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# MONOTOCITY PROPERTIES OF OSCILLATORY SOLUTIONS OF TWO-DIMENSIONAL SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions for the monotonicity of the sequences of the absolute values of all local extrema of components of a two-dimensional systems are obtained.

## 1. Introduction

In this article, we study the system of differential equations

$$\begin{aligned} y_1' &= f_1(t, y_1, y_2) \\ y_2' &= f_2(t, y_1, y_2)\,, \end{aligned} \tag{1.1}$$

where $f_1$ and $f_2$ are continuous on $D = \{(t,u,v) : t \in \mathbb{R}_+ = [0,\infty), u, v \in \mathbb{R}\}$ $\mathbb{R} = (-\infty, \infty)$, and

$$f_1(t,u,v)v > 0 \quad \text{on } D,\ v \neq 0\,, \tag{1.2}$$

$$f_2(t,u,v)u < 0 \quad \text{on } D,\ u \neq 0\,. \tag{1.3}$$

**Definition 1.1.** A function $y = (y_1, y_2) : I = [t_y, \bar{t}_y) \subset \mathbb{R}_+ \to \mathbb{R}^2$ is called a solution of (1.1) if $y_i \in C^1(I)$, $i = 1, 2$, and (1.1) holds on $I$. A solution $y$ is oscillatory on $I$ if there exist two sequences of zeros of $y_1$ and $y_2$ tending to $\bar{t}_y$, and $y_1$, $y_1$ are nontrivial in any left neighbourhood of $\bar{t}_y$.

**Remark 1.2.** The definition of an oscillatory solution $y$ of (1.1) is not restrictive. If $y_1$ has a sequence of zeros tending to $\bar{t}_y$ and $y_1$ is nontrivial in any left neighbourhood of $\bar{t}_y$, then according to (1.2) and (1.3) the same is valid for $y_2$.

Sometimes, solutions are studied on finite intervals since (1.1) may have solutions that cannot be defined in a neighbourhood of $\infty$ (so called noncontinuable solutions, singular solutions of the 2-nd kind, see e.g. [9, 10, 12]).

The prototype of (1.1) is the second-order equation with $p$-Laplacian

$$\big(y^{[1]}\big)' + f\big(t, y, y^{[1]}\big) = 0 \tag{1.4}$$

where

$$y^{[1]}(t) = a(t)\big|y'(t)\big|^p \operatorname{sgn} y'(t) \tag{1.5}$$

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$$\begin{gathered} p>0\,,\quad a\in C^0(\mathbb{R}_+)\,,\quad a>0 \quad \text{on } \mathbb{R}_+\,,\quad f\in C^0(\mathbb{R}_+\times\mathbb{R}^2),\\ f(t,u,v)u>0 \quad \text{on } D\,,\ u\neq 0\,; \end{gathered} \tag{1.6}$$

Equation (1.4) is equivalent to the system

$$\begin{gathered} y_1' = a^{-1/p}(t)|y_2|^{1/p}\operatorname{sgn} y_2\,,\\ y_2' = -f(t,y_1,y_2) \end{gathered} \tag{1.7}$$

with the relation between solutions of (1.4) and (1.7) given by

$$y_1 = y\,,\quad y_2 = y^{[1]}\,.$$

Note, that (1.4) is a special case of (1.1)–(1.3) with

$$f_1(t,u,v) = a^{-1/p}(t)|v|^{1/p}\operatorname{sgn} v\,,\quad f_2(t,u,v) = -f(t,u,v)\,. \tag{1.8}$$

The study of oscillatory solutions of (1.1) or (1.4) is of interest to many authors at the present time; see e.g. [8, 12].

Let $y$ be an oscillatory solution of (1.4) defined on $I=[t_y,\bar t_y)\subset\mathbb{R}_+$ such that it has no accumulation point of zeros on $I$. Then a left neighbourhood of $\bar t_y$ exists such that all zeros of $y$ and $y^{[1]}$ in it can be described by two increasing sequences $\{t_k\}_{k=1}^\infty$ and $\{\tau_k\}_{k=1}^\infty$, respectively. Note, that by virtue of (1.4) and (1.6), $y(\tau_k)$ and $y^{[1]}(t_k)$, $k=1,2,\dots$ are local extrems of $y$ and $y^{[1]}$, respectively (see [6]). Then, the following problem for (1.4) has a long history.

**Problem.** Find sufficient conditions for the sequence $\big\{|y(\tau_k)|\big\}_{k=1}^\infty$ $\big(|y^{[1]}(t_k)|_{k=1}^\infty\big)$ of the absolute values of the local extrema of $y$ (of $y^{[1]}$) to be monotone.

This problem was initiated by Milloux [11] and then it was considered by many authors for linear (e.g. [9]) and special types of nonlinear equations of the form

$$y'' + f(t,y,y') = 0 \tag{1.9}$$

(the first results are given by Bihari [7]); the history of this problem is described more precisely in monograph [5] and paper [1].

Concerning equation (1.4), some results of [6] are summed up in the following theorems. Let $y$ be an oscillatory solution of (1.4) and let $\{t_k\}_{k=1}^\infty$ and $\{\tau_k\}_{k=1}^\infty$ be given as above.

**Theorem 1.3** ([6]). *Let $|f(t,u,v)|$ be non-decreasing with respect to $t$ in $D$ and $a$ be non-decreasing on $\mathbb{R}_+$.*

*(i) Let $f(t,-u,v)=-f(t,u,v)$ on $D$, $f(t,u,v)$ be non-increasing with respect to $v$ on $D\cap\{v\ge 0,u\ge 0\}$, and be non-decreasing with respect to $v$ on $D\cap\{v\le 0,u\ge 0\}$. Then $\big\{|y(\tau_k)|\big\}_{k=1}^\infty$ is non-increasing.*

*(ii) Let $f(t,u,-v)=f(t,u,v)$ on $D$, $|f(t,u,v)|$ be non-decreasing with respect to $v$ on $D\cap\{v\ge 0\}$. Then $\big\{|y^{[1]}(t_k)|\big\}_{k=1}^\infty$ is non-decreasing.*

For a special case of (1.4), the results in Theorem 1.3 are proved under weaker assumptions, the monotonicity with respect to $t$ of $a^{1/p}(t)|f(t,u,v)|$ is supposed instead of the monotonicities of $a$ and $|f(t,u,v)|$.

**Theorem 1.4** ([6]). *Let $f(t,u,v)\equiv r(t)h(u)$, where $r\in C^0(\mathbb{R}_+)$ and $r>0$, let $h\in\mathbb{R}$ be an odd function with $h(u)>0$ for $u>0$, and let $a^{1/p}r\in C^1(\mathbb{R}_+)$ be non-increasing. Then $\big\{|y(\tau_k)|\big\}_{k=1}^\infty$ is non-decreasing and $\big\{|y^{[1]}(t_k)|\big\}_{k=1}^\infty$ is non-increasing.*

**Remark 1.5.** If we change "non-decreasing" to "non-increasing", and "non-increasing" to "non-decreasing", then Theorems 1.3 and 1.4 still hold.

The same Problem is studied for (1.1) in [2]. Our goal is to generalize the results of [2] and of Theorems 1.3 and 1.4 to equation (1.1). We prove them under weaker assumptions and under different ones as well. We will remove the assumption that the oscillatory solution is defined on the interval without accumulation points of zeros of $y_1$; it will be shown that oscillatory solutions of (1.1) have no such points on their definition intervals under the assumptions in our theorems.

In Theorem 1.3, some kind of monotonicity of $f$ with respect to $v$ is used; we show that this assumption is not needed. We are also able to weaken the assumption concerning to the monotonicity with respect to $t$.

The basics of the method of the proofs are used in [1, 5] for (1.9) and in [2] for (1.1). We study a solution $y$ of (1.1) locally on two consecutive quarter-waves using the inverse functions to $y_1$ on each of them.

The structure of zeros of a solution of (1.1) can be complicated (see [3] for equation (1.9)). So, we introduce the following definition (see [4] for (1.1)).

**Definition 1.6.** Let $y$ be a solution of (1.1) defined on $[t_y, \bar{t}_y)$. A number $c \in [t_y, \bar{t}_y)$ is called an $H$-point of $y$ if there are sequences $\{\tau_k\}_{k=1}^{\infty}$ and $\{\bar{\tau}_k\}_{k=1}^{\infty}$ of numbers from $[t_y, \bar{t}_y)$ tending to $c$ such that

$$y_1(\tau_k) = 0\,, \quad y_1(\bar{\tau}_k) \neq 0\,, \quad (\tau_k - c)(\bar{\tau}_k - c) > 0\,, \quad k = 1, 2, \dots$$

In Definition 1.6, it is sufficient to work only with $y_1$, as according to (1.1)–(1.3), $y_1$ has a sequence of zeros tending to $c$ from the left (right) side and $y_1$ is nontrivial in any left (right) neighbourhood of $c$ if and only if the same properties hold for $y_2$. Moreover, if $c$ is an $H$-point of $y$, then

$$y_1(c) = y_2(c) = 0\,. \tag{1.10}$$

Conditions for the nonexistence of $H$-points of a solution (1.1) are given in [4]; for equation (1.9), see also e.g. [10]. On the other side, there exists an equation of the form (1.9) with a solution with infinitely many $H$-points tending to $\infty$, see [3].

**Definition 1.7.** Let $i \in \{1, 2\}$ and $y$ be a solution of (1.1) defined on $[t_y, \bar{t}_y)$. Then $y_i$ has a local extreme at $t = T$ if a neighbourhood $I$ (a right neighbourhood $I$) of $T$ exists such that either $y_i(t) \geq y_i(T)$ or $y_i(t) \leq y_i(T)$ for $t \in I$ in case $T > t_y$ (and $y_i' = 0$ in case $T = t_y$).

## 2. Preliminary results

At first, we give some auxiliary results concerning zeros of a solution of (1.1).

**Lemma 2.1.** *Let $y$ be a solution of* (1.1) *defined on $I$, let $c \in I$ be such that*

$$y_1(c) = y_2(c) = 0\,, \tag{2.1}$$

*and let $y$ be nontrivial in any right (left) neighbourhood of $c$. Then there is a sequence $\{t_k\}_{k=1}^{\infty}$ of zeros of $y_1$ such that $t_k > c$ ($t_k < c$) for $k = 1, 2, \dots$ and $\lim_{k\to\infty} t_k = c$. Hence, $c$ is $H$-point of $y$.*

*Proof.* Suppose $y$ is a nontrivial solution in any right neighbourhood of $c$ and (2.1) holds. Let, contrarily, a right neighbourhood $I$ of $c$ exist such that

$$y_1(t) > 0 \quad \text{for } t \in I\,. \tag{2.2}$$

Then (1.1) ($i = 2$) and (1.3) imply $y_2$ is decreasing on $I$, and due to (2.1), we have $y_2(t) < 0$ on $I$. From this, from (1.1) ($i = 1$), and (1.2), we have $f_1\big(t, y_1(t), y_2(t)\big) < 0$ on $I$ or $y_1$ is decreasing on $I$. As $y_1(c) = 0$, we can conclude $y_1(t) < 0$ on $I$. The contradiction with (2.2) proves the statement.

The case $y_1(t) < 0$ for $t \in I$ can be studied similarly. □

**Lemma 2.2.** *Let a solution $y$ of* (1.1) *be oscillatory on $I = [t_y, \bar t_y) \subset \mathbb{R}_+$ without $H$-points. Then all zeros of either $y_1$ or $y_2$ are simple and isolated, and sequences $\{t_k\}_{k=1}^{\infty}$ and $\{\tau_k\}_{k\in\mathcal{N}_0}$ exist such that either $\mathcal{N}_0 = \{1, 2, \dots\}$ or $\mathcal{N}_0 = \{0, 1, 2, \dots\}$,*

$$t_y \le t_k < \tau_k < t_{k+1} < \bar t_y\,, \quad k = 1, 2, \dots\,, \ \lim_{k\to\infty} t_k = \bar t_y\,, \tag{2.3}$$

$$y_1(t_k) = 0\,, \quad y_1(t) \ne 0 \quad \text{for } t \ne t_k\,,\ t \in I\,,\ k \in \{1, 2, \dots\}\,,$$

$$y_2(\tau_k) = 0\,, \quad y_2(t) \ne 0 \quad \text{for } t \ne \tau_k\,,\ t \in I\,,\ k \in \mathcal{N}_0\,.$$

*Moreover,*

$$\begin{gathered} y_1(t)\, y_2(t) > 0 \quad \text{on } (t_k, \tau_k)\,, \\ y_1(t)\, y_2(t) < 0 \quad \text{on } (\tau_k, t_{k+1})\,,\ k = 1, 2, \dots \end{gathered} \tag{2.4}$$

*Proof.* Note that $y$ is not trivial on $I$ due to $y$ being oscillatory. As $y$ has no $H$-point, Lemma 2.1 implies (2.1) is not valid for any $c \in I$, and according to (1.1) and (1.2) any zero of $y_1$ is simple and isolated. Let $y_2(c) = 0 = y_2'(c)$. Then, according to (1.1) and (1.3), $y_1(c) = 0$, which contradicts the proved part. Hence, any zero of $y_2$ is simple and isolated.

Let $T_1 < T_2$ be successive zeros of $y_1$ and let, for the sake of simplicity, $y_1(t) > 0$ on $(T_1, T_2)$. Then (1.1) and (1.3) imply $y_2$ is decreasing on $(T_1, T_2)$. By Rolle's Theorem $y_1'$ has a zero $T_3 \in (T_1, T_2)$, so we obtain from (1.1) and (1.3) that $y_2(T_3) = 0$ and

$$y_2(t) > 0 \quad \text{on } [T_1, T_3)\,, \quad y_2(t) < 0 \quad \text{on } (T_3, T_2]\,.$$

Inequalities (2.3) and (2.4) follow from this. If $y_1(t) < 0$ on $(T_1, T_2)$, the proof is similar. □

**Lemma 2.3.** *Let the assumptions of Lemma 2.2 hold. Then $\{y_1(\tau_k)\}_{k\in\mathcal{N}_0}$ (resp. $\{y_2(t_k)\}_{k=1}^{\infty}$) is the sequence of all local extrema of $y_1$ (of $y_2$) on $I$.*

*Proof.* By Lemma 2.2, $t_k$ are simple zeros of $y_1$, and $y_1$ changes its sign when $t$ is going through $t_k$; hence, using (1.1) and (1.3), a neighbourhood of $t_k$ exists such that $y_2'(t)\, y_1(t) < 0$ in it. Thus, $y_2$ has a local extreme at $t = t_k$. Similarly, it can be proved that $y_1$ has a local extrum at $t = \tau_k$ using (1.1) and (1.2). □

**Lemma 2.4.** *Let $y$ be a solution of* (1.1) *defined on $[t_y, \bar t_y)$ without $H$-points,*

$$\frac{\big|f_2(t,u,v)\big|}{\big|f_1(t,u,v)\big|} \quad \textit{be non-decreasing with respect to } t \tag{2.5}$$

*on $D$ with $uv < 0$. For any integer $m$, let there be a continuous function $g_m : (0, m] \to (0, \infty)$ such that*

$$\frac{g(|v|)\big|f_2(t,u,v)\big|}{f_1(t,u,v)} \quad \textit{is non-decreasing} \tag{2.6}$$

*with respect to $v$ for $|v| \in \big[\frac{1}{m}, m\big]$, $t \in [0, m]$, and $|u| \le m$ with $uv < 0$.*

*(i) Then the sequence of all positive (the absolute values of all negative) local extrema of $y_1$ is non-increasing.*

*(ii) Let, moreover,*

$$f_1(t,-u,v) = f_1(t,u,v)\,, \quad f_2(t,-u,v) = -f_2(t,u,v) \quad \text{on } D\,. \tag{2.7}$$

*Then the sequence of the absolute values of all local extrema of $y_1$ is non-increasing, i.e., $\{y_1(\tau_k)\}_{k\in\mathcal{N}_0}$ is non-increasing, where $\{\tau_k\}_{k\in\mathcal{N}_0}$ is given by Lemma 2.2.*

*Proof.* Let $y$ be defined on $[t_y, \bar{t}_y)$ without $H$-points. We use the notation from Lemma 2.2, and first we prove case (ii). Let $n-1 \in \mathcal{N}_0$ be fixed and put

$$T_0 = \tau_{n-1}\,, \quad T_1 = t_n\,, \quad T_2 = \tau_n\,, \quad J_0 = [T_0, T_1]\,, \quad J_1 = [T_1, T_2]\,.$$

Suppose, without the loss of generality, $y_2(t) > 0$ on $(T_0, T_2)$ (if $y_2 < 0$ the proof is similar). For convenience, we describe the situation more precisely using (1.1)–(1.3) and (2.4). We have

$$\begin{gathered} y_1 < 0 \text{ is increasing}, \quad y_2 > 0 \text{ is increasing}, \\ f_1\big(t, y_1(t), y_2(t)\big) > 0\,, \quad f_2\big(t, y_1(t), y_2(t)\big) > 0 \quad \text{on } (T_0, T_1), \\ y_1 > 0 \text{ is increasing}, \quad y_2 > 0 \quad \text{is decreasing}, \\ f_1\big(t, y_1(t), y_2(t)\big) > 0\,, \quad f_2\big(t, y_1(t), y_2(t)\big) < 0 \quad \text{on } (T_1, T_2). \end{gathered} \tag{2.8}$$

Define $s_0(z)$, $z \in \big[0, |y_1(T_0)|\big]$ ($s_1(z)$, $z \in [0, y_1(T_2))$ as the inverse function to $|y_1|$ (to $y_1$) on $J_0$ (on $J_1$). Let $\bar{z} = \min\big(|y_1(T_0)|, y_1(T_2)\big)$. We prove that

$$y_2\big(s_0(z)\big) \geq y_2\big(s_1(z)\big) \quad \text{for } z \in [0, \bar{z}]\,. \tag{2.9}$$

Note that $y_2(s_0(0)) = y_2(s_1(0)) > 0$. Assume, to the contrary, that there exists $\tilde{z} \in (0, \bar{z})$ such that

$$y_2\big(s_0(\tilde{z})\big) < y_2\big(s_1(\tilde{z})\big)\,. \tag{2.10}$$

Thus, an integer $m$ exists such that

$$T_2 \leq m\,, \quad 0 < \tilde{z} \leq m\,, \quad y_2\big(s_i(z)\big) \in \big[\tfrac{1}{m}, m\big] \tag{2.11}$$

for $i = 1, 2$, $z \in [0, \tilde{z}]$, and the function

$$\frac{g_m(v)\big|f_2(t,u,v)\big|}{f_1(t,u,v)} \quad \text{is non-decreasing with respect to } v \tag{2.12}$$

for $t \in [0, m]$, $0 < |u| \leq m$ and $v \in \big[\frac{1}{m}, m\big]$. Put

$$G(v) = \int_0^v g_m(\sigma)\,d\sigma\,, \quad H(z) = G\big(y_2(s_0(z))\big) - G\big(y_2(s_1(z))\big)\,.$$

Note that for $z \in (0, \tilde{z}]$,

$$y_2\big(s_0(z)\big) < y_2\big(s_1(z)\big) \Leftrightarrow H(z) < 0\,. \tag{2.13}$$

Furthermore, using (2.5), (2.7), (2.8), we have

$$\begin{aligned} \frac{d}{dz}H(z) &= -\frac{g_m(y_2(s_0))f_2(s_0, -z, y_2(s_0))}{f_1(s_0, -z, y_2(s_0))} - \frac{g_m(y_2(s_1))f_2(s_1, z, y_2(s_1))}{f_1(s_1, z, y_2(s_1))} \\ &\geq -\frac{g_m(y_2(s_0))f_2(s_1, -z, y_2(s_0))}{f_1(s_1, -z, y_2(s_0))} + \frac{g_m(y_2(s_1))f_2(s_1, -z, y_2(s_1))}{f_1(s_1, -z, y_2(s_1))} \end{aligned} \tag{2.14}$$

for $z \in (0, \tilde{z}]$, $s_0 = s_0(z)$, and $s_1 = s_1(z)$. Then (2.11), (2.12), (2.13) and (2.14) imply

$$z \in (0, \tilde{z}]\,, \quad \text{and} \quad H(z) < 0 \Rightarrow \frac{d}{dz}H(z) \geq 0\,. \tag{2.15}$$

As (2.10) and (2.13) imply $H(\tilde{z}) < 0$, we have from (2.15) that

$$H(z) \le H(\tilde{z}) < 0 \quad \text{for } z \in (0, \tilde{z}]\,,$$

which contradicts $H(0) = G\big(y_2(T_1)\big) - G\big(y_2(T_1)\big) = 0$. Hence, (2.9) holds. Furthermore, we prove that

$$|y_1(T_0)| \ge y_1(T_2)\,. \tag{2.16}$$

Assume, to the contrary, that

$$|y_1(T_0)| < y_1(T_2)\,. \tag{2.17}$$

Then $\bar{z} = |y_1(T_0)|$ and (2.9) imply

$$0 = y_2(T_0) = y_2\big(s_0(\bar{z})\big) \ge y_2\big(s_1(\bar{z})\big)\,.$$

From this and from (2.8), $s_1(\bar{z}) = T_2$. As (2.17) implies $|y_1(T_0)| = \bar{z} = y_1(s_1(\bar{z})) < y_1(T_2)$, we have from (2.8) that $s_1(\bar{z}) < T_2$. This contradiction proves(2.16). Hence, as $n$ was arbitrary, (ii) holds.

**Case (i).** The proof is similar to case (i). We study the solution on intervals $[\tau_{n-1}, t_n]$ and $[\tau_{n+1}, t_{n+2}]$ instead of on $J_0$ and $J_1$. Note, that $y_1(\tau_{n-1})$ and $y_1(\tau_{n+1})$ are two consecutive local extrema with the same signs. As $y_2(t)\,y_2(s) > 0$ for $t \in [\tau_{n-1}, t_n)$ and $s \in [\tau_{n+1}, t_{n+2})$, condition (2.7) is not necessary (see (2.14)). □

**Remark 2.5.** If "non-decreasing" and "non-increasing" is replaced by "non-increasing" and "non-decreasing", respectively, with the exception of (2.6), then Lemma 2.4 holds, too. It is important to note that (2.6) must have the given form.

**Lemma 2.6.** *Let $y$ be a solution of* (1.1) *defined on $[t_y, \bar{t}_y)$ without $H$-points,*

$$\Big|\frac{f_2(t,u,v)}{f_1(t,u,v)}\Big| \text{ be non-decreasing with respect to } t \text{ on } D,\ uv > 0. \tag{2.18}$$

*For any integer $m$, assume there is a continuous function $g_m : (0, m] \to (0, \infty)$ such that*

$$\frac{g(|v|)\big|f_2(t,u,v)\big|}{f_1(t,u,v)} \text{ is non-increasing} \tag{2.19}$$

*with respect to $v$ for $v \in (0, m]$, and for $v \in [-m, 0)$, and for any $t \in [0, m]$, $\frac{1}{m} \le |u| \le m$, $uv > 0$.*

*(i) Then the sequence of all positive (the absolute values of all negative) local extrema of $y_2$ is non-decreasing.*

*(ii) If, moreover,*

$$f_1(t,u,-v) = -f_1(t,u,v)\,, \quad f_2(t,u,-v) = f_2(t,u,v)\,, \tag{2.20}$$

*then the sequence of the absolute values of all local extrema of $y_2$ is non-decreasing, i.e. $\big\{|y_2(t_k)|\big\}_{k=1}^{\infty}$ is non-decreasing, where $\{t_k\}_{k=1}^{\infty}$ is given by Lemma 2.2.*

*Proof.* Let $y$ be defined on $[t_y, \bar{t}_y)$ without $H$-points. We use the notation in Lemma 2.2. First we prove case (ii). Let $n \in \{1, 2, \dots\}$ be fixed. Put $T_1 = t_n$, $T_2 = \tau_n$, $T_3 = t_{n+1}$, $J_1 = [T_1, T_2]$ and $J_2 = [T_2, T_3]$. Suppose, without loss of the generality, that $y_1(t) > 0$ on $J_1 \cup J_2$, the proof is similar in case $y_1 < 0$. For convenience, we describe the situation more precisely using (1.1)–(1.3) and (2.4). We have

$$\begin{gathered} y_1>0 \text{ is increasing }, \quad y_2>0 \text{ is decreasing} \\ f_1\big(t,y_1(t),y_2(t)\big)>0\,, \quad f_2\big(t,y_1(t),y_2(t)\big)<0 \quad \text{on } (T_1,T_2), \\ y_1>0 \text{ is decreasing}, \quad y_2<0 \quad \text{is decreasing} \\ f_1\big(t,y_1(t),y_2(t)\big)<0\,, \quad f_2\big(t,y_1(t),y_2(t)\big)<0 \quad \text{on } (T_2,T_3)\,. \end{gathered} \tag{2.21}$$

Let $z\in[0,y_1(T_2)]$. Define $s_1(z)$ and $s_2(z)$ as the inverse functions to $y_1$ on $J_1$ and $J_2$, respectively. We prove that

$$y_2\big(s_1(z)\big)\le \big|y_2\big(s_2(z)\big)\big| \quad \text{for} \quad z\in\big[0,y_1(T_2)\big]\,. \tag{2.22}$$

Assume to the contrary that there exists $\bar z\in\big(0,y_1(T_2)\big)$ such that

$$y_2\big(s_1(\bar z)\big)>\big|y_2\big(s_2(\bar z)\big)\big|\,. \tag{2.23}$$

Then there exist an integer $m$ such that

$$T_3\le m\,, \quad \big[\bar z,y_1(T_2)\big]\subset\big[\tfrac{1}{m},m\big]\,, \tag{2.24}$$

$$\max\big\{y_2\big(s_1(\bar z)\big),\big|y_2\big(s_2(\bar z)\big)\big|\big\}\le m, \tag{2.25}$$

$$\begin{gathered} \frac{g_m(v)|f_2(t,u,v)|}{f_1(t,u,v)} \quad \text{is non-increasing in } v \\ \text{for } t\in[0,m],\ u\in\big[\tfrac{1}{m},m\big], \text{ and } v\in(0,m). \end{gathered} \tag{2.26}$$

Put

$$G(v)=\int_0^v g_m(\sigma)\,d\sigma\,, \quad H(z)=G\big(y_2(s_1(z))\big)-G\big(|y_2(s_2(z))|\big)\,.$$

Note, that

$$y_2\big(s_1(z))\big)-|y_2\big(s_2(z)\big)|>0 \Leftrightarrow H(z)>0 \text{ for } z\in[0,y_1(T_2)]\,. \tag{2.27}$$

Furthermore, using (2.18), (2.20) and (2.21),

$$\begin{aligned} \frac{d}{dz}H(z)&=\frac{g_m(y_2(s_1))f_2(s_1,z,y_2(s_1))}{f_1(s_1,z,y_2(s_1))}+\frac{g_m(y(s_2))f_2(s_2,z,y_2(s_2))}{f_1(s_2,z,y_2(s_2))} \\ &\ge\frac{g_m(y_2(s_1))f_2(s_2,z,y_2(s_1))}{f_1(s_2,z,y_2(s_1))}-\frac{g_m(y(s_2))f_2(s_2,z,|y_2(s_2)|)}{f_1(s_2,z,|y_2(s_2)|)} \end{aligned}$$

for $z\in\big[\bar z,y_1(T_2)\big)$, $s_1=s_1(z)$, and $s_2=s_2(z)$. As $t=s_i(z)$, $u=z$, $v=|y_j(s_i)|$ satisfies (2.26) for $z\in\big[\bar z,y_1(T_2)\big]$, $i=1,2$ and $j=1,2$, (2.25), (2.27), (2.28) imply

$$z\in\big(\bar z,y_1(T_2)\big)\,, \quad H(z)>0\Rightarrow\frac{d}{dz}\,H(z)\ge 0\,.$$

By (2.23) and (2.27), $H(\bar z)>0$ and we can conclude

$$H(z)\ge H(\bar z)>0\,, \quad z\in\big[\bar z,y_1(T_2)\big]$$

which contradicts $H\big(y_1(T_2)\big)=0$. Hence, (2.22) holds and

$$y_2(t_n)=y_2(s_1(0))\le\big|y_2(s_2(0))\big|=|y_2(t_{n+1})|\,.$$

As $n$ was arbitrary, the conclusion holds. Case (i) can be proved from case (ii) as in the proof of Lemma 2.4. □

**Remark 2.7.** If "non-decreasing" and "non-increasing" is replaced by "non-increasing" and "non-decreasing", respectively, with the exception of (2.19), then Lemma 2.6 holds, too. Again (2.19) must have the given form.

**Remark 2.8.** *The results of Lemmas 2.4 (ii) and 2.6 (ii) are proved in* [2] *under stronger assumptions concerning the monotonicity with respect to $t$.*

In Lemmas 2.4 and 2.6 no assumptions are made on functions $f_1$ and $f_2$ with respect to the second variable. The following results are obtained without assumptions on $f_1$ and $f_2$ with respect to the third variable.

**Lemma 2.9.** *Let $y$ be a solution of* (1.1) *defined on $[t_y, \bar{t}_y)$ without $H$-points,*

$$\frac{|f_2(t,u,v)|}{|f_1(t,u,v)|} \text{ be non-decreasing with respect to } t \text{ on } D,\ uv \neq 0.$$

*(i) For any integer $m$, assume there exists a continuous function $g_m : (0, m] \to (0, \infty)$ such that*

$$\frac{g(|u|)|f_1(t,u,v)|}{f_2(t,u,v)} \text{ is non-increasing}$$

*with respect to $u$ for $|u| \in [\frac{1}{m}, m]$, for any $t \in [0, m]$, and $|v| \in \big(0, m\big]$. Then the results of Lemma 2.6 hold.*

*(ii) For any integer $m$, assume there exists a continuous function $\bar{g}_m : (0, m] \to (0, \infty)$ such that*

$$\frac{\bar{g}(|u|)|f_1(t,u,v)|}{f_2(t,u,v)} \quad \text{is non-decreasing} \tag{2.28}$$

*with respect to $u$ for $u \in (0, m]$, and for $u \in [-m, 0)$, for any $t \in [0, m]$, and $|v| \in \big[\frac{1}{m}, m\big]$. Then the results of Lemma 2.4 hold.*

*Proof.* By the transformation

$$z_1(t) = -y_2(t), \quad z_2(t) = y_1(t), \tag{2.29}$$

system (1.1) is equivalent to

$$z_i' = F_i(t, z_1, z_2), \quad i = 1, 2, \tag{2.30}$$

where $F_1(t, z_1, z_2) = -f_2(t, z_2, -z_1)$, $F_2(t, z_1, z_2) = f_1(t, z_2, -z_1)$ in $D$. From (1.2) and (1.3),

$$\begin{gathered} F_1(t, z_1, z_2)z_2 = -f_2(t, z_2, -z_1)z_2 > 0 \quad \text{for } z_2 \neq 0, \\ F_2(t, z_1, z_2)z_1 = f_1(t, z_2, -z_1)z_1 < 0 \quad \text{for } z_1 \neq 0, \end{gathered}$$

Remarks 2.5 and 2.7 can be applied to (2.30). If we use the back transformation (2.29), we can obtain the results of the lemma. Note that case (i) $\big($(ii)$\big)$ follows from Remark 2.5 (Remark 2.7). ∎

The following lemmas give sufficient conditions for the validity of either (2.19) or (2.28).

**Lemma 2.10.** *Let $m$ be an integer and let $\frac{\partial}{\partial v}\frac{f_2(t,u,v)}{f_1(t,u,v)}$ be continuous on $D$ for $uv \neq 0$. Then there is a function $g_m : (0, m] \to \mathbb{R}_+$ such that*

$$J(v) = \frac{g_m(|v|)|f_2(t,u,v)|}{f_1(t,u,v)}$$

*is non-increasing in $v$ for $v \in (0,m]$ and for $v \in [-m,0)$, for any $t \in [0,m]$, and $|u| \in \big[\frac{1}{m}, m\big]$.*

*Proof.* Put $\bar{D} = \big\{(t,u) : t \in [0,m], |u| \in \big[\frac{1}{m}, m\big]\big\}$,

$$g(z) = \exp\Big\{ -\int_z^m \min\big(A_1(\sigma), A_2(-\sigma)\big)\, d\sigma \Big\}, \quad z \in (0,m],$$

with

$$\begin{gathered} B(t,u,v) = -\frac{d}{dv}\Big(\frac{|f_2(t,u,v)|}{f_1(t,u,v)}\Big)\frac{|f_1(t,u,v)|}{|f_2(t,u,v)|}, \\ A_1(z) = \min_{(t,u)\in\bar{D}} B(t,u,z), \quad A_2(-z) = \min_{(t,u)\in\bar{D}} B(t,u,-z). \end{gathered}$$

Let $v \in (0,m]$ and $(t,u) \in \bar{D}$. Then (1.2) implies $f_1(t,u,v) > 0$,

$$\frac{g'(v)}{g(v)} = \min\big(A_1(v), A_2(-v)\big) \le B(t,u,v)$$

or

$$g'(v)\frac{|f_2(t,u,v)|}{f_1(t,u,v)} \le -g(v)\frac{d}{dv}\frac{|f_2(t,u,v)|}{f_1(t,u,v)}$$

and, hence $J'(v) \le 0$.

Let $v \in [-m,0)$. Then (1.2) implies $f_1(t,u,v) < 0$,

$$\begin{aligned} \frac{g'(-v)}{g(-v)} &= -\min\big(A_1(-v), A_2(v)\big) \ge -B(t,u,v) \\ &= -\frac{d}{dv}\Big(\frac{|f_2(t,u,v)|}{f_1(t,u,v)}\Big)\frac{f_1(t,u,v)}{|f_2(t,u,v)|}, \end{aligned}$$

or

$$g'(|v|)\frac{|f_2(t,u,v)|}{f_1(t,u,v)} \le -g(|v|)\frac{d}{dv}\frac{|f_2(t,u,v)|}{f_1(t,u,v)},$$

and so $J'(v) \le 0$. □

The following lemma can be proved similarly as Lemma 2.10.

**Lemma 2.11.** *Let $m$ be an integer and let $\frac{\partial}{\partial u}\frac{f_1(t,u,v)}{f_2(t,u,v)}$ be continuous on $D$ for $uv \ne 0$. Then there is a function $g_m : (0,m] \to \mathbb{R}_+$ such that*

$$J(u) = \frac{g_m(|u|)|f_1(t,u,v)|}{f_2(t,u,v)}$$

*is non-decreasing in $u$ for $u \in (0,m]$ and for $u \in [-m,0)$, for any $t \in [0,m]$, and $|v| \in \big[\frac{1}{m}, m\big]$.*

## 3. Main results

**Theorem 3.1.** *Suppose*

$$\Big|\frac{f_2(t,u,v)}{f_1(t,u,v)}\Big| \text{ is non-decreasing (non-increasing) on } D \text{ for } uv \ne 0$$

*and either*

(i) $\dfrac{\partial}{\partial u}\dfrac{f_2(t,u,v)}{f_1(t,u,v)}$ *is continuous on $D$, $uv \ne 0$, or* *(ii) for any integer $m$, there is a continuous function $g_m : (0, m] \to (0, \infty)$ such that*

$$\frac{g_m(|u|)|f_1(t,u,v)|}{f_2(t,u,v)} \text{ is non-decreasing}$$

*with respect to $u$ for $u \in (0, m]$ and with respect to $u$ for $u \in [-m, 0)$, and for any $t \in [0, m]$ and $|v| \in \left[\frac{1}{m}, m\right]$; or*

*(iii) for any integer $m$, there is a continuous function $\bar{g}_m : (0, m] \to (0, \infty)$ such that*

$$\frac{\bar{g}_m(|v|)|f_2(t,u,v)|}{f_1(t,u,v)} \text{ is non-decreasing}$$

*with respect to $v$ for $|v| \in \left[\frac{1}{m}, m\right]$ and for any $t \in [0, m]$ and $|u| \in (0, m]$.*

*Let $y$ be an oscillatory solution of (1.1) defined on $[t_y, \bar{t}_y) \subset R_+$. Then*

(1) *There exists no $H$-point of $y$, $y$ can not be defined at $t = \bar{t}_y$, and all zeros of $y_1$ can be described by the increasing sequence $\{\tau_k\}_{k=1}^{\infty}$.*
(2) *The sequence of all positive local extrema of $y_1$ is non-increasing (is non-decreasing).*
(3) *The sequence of the absolute values of all negative local extrema of $y_1$ is non-increasing (is non-decreasing).*
(4) *If, moreover,*
$$f_1(t, -u, v) = f_1(t, u, v), \quad f_2(t, -u, v) = -f_2(t, u, v)$$
*on $D$, then the sequence $\big\{|y_1(\tau_k)|\big\}_{k=1}^{\infty}$ of the absolute values of all local extrema of $y_1$ is non-increasing (is non-decreasing).*

*Proof.* As $y$ is oscillatory, $T_y \in [t_y, \bar{t}_y)$ exists such that $y_1(T_y) \neq 0$. Let $[T_y, \bar{T}_y) \subset [t_y, \bar{t}_y)$ be the maximal interval to the right on which $y$ has no $H$-points. We prove that

$$\bar{T}_y = \bar{t}_y \,. \tag{3.1}$$

Assume, to the contrary, that $\bar{T}_y < \bar{t}_y$. Then $\bar{T}_y$ is $H$-point of $y$, and according to (1.10),

$$y_1(\bar{T}_y) = y_2(\bar{T}_y) = 0 \,. \tag{3.2}$$

From this and from Lemma 2.1, $y$ is oscillatory on $[T_y, \bar{T}_y)$. Moreover, Lemmas 2.4, 2.9 (ii) and 2.11 applied to $y$ and the interval $[T_y, \bar{T}_y)$ imply the validity of (1)–(4) in all cases (i)–(iii). Note, that case (i) follows from Lemmas 2.9 (ii) and 2.11, case (ii) from Lemma 2.9 (ii), and case (iii) from Lemma 2.4. But, according to Lemmas 2.4 and 2.6, the sequences of the absolute values of all local extrema of $y_1$ and $y_2$ are monotone and they have the opposite kind of monotonicity. Hence, the only case where (3.2) holds is $y_1(t) \equiv y_2(t) \equiv 0$ in a left neighbourhood of $\bar{T}_y$. But that contradicts $y$ being oscillatory; thus (3.1) holds and $y$ has no $H$-points on $[T_y, \bar{t}_y)$.

If either $t_y = T_y$ or if $y$ has no $H$-points on $[t_y, T_y)$, then the statement follows from Lemmas 2.4, 2.9 (ii) and 2.11. Let $c \in [t_y, T_y)$ be the maximal $H$-point of $y$. Then (1.10) implies

$$y_1(c) = y_2(c) = 0 \,, \tag{3.3}$$

and according to Lemma 2.1, a decreasing sequence $\{\bar{t}_k\}_{k=1}^{\infty}$ exists such that $\bar{t}_k \in (c, T_y]$, $y_1(\bar{t}_k) = 0$, $k = 1, 2, 3, \dots$ and $\lim_{k\to\infty} \bar{t}_k = c$. From this and from (1.1)–(1.3), a sequence $\{\bar{\tau}_k\}_{k=1}^{\infty}$, of zeros of $y_2$ exists such that $\bar{t}_k > \bar{\tau}_k > \bar{t}_{k+1}$ and $\lim_{k\to\infty} \bar{\tau}_k = c$. As the intervals $J_k = [\bar{t}_k, T_y)$ are without $H$-points, we can apply

Lemmas 2.4, 2.9 (ii) and 2.11 on $J_k$. If, for simplicity, $y_1(T_y) > 0$, then the sequence of all local maxima of $y_1$ on $J_k$ is non-increasing and greater or equal to $y_1(T_y)$. Hence, if $k \to \infty$, $\{y_1(\bar\tau_y)\}_{k=1}^{\infty}$ is non-decreasing and $y_1(c) = \lim_{k\to\infty} y_1(\bar\tau_k) \geq y_1(T_y) > 0$. This contradicts (3.3) and proves that $H$-points do not exist on $[t_y, \bar t_y)$, which is impossible. □

The following result can be proved similarly as in Theorem 3.1.

**Theorem 3.2.** *Suppose*

$$\Big|\frac{f_2(t,u,v)}{f_1(t,u,v)}\Big| \text{ is non-decreasing (non-increasing) with respect to } t$$

*on $D$, $uv \neq 0$, and either*

*(i)*

$$\frac{\partial}{\partial v}\frac{f_2(t,u,v)}{f_1(t,u,v)} \text{ is continuous on } D,\ uv \neq 0,$$

*or*

*(ii) for any integer $m$ there is a continuous function $g_m : (0,m] \to (0,\infty)$ such that*

$$\frac{g_m(|u|)|f_1(t,u,v)|}{f_2(t,u,v)} \text{ is non-increasing}$$

*with respect to $u$ for $|u| \in \big[\frac{1}{m}, m\big]$, for any $t \in [0,m]$, and $|v| \in (0,m]$, or*

*(iii) for any integer $m$ there is a continuous function $\bar g_m : (0,m] \to (0,\infty)$ such that*

$$\frac{\bar g_m(|v|)|f_2(t,u,v)|}{f_1(t,u,v)} \text{ is non-increasing}$$

*with respect to $v$ for $v \in (0,m]$ and $v \in [-m,0)$, for any $t \in [0,m]$, and $|u| \in \big[\frac{1}{m}, m\big]$.*

*Let $y$ be an oscillatory solution of (1.1) defined on $[t_y, \bar t_y] \subset R_+$. Then:*

1. *There exists no $H$-point of $y$, $y$ can not be defined at $t = \bar t_y$ and all zeros of $y_2$ can be described by increasing sequence $\{t_k\}_{k=1}^{\infty}$.*
2. *The sequence of all positive local extrema of $y_2$ is non-decreasing (is non-increasing).*
3. *The sequence of the absolute values of all negative local extrema of $y_2$ is non-decreasing (is non-increasing).*
4. *If, moreover,*
$$f_1(t,u,-v) = -f_1(t,u,v)\,, \quad f_2(t,u,-v) = f_2(t,u,v)$$
*on $D$, then the sequence $\big\{|y_2(t_k)|\big\}_{k=1}^{\infty}$ of the absolute values of all local extrema of $y_2$ is non-decreasing (is non-increasing).*

## 4. Applications

We apply our results to equation (1.4).

**Theorem 4.1.** *Suppose $f(t,-u,v) = -f(t,u,v)$ on $D$ and*

$$a^{1/p}(t)|f(t,u,v)| \quad \text{is non-decreasing (non-increasing)}$$

*with respect to $t$ on $D$. Let $y$ be an oscillatory solution of (1.4) defined on $[t_y, \bar t_y)$ and $\{\tau_k\}_{k=1}^{\infty}$ be the increasing sequence of all zeros of $y^{[1]}$ on $[t_y, \bar t_y)$. Let either*

*(i) $\frac{\partial}{\partial u} f(t,u,v)$ be continuous on $D$, $uv \neq 0$, or*

*(ii) For any integer $m$ there is a positive function $g_m \in C^0(0,m]$ such that*

$$g_m(u)f(t,u,v) \text{ is non-decreasing with respect to } u$$

*for $u \in (0,m]$, $t \in [0,m]$ and $|v| \in \left[\frac{1}{m}, m\right]$, or*

*(iii) for any integer $m$ there is a positive function $\bar{g}_m \in C^0(0,m]$ such that*

$$\bar{g}_m(|v|)f(t,u,v)\operatorname{sgn} v \text{ is non-decreasing with respect to } v$$

*for $|v| \in \left[\frac{1}{m}, m\right]$, $t \in [0,m]$ and $u \in (0,m]$.*

*Then $\left\{|y(\tau_k)|\right\}_{k=1}^{\infty}$ is non-increasing (non-decreasing).*

*Proof.* The result follows from Theorem 3.1 since $f_1(t,u,v) = a^{-1/p}(t)|v|^{1/p}\operatorname{sgn} v$ and $f_2(t,u,v) = -f_1(t,u,v)$. If we denote the function $g_m$ from Theorem 3.1 (ii) by $\tilde{g}_m$, then $\tilde{g}_m(z) = 1/g_m(z)$. Similarly, if we denote $\bar{g}_m$ from Theorem 3.1 (iii) by $\bar{g}_m$, then $\bar{g}_m(z) = \bar{g}_m(z)z^{1/p}$. $\square$

**Theorem 4.2.** *Suppose $f(t,u,-v) = f(t,u,v)$ on $D$ and $a^{1/p}(t)|f(t,u,v,)|$ is non-decreasing (non-increasing) with respect to $t$ on $D$. Let $y$ be an oscillatory solution of (1.4) defined on $[t_y, \bar{t}_y)$ and $\{t_k\}_{k=1}^{\infty}$ be the increasing sequence of all zeros of $y$ on $[t_y, \bar{t}_y)$. Let either*

*(i) $\frac{\partial}{\partial v} f(t,u,v)$ be continuous on $D$, $uv \neq 0$, or*

*(ii) for any integer $m$ there is a positive function $g_m \in C^0(0,m]$ such that $g_m(|u|)f(t,u,v)$ is non-increasing with respect to $u$ for $|u| \in \left[\frac{1}{m}, m\right]$, $t \in [0,m]$ and $v \in (0,m]$, or*

*(iii) for any integer $m$ there is a positive function $\bar{g}_m \in C^0(0,m]$ such that $\bar{g}_m(v)|f(t,u,v)|$ is non-increasing with respect to $v$ for $v \in\in (0,m]$, $t \in [0,m]$ and $|u| \in \left[\frac{1}{m}, m\right]$.*

*Then $\left\{|y^{[1]}(t_k)|\right\}_{k=1}^{\infty}$ is non-decreasing (non-increasing).*

The proof of the above theorem is similar to that of Theorem 4.1, using Theorem 3.2.

**Remark 4.3.** *Theorem 1.3 (i) is a special case of Theorem 4.1 (iii) and Theorem 1.3 (ii) follows from Theorem 4.2 (iii).*

Finally, we formulate our results for the equation

$$y^{[1]} + r(t)\,\bar{f}(y)\,h(y^{[1]}) = 0 \tag{4.1}$$

where $p > 0$, $y^{[1]}$ is given by (1.5), $\bar{f} \in C^0(\mathbb{R})$, $h \in C^0(\mathbb{R})$, $\bar{f}(u)u > 0$ for $u \neq 0$, and $h(v) > 0$ on $\mathbb{R}$.

**Corollary 4.4.** *Suppose $a^{1/p}r$ is non-decreasing (non-increasing) on $\mathbb{R}_+$. Let $y$ be an oscillatory solution of (4.1) defined on $[t_y, \bar{t}_y)$ and $\{t_k\}_{k=1}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$ be the increasing sequences of all zeros of $y$ and $y^{[1]}$ on $[t_y, \bar{t}_y)$, respectively.*

*(i) If $\bar{f}(-u) = -\bar{f}(u)$ on $\mathbb{R}$, then the sequence $\left\{|y(\tau_k)|\right\}_{k=1}^{\infty}$ is non-increasing (non-decreasing).*

*(ii) If $h(-v) = h(v)$ on $\mathbb{R}$, then the sequence $\left\{|y^{[1]}(t_k)|\right\}_{k=1}^{\infty}$ is non-decreasing (non-increasing).*

*Proof.* Suppose $a^{-1/p}r$ is non-decreasing. Put $f(t,u,v) = r(t)\,\bar{f}(u)\,h(v)$. Case (i) follows from Theorem 4.1 (ii) with $g_m(u) = (\bar{f}(u))^{-1}$. Case (ii) follows from Theorem 4.2 (iii) with $\bar{g}_m(v) = \frac{1}{h(v)}$. $\square$

**Remark 4.5.** Note that [6, Theorems 3.6 and 3.7] and Theorem 1.4 are special cases of Corollary 4.4 for $h \equiv 1$. Some results in [13, Theorems 4.2 and 4.6] are special cases of Corollary 4.4 (with $p = 1$, $h \equiv 1$).

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