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NON-SMOOTH EXTENSION OF A THREE CRITICAL POINTS THEOREM BY RICCERI WITH AN APPLICATION TO $p(x)$-LAPLACIAN DIFFERENTIAL INCLUSIONS

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Abstract. We extend a smooth Ricceri three critical-points theorem to a non-smooth case. Our approach is based on the non-smooth analysis. As an application, we obtain the existence of at least three critical points for a $p(x)$-Laplacian differential inclusion.

1. Introduction

First, we give some definitions which will be used throughout this paper. If X is a nonempty set and I, Ψ, $\Phi : X \to \mathbb{R}$ are three given functions, for each $\mu > 0$ and $r \in]\inf_X \Phi, \sup_X \Phi[$, we define

$$\begin{aligned} h_1(\mu I+\Psi,\Phi,r) &= \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\mu I(u)+\Psi(u)-\inf_{u\in\Phi^{-1}(]-\infty,r])}(\mu I+\Psi)}{r-\Phi(u)}, \\ h_2(\mu I+\Psi,\Phi,r) &= \sup_{u\in\Phi^{-1}(]r,+\infty[)} \frac{\mu I(u)+\Psi(u)-\inf_{u\in\Phi^{-1}(]-\infty,r])}(\mu I+\Psi)}{r-\Phi(u)}. \end{aligned}$$

When $\Psi+\Phi$ is bounded below, for each $r \in]\inf_X \Phi, \sup_X \Phi[$ such that

$$\inf_{u\in\Phi^{-1}(]-\infty,r])} I(u) < \inf_{u\in\Phi^{-1}(r)} I(u).$$

We define

$$h_3(I,\Psi,\Phi,r) = \inf\Big\{ \frac{\Psi(u)-\gamma+r}{\eta_r - I(u)} : u\in X, \Phi(u)<r, I(u)<\eta_r \Big\},$$

where

$$\gamma = \inf_{u\in X}(\Psi(u)+\Phi(u)), \quad \eta_r = \inf_{u\in\Phi^{-1}(r)} I(u).$$

In the past years, many authors have studied three critical points theorems. We refer to [3] for C^2 functions, to [20] for application in quasilinear elliptic system, and to [16] for C^1 functions. Recently, Ricceri [19] established the following three critical points theorem.

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Theorem 1.1. *Let X be a reflexive real Banach space. $I : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 function bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*. $\Psi, \Phi : X \to \mathbb{R}$ are two C^1 functions with compact derivative. Moreover, assume that there exists $r \in]\inf_X \Phi, \sup_X \Phi[$ such that*

$$h_1(I + \Psi, \Phi, r) < h_2(I + \Psi, \Phi, r)$$

and that, for each $\lambda \in]h_1(I + \Psi, \Phi, r), h_2(I + \Psi, \Phi, r)[$, the function $I + \Psi + \lambda\Phi$ is coercive.

Then, for each compact interval $[a, b] \subset]h_1(I + \Psi, \Phi, r), h_2(I + \Psi, \Phi, r)[$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 function $\Gamma : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the equation

$$I'(u) + \Psi'(u) + \lambda\Phi'(u) + \nu\Gamma'(u) = 0$$

has at least three solutions whose norms are less than ρ.

As pointed out in [19], a natural framework where the above result applies successfully is given by quasilinear equations in bounded domains. This situation occurs, for example, when $X = W_0^{1,p}(\Omega)$ and

$$\begin{gathered}
I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x, \quad \Psi(u) = \int_\Omega \int_0^u f(x,t)\mathrm{d}t\mathrm{d}x, \\
\Phi(u) = \int_\Omega \int_0^u g(x,t)\mathrm{d}t\mathrm{d}x, \quad \Gamma(u) = \int_\Omega \int_0^u h(x,t)\mathrm{d}t\mathrm{d}x, \quad \forall u \in X,
\end{gathered}$$

$f, g, h : \Omega \times \mathbb{R} \to \mathbb{R}$ being three continuous functions with subcritical growth.

However, because of the C^1 assumption on Ψ, Φ and Γ, several other problems that one meets in important concrete setting cannot be treated through Theorem 1.1. For instance, let us mention both variational inequalities and elliptic equations with discontinuous nonlinearities. In fact, Ψ, Φ and Γ usually are locally Lipschitz at most. So the question of providing a non-smooth version of the above results which applies also to these meaning situations spontaneously arises. Our interest in the present paper is to extend Theorem 1.1 into a non-smooth version by adopting the framework of Motreanu-Panagiotopoulos [13].

Recently, smooth critical points have been extended to nonsmooth cases by several authors via different methods. We should mention that Kristály et al [11] extended a Ricceri's multiplicity theorem for the existence of three critical points of nonsmooth functionals. Arcoya and Carmona [2] dealt with the Pucci-Serrin type critical point theorem in [15] to the nondifferentiable type. Li and Shen [12] proved a Pucci-Serrin type three critical points for continuous functionals. These results based on various conditions. All these results enrich the theory of nonsmooth analysis. We think that our abstract results in this direction presented here can be used to study a large number of differential equations with nonsmooth potentials. Furthermore, we improve the results in [11] by omitting the restrictions on the nonsmooth potentials, see Remark 3.2 below.

The rest of the article is organized as follows. Section 2 contains the necessary preliminaries. Section 3 contains the proofs our main results. Section 4 provides an application to a $p(x)$-Laplacian differential inclusion.

2. Preliminaries

Basic notation:

- $|\cdot|_{p(x)}$ is the usual $L^{p(x)}(\Omega)$-norm.
- $\rightharpoonup$ means weak convergence, and $\to$ strong convergence.
- C denotes all the embedding constants (the exact value may be different from line to line).
- $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ its topological dual.

Definition 2.1. A function $I : X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $\nu, \eta \in U$,

$$|I(\nu) - I(\eta)| \leq L\|\nu - \eta\|.$$

Definition 2.2. Let $I : X \to \mathbb{R}$ be a locally Lipschitz function, $u, \nu \in X$. The generalized derivative of I in u along the direction ν is

$$I^0(u;\nu) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau\nu) - I(\eta)}{\tau}.$$

It is easy to see that the function $\nu \mapsto I^0(u;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \{u^* \in X^* : \langle u^*, \nu\rangle_X \leq I^0(u;\nu) \text{ for all } v \in X\}.$$

If $I \in C^1(X)$, then $\partial I(u) = \{I'(u)\}$. Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

A point $u \in X$ is a critical point of I, if $0 \in \partial I(u)$. It is easy to see that, if $u \in X$ is a local minimum of I, then $0 \in \partial I(u)$. For more details we refer the reader to Clarke [4].

Definition 2.3. The locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the non-smooth $(PS)_c$, if for every sequence $\{u_n\}$ in X such that

(i) $\varphi(u_n) \to c$ as $n \to \infty$;
(ii) there exists a sequence $\{\varepsilon_n\}$ in $]0, +\infty[$ with $\varepsilon_n \to 0$ such that

$$\varphi^\circ(u_n; y - u_n) + \varepsilon_n\|y - u_n\| \geq 0 \quad \text{for all } y \in X, n \in \mathbb{N},$$

admits a convergent subsequence.

Definition 2.4. If X is a topological space, a function $\varphi : X \to \mathbb{R}$ is said to be sequentially inf-compact if, for each $r \in \mathbb{R}$, the set $\varphi^{-1}(]-\infty, r])$ is sequentially compact.

Definition 2.5. A mapping $A : X \to X^*$ is of type (S_+) if for every sequence $\{u_n\}$ such that $u_n \rightharpoonup u \in X$ and

$$\limsup\langle A(u_n), u_n - u\rangle \leq 0,$$

one has $u_n \to u$.

In the following, we state some properties of the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ which we call generalized Lebesgue-Sobolev spaces. Set

$$C_+(\bar{\Omega}) = \{h \mid h(x) \in C(\Omega), h(x) > 1, \text{ for any } x \in \bar{\Omega}\}.$$

For $h(x) \in C_+(\bar{\Omega})$, we write

$$h^- = \inf_{x\in\Omega} h(x), \quad h^+ = \sup_{x\in\Omega} h(x).$$

We define, for $p(x) \in C_+(\bar{\Omega})$

$$L^{p(x)}(\Omega) = \Big\{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\mathrm{d}x < \infty\Big\}$$

with the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}\mathrm{d}x \le 1\Big\},$$

then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space. We call it a generalized Lebesgue space.

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We denote $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Then $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces (see [5, 6, 8, 9, 10]).

Proposition 2.6 ([6, 7])**.** (i) If $q(x) \in C_+(\bar{\Omega})$ and $q(x) < p^*(x), \forall x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}$ is compact and it is also continuous for $q(x) \le p^*(x)$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N, \end{cases}$$

(ii) If $p_1(x), p_2(x) \in C_+(\bar{\Omega})$, and $p_1(x) \le p_2(x)$ for all $x \in \bar{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the embedding is continuous.

Proposition 2.7 ([7])**.** *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\mathrm{d}x$. For $u, u_k \in L^{p(x)}(\Omega)$, we have*

(i) *For $u \ne 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;*
(ii) $|u|_{p(x)} < 1(=1, >1) \Leftrightarrow \rho(u) < 1(=1, >1)$;
(iii) *If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
(iv) *If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
(v) $\lim_{k\to\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$;
(vi) $|u_k|_{p(x)} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

Proposition 2.8 ([7])**.** *(i) The space $L^{p(x)}(\Omega)$ is a separable, uniform Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have*

$$\Big|\int_\Omega uv\mathrm{d}x\Big| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)} \le 2|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)}.$$

(ii) There is a constant $C > 0$, such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (ii) of Proposition 2.8, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Proposition 2.9 ([4]). *Let $h : X \to \mathbb{R}$ be locally Lipschitz function. Then*

(i) $(-h)^\circ(u;z) = h^\circ(u;-z)$ *for all* $u, z \in X$;

(ii) $h^\circ(u;z) = \max\{\langle u^*, z\rangle_X : u^* \in \partial h(u)\} \le L\|z\|$ *with L as in Definition 2.1, for all* $u, z \in X$;

(iii) *Let $j : X \to \mathbb{R}$ be a continuously differentiable function Then* $\partial j(u) = \{j'(u)\}$, $j^\circ(u;z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h+j)^\circ(u;z) = h^\circ(u;z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$;

(iv) *(Lebourg's mean value theorem) Let u and v be two points in X. Then, there exists a point ω in the open segment between u and v, and a $u^*_\omega \in \partial h(\omega)$ such that*
$$h(u) - h(v) = \langle u^*_\omega, u - v\rangle_X;$$

(v) *Let Y be a Banach space and $j : Y \to X$ a continuously differentiable function. Then $h \circ j$ is locally Lipschitz and*
$$\partial(h \circ j)(u) \subseteq \partial h(j(y)) \circ j'(y) \text{ for all } y \in Y;$$

(vi) *If $h_1, h_2 : X \to \mathbb{R}$ are locally Lipschitz, then*
$$\partial(h_1 + h_2)(u) \subseteq \partial h_1(u) + \partial h_2(u);$$

(vii) *$\partial h(u)$ is convex and weakly* compact and the set-valued mapping $\partial h : X \to 2^{X^*}$ is weakly* u.s.c.;*

(viii) $\partial(\lambda h)(u) = \lambda \partial h(u)$ *for every* $\lambda \in \mathbb{R}$.

Lemma 2.10. *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function with compact gradient. Then, φ is sequentially weakly continuous.*

Proof. Our assumptions imply that the set-valued mapping $\partial\varphi : X \to \mathbb{R}$ sends bounded sets into relatively compact sets. We proceed by contradiction. Suppose that $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u \in X$, and $\{\varphi(u_n)\}$ does not converge to $\{\varphi(u)\}$. Then, passing to a subsequence, there exists some $\epsilon > 0$ such that
$$|\varphi(u_n) - \varphi(u)| \ge \epsilon \tag{2.1}$$
for all $n \in \mathbb{N}$. Since the sequence $\{u_n\}$ is bounded, there exists $M > 0$ such that $\|u_n - u\| \le M$ for all $n \in \mathbb{N}$. By Proposition 2.9 (iv) there exist some ω_n between u and u_n, and $\omega_n^* \in \partial\varphi(\omega_n)$ such that
$$\varphi(u_n) - \varphi(u) = \langle \omega_n^*, u_n - u\rangle.$$
Note that $\{\omega_n\}$ is bounded as well. Up to a subsequence, we may assume that $\omega_n^* \to \omega^* \in X^*$. So, for n large enough we have
$$\|\omega_n^* - \omega^*\| < \frac{\varepsilon}{2M}, \quad |\langle \omega^*, u_n - u\rangle| < \frac{\epsilon}{2},$$
which means
$$|\varphi(u_n) - \varphi(u)| \le \|\omega_n^* - \omega^*\|_* \|u_n - u\| + |\langle \omega^*, u_n - u\rangle| < \epsilon,$$
contradicting (2.1). □

For the convenience of the reader, we recall two results which are crucial in our further investigations. The first result is due to Ricceri [18] which ensures the existence of two local minima for a parametric function defined on a Banach space. Note that no smoothness assumption is required on the function. We denote by $\overline{(A)}_w$ the closure of A in the weak topology.

Theorem 2.11. *Let X be a reflexive Banach space, and $J, H : X \to \mathbb{R}$ two sequentially weakly lower semi-continuous functions, with J continuous. Assume that there is $\sigma > \inf_X J$ such that the set $\overline{(J^{-1}(]-\infty,\sigma[))}_w$ is bounded and disconnected in the weak topology. Then, there exists $\theta > 0$ such that, for each $\nu \in [0,\theta]$, the function $J + \nu H$ has at least two local minima lying in $J^{-1}(]-\infty,\sigma[)$.*

The second main tool in our argument is the zero-altitude mountain pass theorem for locally Lipschitz functions, due to Motreanu-Varga [14].

Theorem 2.12. *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function satisfying $(PS)_c$ for all $c \in \mathbb{R}$. If there exist $u_1, u_2 \in X, u_1 \neq u_2$ and $r \in (0, \|u_2 - u_1\|)$ such that*

$$\inf\{\varphi(u) : \|u - u_1\| = r\} \geq \max\{\varphi(u_1), \varphi(u_2)\},$$

and we denote Γ the family of continuous paths $\gamma : [0,1] \to X$ joining u_1 and u_2, then

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s)) \geq \max\{\varphi(u_1), \varphi(u_2)\}$$

is a critical value for E and $K_c \setminus \{u_1, u_2\} \neq \emptyset$.

3. The main results

This section is devoted to the statement and proof of our main results.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $I \in C^1(X,\mathbb{R})$ a sequentially weakly lower semicontinuous function, bounded on any bounded subset of X, such that I' is of type $(S)_+$. $\Psi, \Phi : X \to \mathbb{R}$ are two locally Lipschitz functions with compact gradient. Moreover, assume that there exists $r \in]\inf_X \Phi, \sup_X \Phi[$ such that*

$$h_1(I + \Psi, \Phi, r) < h_2(I + \Psi, \Phi, r)$$

and that for each $\lambda \in]h_1(I + \Psi, \Phi, r), h_2(I + \Psi, \Phi, r)[$, the function $I + \Psi + \lambda\Phi$ is coercive.

Then, for each compact interval $[a, b] \subset]h_1(I + \Psi, \Phi, r), h_2(I + \Psi, \Phi, r)[$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every locally Lipschitz function $H : X \to \mathbb{R}$ with compact gradient, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the function $I(u) + \Psi(u) + \lambda\Phi(u) + \nu H(u)$ has at least three critical points whose norms are less than ρ.

Remark 3.2. In [11], Kristály et al. proved a non-smooth three critical points theorem (see [11, Theorem 2.1]). While in our paper we improved [11, Theorem 2.1]. Since the inequality $h_1(I, \Phi, r) < h_2(I, \Phi, r)$ is equivalent to

$$\sup_{\lambda\in\Lambda} \inf_{u\in X}[I(u) + \lambda(\Phi(u) - r)] < \inf_{u\in X} \sup_{\lambda\in\Lambda}[I(u) + \lambda(\Phi(u) - r)].$$

Proof of Theorem 3.1. Although the proof is similar as that in [19], our proof is based on the non-smooth analysis and we will see that it is more complicated to prove the third critical point. The difficulty is caused by the lack of differentiability of the potential function F. From Lemma 2.10, we know that a locally Lipschitz function with compact gradient is sequentially weakly continuous, and so in particular, it is bounded on each bounded subset of X, due to the reflexivity of X. Set $\lambda \in]h_1(I + \Psi, \Phi, r), h_2(I + \Psi, \Phi, r)[$. Note that the function $I + \Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous and coercive, and the set $\Phi^{-1}(]-\infty, r])$ is sequentially weakly closed, the set, denoted by N_1, of all global minima of the restriction of $I+\Psi+\lambda\Phi$ to $\Phi^{-1}(]-\infty,r])$ is nonempty. Fix $\tilde{u}\in N_1$. We assert that $\Phi(\tilde{u})<r$. Proceeding by contradiction, assume that $\Phi(\tilde{u})=r$. Since $\lambda>h_1(I+\Psi,\Phi,r)$, there exists $u_1\in\Phi(]-\infty,r[)$ such that

$$\frac{I(u_1)+\Psi(u_1)-\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi)}{r-\Phi(u_1)}<\lambda.$$

Thus

$$I(u_1)+\Psi(u_1)-\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi)<\lambda(r-\Phi(u_1))$$

and so

$$I(u_1)+\Psi(u_1)+\lambda\Phi(u_1)<\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi)+\lambda r=I(\tilde{u})+\Psi(\tilde{u})+\lambda\Phi(\tilde{u}),$$

which contradicts the fact $\tilde{u}\in N_1$. Likewise, recall that the set $\Phi^{-1}([r,+\infty[)$ is sequentially weakly closed, the set of all global minima, denoted by N_2, of the restriction of $I+\Psi+\lambda\Phi$ to $\Phi^{-1}([r,+\infty[)$ is nonempty. Set $\hat{u}\in N_2$. We claim that $\Phi(\hat{u})>r$. Proceeding by contradiction, suppose that $\Phi(\hat{u})=r$. Since $\lambda<h_2(I+\Psi,\Phi,r)$, there exists $u_2\in\Phi^{-1}(]r,+\infty[)$ such that

$$\frac{I(u_2)+\Psi(u_2)-\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi)}{r-\Phi(u_2)}>\lambda.$$

Hence

$$I(u_2)+\Psi(u_2)-\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi)<\lambda(r-\Phi(u_2))$$

and so

$$I(u_2)+\Psi(u_2)+\lambda\Phi(u_2)<\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi)+\lambda r\le I(\hat{u})+\Psi(\hat{u})+\lambda\Phi(\hat{u}),$$

which contradicts the fact $\hat{u}\in N_2$. Now, set

$$a_\lambda=\max\Big\{\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi+\lambda\Phi),\inf_{\Phi^{-1}([r,+\infty[)}(I+\Psi+\lambda\Phi)\Big\}.$$

If $a_\lambda=\inf_{\Phi^{-1}(]-\infty,r])}(I+\Psi+\lambda\Phi)$, then we obtain

$$(I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_\lambda])=N_1\cup((I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_\lambda])\cap\Phi^{-1}([r,+\infty[)).$$

While, if $a_\lambda=\inf_{\Phi^{-1}([r,+\infty[)}(I+\Psi+\lambda\Phi)$, we derive

$$(I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_\lambda])=N_2\cup((I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_\lambda])\cap\Phi^{-1}(]-\infty,r])).$$

From the Eberlein-Smulian theorem, the set $(I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_\lambda])$ is weakly compact being sequentially weakly compact. Furthermore, for what seen above, the same set turns out to be the union of two nonempty, weakly closed and disjoint sets. So it is disconnected in the weak topology. Now, set any compact interval $[a,b]\subset]h_1(I+\Psi,\Phi,r),h_2(I+\Psi,\Phi,r)[$. It is obvious that the function $\lambda\to a_\lambda$ is upper semicontinuous in $]h_1(I+\Psi,\Phi,r),h_2(I+\Psi,\Phi,r)[$. Consequently

$$\sigma=\sup_{\lambda\in[a,b]}a_\lambda<+\infty.$$

We obtain

$$\begin{aligned}&\cup_{\lambda\in[a,b]}(I+\Psi+\lambda\Phi)^{-1}(]-\infty,\sigma+1])\\&=(I+\Psi+a\Phi)^{-1}(]-\infty,\sigma+1])\cup(I+\Psi+b\Phi)^{-1}(]-\infty,\sigma+1]).\end{aligned}$$

Obviously, the right-hand side set is bounded and so there exists some $\eta > 0$ such that

$$\cup_{\lambda\in[a,b]}(I+\Psi+\lambda\Phi)^{-1}(]-\infty,\sigma+1]) \subseteq B_\eta,$$

where $B_\eta = \{u \in X : \|u\| < \eta\}$. Now, put

$$\tilde{c} = \sup_{B_\eta}(I+\Psi) + \max\{|a|,|b|\}\sup_{B_\eta}|\Phi|$$

and fix $\rho > \eta$ such that

$$\cup_{\lambda\in[a,b]}\,(I+\Psi+\lambda\Phi)^{-1}(]-\infty,\tilde{c}+2]) \subseteq B_\rho. \tag{3.1}$$

Set $H : X \to \mathbb{R}$ be a locally Lipschitz function with compact gradient. We choose a bounded function $g \in C^1(\mathbb{R},\mathbb{R})$, $g(t) \in [-M, M], g'(t) \in [0, 1]$, $M > \sup_{B_\rho}|H|$ and $g(t) = t$ for all $t \in [-\sup_{B_\rho}|H|, \sup_{B_\rho}|H|]$. Let

$$\tilde{H}(u) = g(H(u)) \quad \text{for all } u \in X.$$

Clearly $\tilde{H} : X \to \mathbb{R}$ is a locally Lipschitz function and $\tilde{H}(u) = H(u)$ for all $u \in B_\rho$. From the chain rule, we obtain

$$\partial\tilde{H}(u) \subseteq g'(H(u))\partial H(u)$$

for all $u \in X$. Now we show that $\partial\tilde{H}(u) : X \to 2^{X^*}$ is a compact set-valued mapping. Let $\{u_n\}$ be a bounded sequence in X and $u_n^* \in \partial\tilde{H}(u_n)$ for every $n \in \mathbb{N}$. Then there exists a sequence $\{w_n^*\}$ in X^* such that for all $n \in \mathbb{N}$ we have $w_n^* \in \partial H(u_n)$ and

$$u_n^* = g'(H(u_n))w_n^*.$$

Note that $\partial H(u)$ is compact. Passing to a subsequence, we have $w_n^* \to w^* \in X^*$ and $g'(H(u_n)) \to d \in [0, 1]$ (from Bolzano-Weirstrass theorem). Hence $u_n^* \to dw^*$. Fix $\lambda \in [a, b]$. Recall that there exists $c_\lambda \in]a_\lambda, a_\lambda + 1[$ such that the set $\overline{((I+\Psi+\lambda\Phi)^{-1}(]-\infty,c_\lambda[)_\omega}$ is disconnected in the weak topology. Indeed, otherwise for any decreasing sequence $\{a_n\}$ in $]a_\lambda, a_\lambda + 1[$ with $\lim_{n\to\infty} a_n = a_\lambda$, from Lemma 2.10 we have that the function $I + \Psi + \lambda\Phi$ is weakly lower semicontinuous. Then, we obtain

$$(I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_\lambda]) = \cap_{n\in\mathbb{N}}\overline{((I+\Psi+\lambda\Phi)^{-1}(]-\infty,a_n[))_\omega}$$

and so the set on the left-hand side would be connected in the weak topology, contrary to what seen above. Hence, we can use Theorem 2.11 to obtain $\theta > 0$ such that for each $\nu \in [0, \theta]$ the function $I + \Psi + \lambda\Phi + \nu\tilde{H}$ has at least two local minima, denoted by u_1, u_2, lying in B_η. Further, put

$$\delta = \min\big\{\theta, \frac{1}{M}\big\}$$

and choose $\nu \in [0, \delta]$, we will prove that the function

$$\phi = I + \Psi + \lambda\Phi + \nu\tilde{H}$$

possesses at least three critical points lying in B_ρ. With this aim in mind, we show that ϕ satisfies the non-smooth $(PS)_c$. Let $\{u_n\}$ be a sequence in $X, \forall y \in X$, such that

$$\phi(u_n) \to c, \tag{3.2}$$

$$\phi^\circ(u_n)(u_n; y-u_n) + \varepsilon_n\|y-u_n\| \geq 0 \tag{3.3}$$

with $\varepsilon_n \to 0$ and $n \to \infty$. Observe that $\tilde{H}$ is bounded, i.e.,

$$\sup_{u\in X} |\tilde{H}(u)| \le M. \tag{3.4}$$

Note that $I + \Psi + \lambda\Phi$ is coercive. It follows that ϕ is also coercive from (3.4). Then $\{u_n\}$ is a bounded sequence. Passing to a subsequence, we have $u_n \rightharpoonup u \in X$. Put $R > 0$ such that

$$\|u_n - u\| \le R$$

for all $n \in \mathbb{N}$. Chose sequences $\{\xi_n^1\}$, $\{\xi_n^2\}$, $\{\xi_n^3\}$ in X^* such that $\xi_n^1 \in \partial\Psi(x, u_n)$, $\xi_n^2 \in \partial\Phi(x, u_n)$, $\xi_n^3 \in \partial\tilde{H}(x, u_n)$ and

$$\Psi^\circ(u_n; u - u_n) = \langle \xi_n^1, u - u_n\rangle, \quad \Phi^\circ(u_n; u - u_n) = \langle \xi_n^2, u - u_n\rangle,$$
$$\tilde{H}^\circ(u_n; u - u_n) = \langle \xi_n^3, u - u_n\rangle.$$

From the compactness of $\partial\Psi$, $\partial\Phi$ and $\partial\tilde{H}$, up to a subsequence, we have $\xi_n^1 \to \xi^1 \in X^*$, $\xi_n^2 \to \xi^2 \in X^*$ and $\xi_n^3 \to \xi^3 \in X^*$. By (3.3), we obtain

$$\begin{aligned} &\langle I'(u_n), u - u_n\rangle + \Psi^\circ(u_n, u - u_n) + \lambda\Phi^\circ(u_n, u - u_n) \\ &+ \nu\tilde{H}^\circ(u_n, u - u_n) + \varepsilon_n\|u - u_n\| \ge 0. \end{aligned} \tag{3.5}$$

Fix $\varepsilon > 0$. From what was stated above, we have

$$\begin{aligned} &\|\xi_n^1 - \xi^1\|_* < \frac{\varepsilon}{5R}, \ \|\xi_n^2 - \xi^2\|_* < \frac{\varepsilon}{5\lambda R}, \ \|\xi_n^3 - \xi^3\|_* < \frac{\varepsilon}{5\nu R} \\ &\varepsilon_n < \frac{\varepsilon}{5R}, \ \langle \xi^1 + \lambda\xi^2 + \nu\xi^3, u - u_n\rangle < \frac{\varepsilon}{5R} \end{aligned}$$

for $n \in \mathbb{N}$ big enough. Then, by virtue of (3.5) we can obtain

$$\langle I'(u_n), u_n - u\rangle < \varepsilon$$

for n large enough. This means that

$$\limsup_n \langle I'(u_n), u_n - u\rangle \le 0.$$

Recall that I' is of type $(S)_+$. So $u_n \to u$ in X; i.e., ϕ satisfies the non-smooth $(PS)_c$. Since u_1, u_2 are local minima of ϕ we apply Theorem 2.12 to obtain

$$c_{\lambda,\nu} = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \phi(\gamma(s)) \ge \max\{\phi(u_1), \phi(u_2)\}$$

is a critical value of ϕ, where Γ is the family of continuous paths $\gamma : [0, 1] \to X$ combining u_1 and u_2. Hence, there exists $u_3 \in X$ such that

$$c_{\lambda,\nu} = \phi(u_3) \quad \text{and} \quad 0 \in \partial\phi(u_3).$$

If we consider the path $\gamma \in \Gamma$, given by $\gamma(s) = u_1 + s(u_2 - u_1) \subset B_\eta$, then we have

$$\begin{aligned} c_{\lambda,\nu} &\le \sup_{s\in[0,1]} \big(I(\gamma(s)) + \Psi(\gamma(s)) + \lambda\Phi(\gamma(s)) + \nu\tilde{H}(\gamma(s))\big) \\ &\le \sup_{B_\eta}(I + \Psi) + \max\{|a|, |b|\} \sup_{B_\eta} |\Phi| + \delta \sup_{u\in X} |\tilde{H}| \\ &\le \tilde{c} + 1. \end{aligned}$$

Consequently, we derive

$$I(u_3) + \Psi(u_3) + \lambda\Phi(u_3) \le \tilde{c} + 2.$$

From (3.1) we have $u_3 \in B_\rho$. Therefore, u_i $(i = 1, 2, 3)$ are critical points for ϕ, all belong to the ball B_ρ. It remains to prove that these elements are critical points not only for ϕ, but also for $E = I(u)+\Psi(u)+\lambda\Phi(u)+\nu H(u)$ (removing the truncation). For every $u_i \in X$, there exists $\xi_i^3 \in \partial H(u_i)$ such that

$$H^\circ(u_i; u-u_i) = \langle g'(H(u_i))\xi_i^3, u-u_i\rangle = \langle \xi_i^3, u-u_i\rangle$$

(since $|g(u_i)| \le \sup_{B_\rho} |H|$ and $g'(H(u_i)) = 1$). So

$$\begin{aligned}
0 &\le \langle I'(u_i), u-u_i\rangle + \Psi^\circ(u_i, u-u_i) + \lambda\Phi^\circ(u_i, u-u_i) + \nu\tilde{H}^\circ(u_i, u-u_i) \\
&= \langle I'(u_i), u-u_i\rangle + \Psi^\circ(u_i, u-u_i) + \lambda\Phi^\circ(u_i, u-u_i) + \nu\langle \xi_i^3, u-u_i\rangle \\
&\le \langle I'(u_i), u-u_i\rangle + \Psi^\circ(u_i, u-u_i) + \lambda\Phi^\circ(u_i, u-u_i) + \nu H^\circ(u_i, u-u_i).
\end{aligned}$$

This completes the proof. □

Let us recall [19, Theorem 2], where $h_1 = 0$ and $h_2 > 0$.

Theorem 3.3. *Let X be a topological space and $I, \Psi, \Phi : X \to \mathbb{R}$ be three sequentially lower semicontinuous functions, with I also sequentially inf-compact, satisfying the following conditions:*

(i) $\inf_{u\in X}(\mu I(u) + \Psi(u)) = -\infty$ *for all* $\mu > 0$*;*
(ii) $\inf_{u\in X}(\Psi(u) + \Phi(u)) > -\infty$*;*
(iii) *there exists* $r \in]\inf_X \Phi, \sup_X \Phi[$ *such that*

$$\inf_{u\in\Phi^{-1}(]-\infty,r])} I(u) < \inf_{u\in\Phi^{-1}(r)} I(u).$$

Under such hypotheses, for each $\mu > \max\{0, h_3(I, \Psi, \Phi, r)\}$*, one has*

$$h_1(\mu I + \Psi, \Phi, r) = 0, \quad h_2(\mu I + \Psi, \Phi, r) > 0.$$

Based on Theorems 3.1 and 3.3, we have the following result.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $I \in C^1(X, \mathbb{R})$ a sequentially weakly lower semicontinuous function, bounded on any bounded subset of X, such that I' is of type $(S)_+$. Ψ and $\Phi : X \to \mathbb{R}$ are two locally Lipschitz functions with compact gradient. Assume also that the function $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$ and that*

$$\liminf_{\|u\|\to+\infty} \frac{\Psi(u)}{I(u)} = -\infty. \tag{3.6}$$

Then, for each $r > \sup_N \Phi$, where N is the set of all global minima of I, for each $\mu > \max\{0, h_3(I, \Psi, \Phi, r)\}$ and each compact interval $[a, b] \subset]0, h_2(\mu I + \Psi, \Phi, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every locally Lipschitz function $H : X \to \mathbb{R}$ with compact gradient, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the function $\mu I(u) + \Psi(u) + \lambda\Phi(u) + \nu H(u)$ has at least three critical points in X whose norms are less than ρ.

Proof. It is obvious that (3.6) is equivalent to the fact that the function $\mu I + \Psi$ is unbounded below for all $\mu > 0$. Likewise it is obvious that $\sup_X \Phi = +\infty$. Clearly, our hypotheses mean that N is non-empty and bounded. As a consequence, Φ is bounded in N. Set $r > \sup_N \Phi$. Note that $\Phi^{-1}(r)$ is non-empty and sequentially weakly closed. Then there exists $\bar{u} \in \Phi^{-1}(r)$ such that

$$I(\bar{u}) = \inf_{u\in\Phi^{-1}(r)} I(u).$$

The choice of r means that $\bar{u} \notin N$. So we deduce that

$$\inf_{u\in\Phi^{-1}(]-\infty,r])} I(u) < \inf_{u\in\Phi^{-1}(r)} I(u).$$

If we endow X with the weak topology, all the hypotheses of Theorem 3.3 are satisfied, and the conclusion can be deduced from Theorem 3.1. $\square$

4. Application

In this section, we will apply Theorem 3.4 to obtain the existence and multiplicity of solutions for the following $p(x)-$Laplacian differential inclusion.

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u\in\epsilon\partial F(x,u)-\lambda\partial G(x,u)+\nu\partial K(x,u) \\ \text{for a. a. } x\in\Omega, \\ u|_{\partial\Omega}=0, \end{gathered} \tag{4.1}$$

where Ω is a bounded set in $\mathbb{R}^N$, $p(x)>1$, $p(x)\in C(\bar{\Omega})$, $\partial F(x,\cdot)(\partial G(x,\cdot),\partial K(x,\cdot))$ is the Clarke sub-differential of $F(x,\cdot)(G(x,\cdot),K(x,\cdot))$.

Let $X=W_0^{1,p(x)}(\Omega)$, and define $I(u),\Psi(u),\Phi(u),H(u):X\mapsto\mathbb{R}$ by

$$\begin{gathered} I(u)=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x,\quad \Psi(u)=-\mathscr{F}(u), \\ \mathscr{F}(u)=\int_\Omega F(x,u)\mathrm{d}x,\quad \Phi(u)=\int_\Omega G(x,u)\mathrm{d}x,\quad H(u)=\int_\Omega K(x,u)\mathrm{d}x \end{gathered}$$

for all $u\in X$. For each $r\in]\inf_X\Phi,\sup_X\Phi[$, set

$$h_3^*(I,\Psi,\Phi,r)=\inf\Big\{\frac{\Psi(u)-\hat{\gamma}+r}{\hat{\eta}_r-I(u)}:u\in X,\Phi(u)<r,I(u)<\hat{\eta}_r\Big\},$$

where

$$\hat{\gamma}=\int_\Omega\inf_{u\in\mathbb{R}}(G(x,u)-F(x,u))\mathrm{d}x,\quad \hat{\eta}_r=\inf_{u\in\Phi^{-1}(r)}I(u).$$

For each $\epsilon\in\big]0,\frac{1}{\max\{0,h_3^*(I,\Psi,\Phi,r)\}}\big[$, let

$$h_2^*(I+\Psi,\Phi,r)=\sup_{u\in\Phi^{-1}(]r,+\infty[)}\frac{I(u)+\epsilon\Psi(u)-\inf_{\Phi^{-1}(]-\infty,r])}(I+\epsilon\Psi)}{r-\Phi(u)}.$$

To discuss problem (4.1), we need the following hypotheses:

(F1) for all $u\in\mathbb{R}$, $\Omega\ni x\mapsto F(x,u)$ is measurable;
(F2) for a.a. $x\in\Omega$, $\mathbb{R}\ni u\mapsto F(x,u)$ is locally Lipschitz;
(F3) $|\xi^1|\le k_1(1+|u|^{q_1(x)-1})$ for a.a. $x\in\Omega$ and every $u\in\mathbb{R}$, $\xi^1\in\partial F(x,u)$ $(k_1>0,p(x)<q_1(x)<p^*(x))$;
(F4)
$$\lim_{|u|\to+\infty}\frac{\inf_{x\in\Omega}F(x,u)}{u^{p^+}}=+\infty \text{ and } \lim_{|u|\to+\infty}\frac{\sup_{x\in\Omega}F(x,u)}{|u|^{\alpha^-}}<+\infty,$$
where $p^+<\alpha^-\le\alpha^+<p^*(x)$;
(G1) for all $u\in\mathbb{R}$, $\Omega\ni x\mapsto G(x,u)$ is measurable;
(G2) for a.a. $x\in\Omega$, $\mathbb{R}\ni u\mapsto G(x,u)$ is locally Lipschitz;
(G3) $|\xi^2|\le k_2(1+|u|^{q_2(x)-1})$ for a.a. $x\in\Omega$ and every $u\in\mathbb{R}$, $\xi^2\in\partial G(x,u)$ $(k_2>0,p(x)<q_2(x)<p^*(x))$;
(G4)
$$\lim_{|u|\to+\infty}\frac{\inf_{x\in\Omega}G(x,u)}{|u|^{\alpha^+}}=+\infty,$$
where $p^+<\alpha^-\le\alpha^+<p^*(x)$;
(K1) for all $u\in\mathbb{R}$, $\Omega\ni x\mapsto K(x,u)$ is measurable;

(K2) for a.a. $x \in \Omega$, $\mathbb{R} \ni u \mapsto K(x,u)$ is locally Lipschitz;
(K3) $|\xi^3| \le k_3(1+|u|^{q_3(x)-1})$ for a.a. $x \in \Omega$ and every $u \in \mathbb{R}$, $\xi^3 \in \partial K(x,u)$ $(k_3 > 0, p(x) < q_3(x) < p^*(x))$.

Definition 4.1. We say that $u \in X$ is a solution of problem (4.1) if there exist $\xi^1 \in \partial F(x,u)$, $\xi^2 \in \partial G(x,u)$ and $\xi^3 \in \partial K(x,u)$ for a.a. $x \in \Omega$ such that for all $v \in X$ we have

$$\begin{aligned}&\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega |u|^{p(x)-2}u \cdot v \mathrm{d}x\\&- \epsilon \int_\Omega \xi^1 v \mathrm{d}x + \lambda \int_\Omega \xi^2 v \mathrm{d}x - \nu \int_\Omega \xi^3 v \mathrm{d}x = 0.\end{aligned}$$

The proof of the following lemma can be found in [6, 7].

Lemma 4.2. *$I \in C^1(X,\mathbb{R})$ and its gradient, defined for every $u, v \in X$ by*

$$\langle I'(u), v\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v + |u|^{p(x)-2}u \cdot v)\mathrm{d}x,$$

is of type $(S)_+$.

The next lemma displays some properties of $\mathscr{F}(u)$.

Lemma 4.3. *If hypotheses* (F1)–(F3) *hold, then $\mathscr{F} : X \to \mathbb{R}$ is a locally Lipschitz function with compact gradient.*

Proof. We firstly prove that $\mathscr{F}$ is locally Lipschitz. Let $u, v \in X$. Apply the Lebourg's mean value theorem, Proposition 2.6 and the Holder inequality to obtain

$$\begin{aligned}&|\mathscr{F}(u) - \mathscr{F}(v)|\\&\le \int_\Omega |F(x,u(x)) - F(x,v(x))|\mathrm{d}x\\&\le \int_\Omega k_1(1+|u(x)|^{q_1(x)-1} + 1 + |v(x)|^{q_1(x)-1})|u(x)-v(x)|\mathrm{d}x\\&\le k_1C|u-v|_{p(x)} + k_1(|u|_{q_1(x)}^{q_1^- -1} + |u|_{q_1(x)}^{q_1^+ -1} + |v|_{q_1(x)}^{q_1^- -1} + |v|_{q_1(x)}^{q_1^+ -1})|u-v|_{q_1(x)}\\&\le k_1C\|u-v\| + k_1C(\|u\|^{q_1^- -1} + \|u\|^{q_1^+ -1} + \|v\|^{q_1^- -1} + \|v\|^{q_1^+ -1})\|u-v\|.\end{aligned}$$

From the above computation, it is obvious that $\mathscr{F}$ is locally Lipschitz.

Now, we prove that $\partial\mathscr{F}$ is compact. Choosing $u \in X$, $u^* \in \partial\mathscr{F}(u)$, we obtain for every $v \in X$

$$\langle u^*, v\rangle \le \mathscr{F}^\circ(u;v) \tag{4.2}$$

and $\mathscr{F}^\circ(u;\cdot) : L^r(\Omega) \to \mathbb{R}$ is a subadditive function (see Proposition 2.9). Furthermore, $u^* \in X^*$ is continuous also with respect to the topology induced on X by the norm $|\cdot|_r$. Indeed, setting $L > 0$ a Lipschitz constant for $\mathscr{F}$ in a neighborhood of u, for all $z \in X$ from Proposition 2.9 (ii) we obtain

$$\langle u^*, z\rangle \le L|z|_r,\ \langle u^*, -z\rangle \le L|-z|_r.$$

So

$$\langle u^*, z\rangle \le L|z|_r.$$

Hence, from the Hahn-Banach Theorem, u^* can be extended to an element of the dual $L^r(\Omega)$ (complying with (4.2)) for every $v \in L^r(\Omega)$, this means that we can represent u^* as an element of $L^{r'}(\Omega)$ and write for every $v \in L^r(\Omega)$

$$\langle u^*, v\rangle = \int_\Omega u^*(x)v(x)\mathrm{d}x. \tag{4.3}$$

Set $\{u_n\}$ be a sequence in X such that $\|u_n\| \le M$ for all $n \in \mathbb{N}$ ($M > 0$) and take $u^*_{F_n} \in \partial\mathscr{F}(u_n)$ for all $n \in \mathbb{N}$. From (F_3) and (4.3) we have

$$\begin{aligned}
\langle u^*_{F_n}, v\rangle &= \int_\Omega u^*_{F_n} v(x)\mathrm{d}x \le \int_\Omega |u^*_{F_n}||v(x)|\mathrm{d}x \\
&\le \int_\Omega k_1(1 + |u_n(x)|^{q_1(x)-1})|v(x)|\mathrm{d}x \\
&\le k_1C(1 + \|u_n\|^{q_1^+-1} + \|u_n\|^{q_1^--1})\|v\| \\
&\le k_1C(1 + M^{q_1^+-1} + M^{q_1^--1})\|v\|
\end{aligned}$$

for all $n \in \mathbb{N}$, $u \in X$. Hence

$$\|u^*_{F_n}\|_{X^*} \le k_1C(1 + M^{q_1^+-1} + M^{q_1^--1}),$$

i.e., the sequence $\{u^*_{F_n}\}$ is bounded. So, passing to a subsequence, we have $u^*_{F_n} \rightharpoonup u^*_F \in X^*$. We will prove that $\{u^*_{F_n}\} \subset X^*$ has a strong convergence. We proceed by contradiction. Assume that there exists some $\varepsilon > 0$ such that

$$\|u^*_{F_n} - u^*_F\|_{X^*} > \varepsilon$$

for all $n \in \mathbb{N}$ and hence for all $n \in \mathbb{N}$ there is a $v_n \in B(0,1)$ such that

$$\langle u^*_{F_n} - u^*_F, v_n\rangle > \varepsilon. \tag{4.4}$$

Noting that $\{v_n\}$ is a bounded sequence and passing to a subsequence, one has

$$v_n \rightharpoonup v \in X, \quad |v_n - v|_{p(x)} \to 0, \quad |v_n - v|_{q_1(x)} \to 0.$$

So, for n big enough, we have

$$\begin{gathered}
|\langle u^*_{F_n} - u^*_F, v\rangle| < \frac{\varepsilon}{4}, \quad |\langle u^*_F, v_n - v\rangle| < \frac{\varepsilon}{4}, \\
|v_n - v|_{p(x)} < \frac{\varepsilon}{4k_1C}, \quad |v_n - v|_{q_1(x)} < \frac{\varepsilon}{4k_1C(M^{q^+-1} + M^{q^--1})}.
\end{gathered}$$

Then

$$\begin{aligned}
\langle u^*_{F_n} - u^*_F, v_n\rangle &= \langle u^*_{F_n} - u^*_F, v\rangle + \langle u^*_{F_n}, v_n - v\rangle - \langle u^*_F, v_n - v\rangle \\
&\le \frac{\varepsilon}{2} + \int_\Omega |u^*_{F_n}||v_n(x) - v(x)|\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + k_1\int_\Omega (1 + |u_n|^{q_1(x)-1})|v_n(x) - v(x)|\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + k_1C|v_n - v|_{p(x)} + k_1(|u_n|^{q_1^+-1}_{q_1(x)} + |u_n|^{q_1^--1}_{q_1(x)})|v_n - v|_{q_1(x)} \\
&\le \frac{\varepsilon}{2} + k_1C|v_n - v|_{p(x)} + k_1C(M^{q_1^+-1} + M^{q_1^--1})|v_n - v|_{q_1(x)} \le \varepsilon,
\end{aligned}$$

which contradicts to (4.4). $\square$

Analogously, we can obtain the following properties of the functions $\Phi(u)$ and $H(u)$.

Lemma 4.4. *If* (G1)–(G3), (K1)–(K3) *hold, then* $\Phi(u)$, $H(u) : X \to \mathbb{R}$ *are locally Lipschitz functions with compact gradient.*

Now we state our main results.

Theorem 4.5. *If* (F1)–(F4), (G1)–(G4), (K1)–(K3) *hold, then for all* $r > 0$, $\epsilon \in \left]0, \frac{1}{\max\{0, h_3^*(I,\Psi,\Phi,r)\}}\right[$ *and all compact interval* $[a,b] \subset]0, h_2^*(I+\Psi,\Phi,r)[$, *there exist numbers* $\rho > 0$ *and* $\delta > 0$ *such that for all* $\lambda \in [a,b]$ *and all* $\nu \in [0,\delta]$, *problem* (4.1) *has at least three weak solutions whose norms in* X *are less than* ρ.

Contrary to most of the known results, we do not make any hypothesis on the behavior of the involved nonlinearities at the origin in Theorem 4.5. So our results are more interesting.

Proof of Theorem 4.5. We use Theorem 3.4 in this proof. We observe that X is a reflexive Banach space, $I \in C^1(X,\mathbb{R})$ is continuous and convex, and hence weakly lower semicontinuous and obviously bounded on any bounded subset of X. From Lemma 4.2, I' is of type (S_+). Furthermore, it follows from Lemmas 4.3 and 4.4 that Φ, Ψ and H are locally Lipschitz functions with compact gradient. So we only need to prove that the function $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$ and $\liminf_{\|u\|\to+\infty} \frac{\Psi(u)}{I(u)} = -\infty$. We firstly prove that $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$. By (F3) and (F4) there exists a constant $c_1 > 0$ such that

$$F(x,u) \le c_1(1 + |u|^{\alpha(x)}) \tag{4.5}$$

for a.a. $x \in \Omega$. Moreover, from (G3) and (G4), we also have that for all $c_2 > 0$ there exists a constant $c_3 > 0$ such that

$$G(x,u) \ge c_2|u|^{\alpha(x)} - c_3 \tag{4.6}$$

for a.a. $x \in \Omega$. From (4.5) and (4.6), noting that $\lambda > 0$ and choosing $c_2 > \frac{c_1}{\lambda}$ we obtain that

$$\begin{aligned}
\Psi + \lambda\Phi &= \int_\Omega [\lambda G(x,u) - F(x,u)]\mathrm{d}x \\
&\ge \int_\Omega [\lambda(c_2|u|^{\alpha(x)} - c_3) - c_1(1 + |u|^{\alpha(x)})]\mathrm{d}x \\
&= \int_\Omega [(\lambda c_2 - c_1)|u|^{\alpha(x)} - \lambda c_3 - c_1]\mathrm{d}x \to +\infty \quad \text{as } |u| \to +\infty,
\end{aligned}$$

which means that $\Psi + \lambda\Phi$ is bounded below.

Next, we prove that

$$\liminf_{\|u\|\to+\infty} \frac{\Psi(u)}{I(u)} = -\infty. \tag{4.7}$$

From [1] we can find a $\beta > 0$ and a function $\theta(x) \in X$, positive in Ω, such that

$$\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\mathrm{d}x = \beta \int_\Omega |\theta(x)|^{p(x)}\mathrm{d}x.$$

To obtain (4.7), it is sufficient to show that

$$\lim_{k\to+\infty} \frac{\mathscr{F}(k\theta)}{I(k\theta)} = +\infty. \tag{4.8}$$

For this purpose, let us fix two numbers M_1, M_2 with $0 < 2M_1 < M_2$. From (F4), there exists a large constant $m_1 > 0$. When $|u| > m_1$ we have

$$F(x,u) \geq M_2 c_3 u^{p^+}$$

for a.a. $x \in \Omega$, where $c_3 = \frac{\beta \max\{|\theta|_{p(x)}^{p^+}, |\theta|_{p(x)}^{p^-}\}}{|\theta|_{p^+}^{p^+}}$. For each $k \in \mathbb{N}$, put

$$\Omega_k = \big\{x \in \Omega : \theta(x) \geq \frac{m_1}{k}\big\}.$$

It is obvious that the sequence $\{\int_{\Omega_k} |\theta(x)|^{p^+} \mathrm{d}x\}$ is non-decreasing and converges to $\int_\Omega |\theta(x)|^{p^+} \mathrm{d}x$. Set $\hat{k} \in \mathbb{N}$ such that

$$\int_{\Omega_{\hat{k}}} |\theta(x)|^{p^+} \mathrm{d}x > \frac{2M_1}{M_2} \int_\Omega |\theta(x)|^{p^+} \mathrm{d}x.$$

From (F1)–(F3), there is a constant $c_4 > 0$ such that

$$\sup_{\Omega \times [0,m_1]} |F(x,u)| < c_4.$$

For all $k \in \mathbb{N}$ satisfying

$$k > \max\Big\{\hat{k}, \Big(\frac{|\Omega| \sup_{\Omega\times[0,m_1]} |F(x,u)|}{M_1 \min\{|\theta(x)|_{p(x)}^{p^+}, |\theta(x)|_{p(x)}^{p^-}\}}\Big)^{\frac{1}{p^+}}\Big\},$$

we obtain

$$\begin{aligned}
&\lim_{k\to+\infty} \frac{\mathscr{F}(k\theta)}{I(k\theta)} \\
&= \lim_{k\to+\infty} \frac{\int_{\Omega_k} F(x,k\theta(x))\mathrm{d}x + \int_{\Omega\setminus\Omega_k} F(x,k\theta(x))\mathrm{d}x}{I(k\theta)} \\
&\geq \lim_{k\to+\infty} \frac{k^{p^+} M_2 c_3 \int_{\Omega_k} |\theta(x)|^{p^+} \mathrm{d}x + \int_{\Omega\setminus\Omega_k} F(x,k\theta(x))\mathrm{d}x}{k^{p^+}\beta \int_\Omega |\theta(x)|^{p(x)}\mathrm{d}x} \\
&\geq \frac{2M_1 c_3 \int_\Omega |\theta(x)|^{p^+}\mathrm{d}x}{\beta \max\{|\theta(x)|_{p(x)}^{p^+}, |\theta(x)|_{p(x)}^{p^-}\}} + \lim_{k\to+\infty} \frac{\int_{\Omega\setminus\Omega_k} F(x,k\theta(x))\mathrm{d}x}{k^{p^+}\beta\int_\Omega |\theta(x)|^{p(x)}\mathrm{d}x} \\
&\geq \frac{2M_1 c_3 |\theta(x)|_{p^+}^{p^+}}{\beta \max\{|\theta(x)|_{p(x)}^{p^+}, |\theta(x)|_{p(x)}^{p^-}\}} - \lim_{k\to+\infty} \frac{|\Omega| \sup_{\Omega\times[0,m_1]} |F(x,k\theta(x))|}{k^{p^+}\beta \min\{|\theta(x)|_{p(x)}^{p^+}, |\theta(x)|_{p(x)}^{p^-}\}} \\
&\geq 2M_1 - M_1 = M_1 \to +\infty \ (\text{as } \mathrm{M}_1 \to +\infty).
\end{aligned}$$

Hence, the proof is complete. □

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