b_{2n-1}}}\Big\}, \tag{2.21}$$

for $n \in \mathbb{N}$.

Using that

$$\min_{i\in\mathbb{N}} x_n^{(i)} = \min_{1\le i\le k} x_n^{(i)} \ge 1, \quad \text{for } n \in \mathbb{N},$$

(2.18) and (2.19), in (2.20) and (2.21), the boundedness of $(x_n^{(1)})_{n\ge -1}$ follows, which along with the cyclicity of system (1.6) implies the boundedness of $(x_n^{(i)})_{n\ge -1}$ for every $1\le i\le k$, as claimed. □

Remark 2.4. Note that from the proofs of Theorems 2.2 and 2.3 is seen that the value of superscript $i\in\mathbb{N}$ in $x_n^{(i)}$ does not influence at the many points. This is why we introduced the convention in (1.5). In fact, an important fact in the proof of Theorem 2.2 is estimate (2.1), while the inequality $\min_{1\le i\le k} x_n^{(i)}\ge 1$, $n\in\mathbb{N}$, is the corresponding important fact in the proof of Theorem 2.3.

Theorem 2.5. *If $\alpha>0$, $p^2\ge 4q>4$, or $p>1+q$, $q\le 1$, or $p=q+1=2$, then system* (1.6) *has positive unbounded solutions.*

Proof. Since from (1.6) we have

$$x_{n+1}^{(i)}\ge \frac{(x_n^{(i+1)})^p}{(x_{n-1}^{(i+2)})^q},\quad i=1,\dots,k,\ n\in\mathbb{N}_0, \tag{2.22}$$

the same arguments as in the proof of [57, Theorem 2] can be used. So, we will only sketch the proof for the completeness.

Let $y_n=\ln\prod_{i=1}^k x_n^{(i)}$, $n\ge -1$, then by using (2.22), we obtain

$$y_{n+1}-py_n+qy_{n-1}\ge 0,\quad n\in\mathbb{N}_0. \tag{2.23}$$

If $p^2\ge 4q>4$, then the polynomial $\lambda^2-p\lambda+q$ has two zeroes λ_1 and λ_2, such that $\lambda_1>1$ and $\lambda_2>0$, while if $p>1+q$, $q\le 1$, then $\lambda_1>1>\lambda_2>0$.

From (2.23) and some simple iterations we obtain

$$\frac{\prod_{i=1}^k x_{n+1}^{(i)}}{(\prod_{i=1}^k x_n^{(i)})^{\lambda_1}}\ge\Big(\frac{\prod_{i=1}^k x_0^{(i)}}{(\prod_{i=1}^k x_{-1}^{(i)})^{\lambda_1}}\Big)^{\lambda_2^{n+1}},\quad n\in\mathbb{N}_0. \tag{2.24}$$

If $x_{-1}^{(i)},x_0^{(i)}$, $i=1,\dots,k$, are chosen, for example, such that

$$\prod_{i=1}^k x_0^{(i)}>1\quad\text{and}\quad \prod_{i=1}^k x_0^{(i)}\ge\Big(\prod_{i=1}^k x_{-1}^{(i)}\Big)^{\lambda_1}, \tag{2.25}$$

then by employing (2.24) and (2.25), it is obtained

$$\prod_{i=1}^k x_n^{(i)}\ge\Big(\prod_{i=1}^k x_0^{(i)}\Big)^{\lambda_1^n}\to+\infty\quad\text{as } n\to\infty, \tag{2.26}$$

which along with the inequality between the arithmetic and geometric means gives $\|\vec{x}_n\|_2\to+\infty$ as $n\to\infty$, showing the existence of unbounded solutions in these two cases. If $p=q+1=2$, then we can get unbounded solutions by choosing the initial values satisfying the strict inequalities in (2.25). □

3. Global attractivity

For the case $p=q$, we give here some sufficient conditions which guaranty the global attractivity of all positive solutions of system (1.6).

Theorem 3.1. *If $p\in(0,1)$ and $\alpha\in(0,1)$, then every positive solution of* (1.6) *converges to the k-dimensional vector* $(1,\dots,1)$.

Proof. Using (2.1) in the equality

$$x_{n+1}^{(i)} = \max\Big\{\alpha, \frac{\alpha^p}{(x_{n-1}^{(i+2)})^p}, \frac{1}{((x_{n-1}^{(i+2)})^{1-p}(x_{n-2}^{(i+3)})^p)^p}\Big\}, \quad i = 1, \dots, k,\ n \in \mathbb{N}, \tag{3.1}$$

which is obtained by iterating the relations in (1.6), we obtain

$$\alpha \le x_{n+1}^{(i)} \le \max\Big\{\alpha, 1, \frac{1}{\alpha^p}\Big\} = \frac{1}{\alpha^p}, \quad \text{for } n \ge 3. \tag{3.2}$$

We write the equations in (1.6) as follows

$$\frac{x_{n+1}^{(i)}}{x_n^{(i+1)}} = \max\Big\{\frac{\alpha}{x_n^{(i+1)}}, \frac{1}{(x_n^{(i+1)})^{1-p}(x_{n-1}^{(i+2)})^p}\Big\}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

for $i = 1, \dots, k$. Using (2.1) and (3.2) in (3.3), and since $p \in (0, 1)$, we obtain

$$\alpha^p \le \frac{x_{n+1}^{(i)}}{x_n^{(i+1)}} \le \frac{1}{\alpha}, \quad \text{for } n \ge 5; \tag{3.4}$$

for $i = 1, \dots, k$.

Using $p, \alpha \in (0, 1)$ and (3.4) in (1.6), we obtain

$$\alpha^{p^2} \le x_{n+1}^{(i)} \le \frac{1}{\alpha^p}, \quad \text{for } n \ge 6; \tag{3.5}$$

for $i = 1, \dots, k$.

From (2.1), (3.3) and (3.5) it follows that

$$\alpha^p \le \frac{x_{n+1}^{(i)}}{x_n^{(i+1)}} \le \frac{1}{\alpha^{p^2}}, \quad \text{for } n \ge 8, \tag{3.6}$$

for $i = 1, \dots, k$. Using (3.6) in (1.6) it follows that

$$\alpha^{p^2} \le x_{n+1}^{(i)} \le \frac{1}{\alpha^{p^3}}, \quad \text{for } n \ge 9,$$

for $i = 1, \dots, k$.

Assume that for some $m \in \mathbb{N}$

$$\alpha^{p^{2m}} \le \min_{1\le i\le k} x_{n+1}^{(i)} \le \max_{1\le i\le k} x_{n+1}^{(i)} \le \frac{1}{\alpha^{p^{2m+1}}}, \tag{3.7}$$

for $n \ge 6m + 3$, and

$$\alpha^{p^{2m+2}} \le \min_{1\le i\le k} x_{n+1}^{(i)} \le \max_{1\le i\le k} x_{n+1}^{(i)} \le \frac{1}{\alpha^{p^{2m+1}}}, \tag{3.8}$$

for $n \ge 6m + 6$. Then, an induction argument shows that (3.7) and (3.8) hold for every $m \in \mathbb{N}_0$.

Letting $m \to \infty$ in (3.7), (3.8) and using $p \in (0, 1)$, we obtain

$$\lim_{n\to\infty} \|(x_n^{(1)}, \dots, x_n^{(k)}) - (1, \dots, 1)\|_2 = 0,$$

as desired. □

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