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COMPUTATION OF FOCAL VALUES AND STABILITY ANALYSIS OF 4-DIMENSIONAL SYSTEMS

BO SANG, QIN-LONG WANG, WEN-TAO HUANG

Abstract. This article presents a recursive formula for computing the n-th singular point values of a class of 4-dimensional autonomous systems, and establishes the algebraic equivalence between focal values and singular point values. The formula is linear and then avoids complicated integrating operations, therefore the calculation can be carried out by computer algebra system such as Maple. As an application of the formula, bifurcation analysis is made for a quadratic system with a Hopf equilibrium, which can have three small limit cycles around an equilibrium point. The theory and methodology developed in this paper can be used for higher-dimensional systems.

1. Preliminaries

Consider a C^k-smooth system ($k \geq 2$)

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x,y),\\ \frac{dy}{dt} &= By + g(x,y), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^{n_c}, y \in \mathbb{R}^{n_s}$, A and B are constant matrices, and $f(x,y), g(x,y)$ are functions with

$$f(0,0) = 0, g(0,0) = 0, Df(0,0) = 0, Dg(0,0) = 0.$$

Suppose that A has n_c critical eigenvalues (i.e. eigenvalues with $\operatorname{Re}\lambda = 0$) and all n_s eigenvalues of B satisfy $\operatorname{Re}\lambda < 0$. According to the Center Manifold Theorem (see e.g. [5]), there exists a (local) center manifold $y = h(x)$ with $h(0) = 0, Dh(0) = 0$, and system (1.1) is topologically equivalent near $(0,0)$ to the system

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x,h(x)),\\ \frac{dy}{dt} &= By. \end{aligned} \tag{1.2}$$

The first equation in (1.2) is called the restriction of system (1.1) to its center manifold at the origin. Thus, the dynamics of (1.1) near a non-hyperbolic equilibrium

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are determined by this restriction, since the second equation in (1.2) is linear and has exponentially decaying solutions.

If A has a simple pair of purely imaginary eigenvalues $\pm\omega i$ ($\omega > 0$), system (1.1) undergoes a Hopf bifurcation or a multiple Hopf bifurcation under proper perturbations of parameters. The computation of focal values (Lyapunov coefficients) plays an important role in the study of small-amplitude limit cycles appeared in these bifurcations (see [1, 2, 4, 9, 10, 12, 14, 15] and reference therein). For system (1.1), the projection method was used for computing the first and the second focal values (see [5]); a perturbation technique based on multiple time scales was used for computing focal values (see [13]). For a class of 3-dimensional systems, a recursive formula was presented for computing focal values (see [15]). It should be noted that the theory and methodology described in [11, 15] can be applied to N-dimensional systems, where $N \geq 4$.

2. Focal values of a class of 4-dimensional systems

Consider a class of analytic systems in $\mathbb{R}^4$,

$$
\begin{aligned}
\frac{dx_1}{dt} &= -x_2 + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} \tilde{a}_{j_1,j_2,j_3,j_4} x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4} = X_1(x_1,x_2,x_3,x_4),\\
\frac{dx_2}{dt} &= x_1 + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} \tilde{b}_{j_1,j_2,j_3,j_4} x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4} = X_2(x_1,x_2,x_3,x_4),\\
\frac{dx_3}{dt} &= \mu x_3 - \omega x_4 + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} \tilde{c}_{j_1,j_2,j_3,j_4} x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4}\\
&= X_3(x_1,x_2,x_3,x_4),\\
\frac{dx_4}{dt} &= \omega x_3 + \mu x_4 + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} \tilde{d}_{j_1,j_2,j_3,j_4} x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4}\\
&= X_4(x_1,x_2,x_3,x_4),
\end{aligned}
\tag{2.1}
$$

with $\mu < 0$, $\omega \geq 0$, $x_1, x_2, x_3, x_4, \tilde{a}_{j_1j_2j_3j_4}, \tilde{b}_{j_1j_2j_3j_4}, \tilde{c}_{j_1j_2j_3j_4}, \tilde{d}_{j_1j_2j_3j_4}$ in $\mathbb{R}$, and j_1, j_2, j_3, j_4 in $\mathbb{N}$. Our motivation for the study of system (2.1) is that a physical model of airfoil with cubic nonlinearity can be transformed into a special case of system (2.1) via a linear transformation, see [6, 16].

The Jacobian matrix of system (2.1) has eigenvalues $\pm i, \mu\pm\omega i$ at the equilibrium $(x_1,x_2,x_3,x_4) = (0,0,0,0)$. Then by the Center Manifold Theorem system (2.1) has a (local) center manifold tangent to the (x_1,x_2) plane at the origin. Moreover this center manifold can be represented as

$$
\begin{aligned}
x_3 &= g_1(x_1,x_2),\\
x_4 &= g_2(x_1,x_2),
\end{aligned}
\tag{2.2}
$$

where $g_1(0,0) = g_2(0,0) = 0, Dg_1(0,0) = Dg_2(0,0) = 0$, and the dynamics of (2.1) restricted to the center manifold are given by

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 + \sum_{j_1+j_2=2}^{\infty} \tilde{a}_{j_1,j_2} x_1^{j_1} x_2^{j_2} = X_1(x_1,x_2,g_1(x_1,x_2),g_2(x_1,x_2)), \\ \frac{dx_2}{dt} &= x_1 + \sum_{j_1+j_2=2}^{\infty} \tilde{b}_{j_1,j_2} x_1^{j_1} x_2^{j_2} = X_2(x_1,x_2,g_1(x_1,x_2),g_2(x_1,x_2)), \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} \frac{dg_1(x_1,x_2)}{dt}\Big|_{(2.3)} &= X_3(x_1,x_2,g_1(x_1,x_2),g_2(x_1,x_2)), \\ \frac{dg_2(x_1,x_2)}{dt}\Big|_{(2.3)} &= X_4(x_1,x_2,g_1(x_1,x_2),g_2(x_1,x_2)). \end{aligned}$$

From [8], we know that for system (2.3) one can derive successively and uniquely the terms of the following formal series

$$\tilde{F}(x_1,x_2) = x_1^2 + x_2^2 + h.o.t. \tag{2.4}$$

such that

$$\begin{aligned} \frac{d\tilde{F}}{dt}\Big|_{(2.3)} &= \frac{\partial \tilde{F}}{\partial x_1} X_1(x_1,x_2,g_1(x_1,x_2),g_2(x_1,x_2)) \\ &\quad + \frac{\partial \tilde{F}}{\partial x_2} X_2(x_1,x_2,g_1(x_1,x_2),g_2(x_1,x_2)), \\ &= \sum_{n=1}^{\infty} V_n (x_1^2+x_2^2)^{n+1}, \end{aligned} \tag{2.5}$$

where V_n are called the nth focal values of system (2.3) and the original system (2.1), and the acronym h.o.t. stands for higher-order terms.

By change of variables

$$\begin{gathered} x_1 = \frac{1}{2}(x+y), \quad x_2 = -\frac{i}{2}(x-y), \quad x_3 = \frac{1}{2}(z+u), \\ x_4 = -\frac{i}{2}(z-u), \quad t = -iT, \end{gathered} \tag{2.6}$$

the original system (2.1) can be transformed into the following complex system

$$\begin{aligned} \frac{dx}{dT} &= x + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} a_{j_1,j_2,j_3,j_4} x^{j_1} y^{j_2} z^{j_3} u^{j_4} = X(x,y,z,u), \\ \frac{dy}{dT} &= -y + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} b_{j_1,j_2,j_3,j_4} x^{j_1} y^{j_2} z^{j_3} u^{j_4} = Y(x,y,z,u), \\ \frac{dz}{dT} &= (\omega - \mu i)z + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} c_{j_1,j_2,j_3,j_4} x^{j_1} y^{j_2} z^{j_3} u^{j_4} = Z(x,y,z,u), \\ \frac{du}{dT} &= -(\omega + \mu i)u + \sum_{j_1+j_2+j_3+j_4=2}^{\infty} d_{j_1,j_2,j_3,j_4} x^{j_1} y^{j_2} z^{j_3} u^{j_4} = U(x,y,z,u). \end{aligned} \tag{2.7}$$

And, applying the transformation

$$x_1 = \frac{1}{2}(x+y), x_2 = -\frac{i}{2}(x-y), t = -iT, \tag{2.8}$$

the restriction system (2.3) can be transformed into the following complex system

$$\begin{aligned}\frac{dx}{dT} &= x + \sum_{j_1+j_2=2}^{\infty} a_{j_1,j_2} x^{j_1} y^{j_2} = X(x,y,z(x,y),u(x,y)),\\ \frac{dy}{dT} &= -y + \sum_{j_1+j_2=2}^{\infty} b_{j_1,j_2} x^{j_1} y^{j_2} = Y(x,y,z(x,y),u(x,y)),\end{aligned} \tag{2.9}$$

where

$$\begin{aligned} z(x,y) &= x_3 + ix_4 \\ &= g_1(x_1,x_2) + ig_2(x_1,x_2) \\ &= g_1(\frac{x+y}{2}, -\frac{i}{2}(x-y)) + ig_2(\frac{x+y}{2}, -\frac{i}{2}(x-y)),\end{aligned} \tag{2.10}$$

$$\begin{aligned} u(x,y) &= x_3 - ix_4 \\ &= g_1(x_1,x_2) - ig_2(x_1,x_2) \\ &= g_1(\frac{x+y}{2}, -\frac{i}{2}(x-y)) - ig_2(\frac{x+y}{2}, -\frac{i}{2}(x-y)).\end{aligned} \tag{2.11}$$

with

$$\begin{aligned} \frac{dz(x,y)}{dT}\Big|_{(2.9)} &= Z(x,y,z(x,y),u(x,y)),\\ \frac{du(x,y)}{dT}\Big|_{(2.9)} &= U(x,y,z(x,y),u(x,y)).\end{aligned} \tag{2.12}$$

Lemma 2.1 ([7]). *For system* (2.9), *one can derive successively and uniquely the terms of the formal series*

$$G(x,y) = xy + \sum_{\alpha+\beta=3}^{\infty} C_{\alpha,\beta} x^{\alpha} y^{\beta},$$

such that

$$\begin{aligned} \frac{dG}{dT}\Big|_{(2.9)} &= \frac{\partial G}{\partial x} X(x,y,z(x,y),u(x,y)) + \frac{\partial G}{\partial y} Y(x,y,z(x,y),u(x,y)) \\ &= \sum_{n=1}^{\infty} W_n^{(2)} (xy)^{n+1},\end{aligned} \tag{2.13}$$

where $C_{\alpha,\beta}$ are determined by the recursive formula (see [7]*), and $W_n^{(2)}$ are called the nth singular point values of system* (2.9).

Lemma 2.2 ([7]). *If $W_1^{(2)} = W_2^{(2)} = \dots = W_{n-1}^{(2)} = 0$, we have*

$$V_n = iW_n^{(2)}, \tag{2.14}$$

where V_n is the nth focal value of system (2.3)*, and $W_j^{(2)}$ are the jth singular point values of system* (2.9)*, $j = 1, 2, \dots, n-1, n$.*

Theorem 2.3. *For system* (2.7)*, we can derive successively and uniquely the terms of the formal series*

$$F(x,y,z,u) = xy + \sum_{\alpha+\beta+\gamma+\delta=3}^{\infty} C_{\alpha,\beta,\gamma,\delta} x^{\alpha} y^{\beta} z^{\gamma} u^{\delta},$$

$$\triangleq \sum_{\alpha+\beta+\gamma+\delta=2}^{\infty} C_{\alpha,\beta,\gamma,\delta} x^\alpha y^\beta z^\gamma u^\delta,$$

such that

$$\frac{dF}{dT}\Big|_{(2.7)} = \frac{\partial F}{\partial x}X + \frac{\partial F}{\partial y}Y + \frac{\partial F}{\partial z}Z + \frac{\partial F}{\partial u}U = \sum_{n=1}^{\infty} W_n (xy)^{n+1}, \tag{2.15}$$

where $C_{\alpha,\beta,\gamma,\delta}$ *are determined by the recursive formula*

$$\begin{aligned} C_{\alpha,\beta,\gamma,\delta} &= \frac{1}{\beta-\alpha-\gamma(\omega-\mu i)+\delta(\omega+\mu i)} \\ &\quad\times \sum_{j_1+j_2+j_3+j_4=3}^{\alpha+\beta+\gamma+\delta+2} \Big[(\alpha-j_1+1)a_{j_1,j_2-1,j_3,j_4} + (\beta-j_2+1)b_{j_1-1,j_2,j_3,j_4} \\ &\quad + (\gamma-j_3)c_{j_1-1,j_2-1,j_3+1,j_4} + (\delta-j_4)d_{j_1-1,j_2-1,j_3,j_4+1}\Big] \\ &\quad\times C_{\alpha-j_1+1,\beta-j_2+1,\gamma-j_3,\delta-j_4} \end{aligned} \tag{2.16}$$

with $C_{\alpha,\alpha,0,0}=0$, W_n *are determined by the recursive formula*

$$W_n = \sum_{j_1+j_2=3}^{2n+4} [(n-j_1+2)a_{j_1,j_2-1,0,0} + (n-j_2+2)b_{j_1-1,j_2,0,0}]C_{n-j_1+2,n-j_2+2,0,0}. \tag{2.17}$$

Proof. Note that

$$\begin{aligned} \frac{dF}{dT}\Big|_{(2.7)} &= \frac{\partial F}{\partial x}X + \frac{\partial F}{\partial y}Y + \frac{\partial F}{\partial z}Z + \frac{\partial F}{\partial u}U \\ &= \sum_{\alpha+\beta+\gamma+\delta\geq 2} [\alpha-\beta+\gamma(\omega-\mu i)-\delta(\omega+\mu i)]C_{\alpha,\beta,\gamma,\delta}x^\alpha y^\beta z^\gamma u^\delta \\ &\quad + \sum_{\alpha+\beta+\gamma+\delta\geq 2}\ \sum_{j_1+j_2+j_3+j_4\geq 3} \Big(\alpha a_{j_1,j_2-1,j_3,j_4} + \beta b_{j_1-1,j_2,j_3,j_4} \\ &\quad + \gamma c_{j_1-1,j_2-1,j_3+1,j_4} + \delta d_{j_1-1,j_2-1,j_3,j_4+1}\Big) \\ &\quad \times C_{\alpha,\beta,\gamma,\delta} x^{\alpha+j_1-1} y^{\beta+j_2-1} z^{\gamma+j_3} u^{\delta+j_4} \\ &= \sum_{\alpha+\beta+\gamma+\delta\geq 2} x^\alpha y^\beta z^\gamma u^\delta \Big\{[\alpha-\beta+\gamma(\omega-\mu i)-\delta(\omega+\mu i)]C_{\alpha,\beta,\gamma,\delta} \\ &\quad + \sum_{j_1+j_2+j_3+j_4\geq 3} \Big[(\alpha-j_1+1)a_{j_1,j_2-1,j_3,j_4} + (\beta-j_2+1)b_{j_1-1,j_2,j_3,j_4} \\ &\quad + (\gamma-j_3)c_{j_1-1,j_2-1,j_3+1,j_4} + (\delta-j_4)d_{j_1-1,j_2-1,j_3,j_4+1}\Big] \\ &\quad \times C_{\alpha-j_1+1,\beta-j_2+1,\gamma-j_3,\delta-j_4}\Big\}. \end{aligned}$$

Comparing the above power series with the right side of (2.15), we can obtain the recursive formulas (2.16) and (2.17). □

Theorem 2.4. *If* $W_1 = W_2 = \dots = W_{n-1} = 0$, *then*

$$W_n^{(2)} = W_n, \tag{2.18}$$

where W_j are the jth singular point values of system (2.7)*, and $W_n^{(2)}$ is the nth singular point value of system*(2.9)*.*

Proof. Suppose that $W_1 = W_2 = \dots = W_{n-1} = 0$, there exists a unique polynomial

$$F_{2n+2}(x, y, z, u) = xy + \sum_{\alpha+\beta+\gamma+\delta=3}^{2n+2} C_{\alpha,\beta,\gamma,\delta} x^\alpha y^\beta z^\gamma u^\delta, \tag{2.19}$$

such that

$$\frac{dF_{2n+2}}{dT}\Big|_{(2.7)} = \frac{\partial F_{2n+2}}{\partial x} X + \frac{\partial F_{2n+2}}{\partial y} Y + \frac{\partial F_{2n+2}}{\partial z} Z + \frac{\partial F_{2n+2}}{\partial u} U = W_n (xy)^{n+1} + \dots. \tag{2.20}$$

Let

$$G_{2n+2}(x, y) = F_{2n+2}(x, y, z(x, y), u(x, y)), \tag{2.21}$$

where $z(x, y), u(x, y)$ are determined by conditions (2.10) and (2.11). We have

$$\begin{aligned}
\frac{dG_{2n+2}}{dT}\Big|_{(2.9)} &= \frac{\partial G_{2n+2}}{\partial x} X(x, y, z(x, y), u(x, y)) + \frac{\partial G_{2n+2}}{\partial y} Y(x, y, z(x, y), u(x, y)) \\
&= \Big(\frac{\partial F_{2n+2}}{\partial x} + \frac{\partial F_{2n+2}}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_{2n+2}}{\partial u}\frac{\partial u}{\partial x}\Big) X(x, y, z(x, y), u(x, y)) \\
&\quad + \Big(\frac{\partial F_{2n+2}}{\partial y} + \frac{\partial F_{2n+2}}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_{2n+2}}{\partial u}\frac{\partial u}{\partial y}\Big) Y(x, y, z(x, y), u(x, y)) \\
&= \frac{\partial F_{2n+2}}{\partial x} X(x, y, z(x, y), u(x, y)) + \frac{\partial F_{2n+2}}{\partial y} Y(x, y, z(x, y), u(x, y)) \\
&\quad + \frac{\partial F_{2n+2}}{\partial z}\Big(\frac{\partial z}{\partial x} X(x, y, z(x, y), u(x, y)) + \frac{\partial z}{\partial y} Y(x, y, z(x, y), u(x, y))\Big) \\
&\quad + \frac{\partial F_{2n+2}}{\partial u}\Big(\frac{\partial u}{\partial x} X(x, y, z(x, y), u(x, y)) + \frac{\partial u}{\partial y} Y(x, y, z(x, y), u(x, y))\Big) \\
&= \frac{\partial F_{2n+2}}{\partial x} X(x, y, z(x, y), u(x, y)) + \frac{\partial F_{2n+2}}{\partial y} Y(x, y, z(x, y), u(x, y)) \\
&\quad + \frac{\partial F_{2n+2}}{\partial z}\frac{dz(x, y)}{dT}\Big|_{(2.9)} + + \frac{\partial F_{2n+2}}{\partial u}\frac{du(x, y)}{dT}\Big|_{(2.9)} \\
&= \frac{\partial F_{2n+2}}{\partial x} X(x, y, z(x, y), u(x, y)) + \frac{\partial F_{2n+2}}{\partial y} Y(x, y, z(x, y), u(x, y)) \\
&\quad + \frac{\partial F_{2n+2}}{\partial z} Z(x, y, z(x, y), u(x, y)) + \frac{\partial F_{2n+2}}{\partial u} U(x, y, z(x, y), u(x, y)) \\
&= \frac{dF_{2n+2}}{dT}\Big|_{(2.7), z=z(x,y), u=u(x,y)} \\
&= W_n (xy)^{n+1} + \dots.
\end{aligned}$$

Hence, the nth singular point value $W_n^{(2)}$ of system (2.9) satisfies $W_n^{(2)} = W_n$. $\square$

Summarizing the results of Lemma 2.2 and Theorem 2.4 we obtain the following Corollary.

Corollary 2.5. *If $W_1 = W_2 = \dots = W_{n-1} = 0$, then*

$$V_n = iW_n, \tag{2.22}$$

where V_n is the nth focal value of system (2.1)*, and W_j are the jth singular point values of system* (2.7)*, $j = 1, 2, \dots, n-1, n$.*

3. Stability analysis and degenerate Hopf bifurcation of 4-dimensional systems

As an application of the results obtained in Section 2, we consider a class of 4-dimensional quadratic systems

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 - a_1x_1x_3 + a_1{x_1}^2, \\ \frac{dx_2}{dt} &= x_1 + a_2{x_2}^2 + a_3x_1x_4, \\ \frac{dx_3}{dt} &= -x_3 - 3\,x_4 + a_2x_1x_2, \\ \frac{dx_4}{dt} &= 3\,x_3 - x_4 + a_4x_2x_3. \end{aligned} \tag{3.1}$$

Applying the recursive formulas (2.16), (2.17) and the relation (2.22), we obtain the first four focal values V_1, V_2, V_3, V_4 of system (3.1), where

$$V_1 = -\frac{a_2(-9\,a_3 + 4\,a_1)}{104},$$

$$\begin{aligned} V_2 = \Big(&a_2a_3\big(16288820\,{a_2}^2 - 609030\,a_2a_3 - 342086\,a_2a_4 - 95543550\,{a_3}^2 \\ &- 890367\,a_3a_4 + 235464\,{a_4}^2\big)\Big)/26166400, \end{aligned}$$

$$\begin{aligned} V_3 = -\frac{a_2a_1}{50958140848561524579655680 0}\Big(&22471616049868116382876934 4{a_1}^4 \\ &+ 23391560818512386519330256a_2{a_1}^3 \\ &+ 108275873757375918099514632a_4{a_1}^3 \\ &- 519476645118198963455620296{a_2}^2{a_1}^2 \\ &+ 308977993201985418967419 88{a_1}^2a_2a_4 \\ &- 260885585926993858543534 8{a_2}^3a_1 \\ &- 107827566965381888322448434a_1{a_2}^2a_4 \\ &+ 30378631999445958792929297 4{a_2}^4 \\ &- 22649516190250502017478481{a_2}^3a_4\Big), \end{aligned}$$

$$\begin{aligned} V_4 = \big(&a_1a_2^3/1233873083997871085151833704681793663152993586503426 67 \\ &60594157221033425381170985060953777258880532456181966071998 39 \\ &4040115200000\big)V_{4,1} \end{aligned}$$

and

$$\begin{aligned} V_{4,1} = {}&151090339549302141420998778418569305630254453586345260252801201492 \\ &1956725316080643463518784097032674882123877575141343312126073 6a_1^4 \\ &+ 690237652936743257980025265621054430785609011059037070857845630 88 \\ &5126271156862905782616567832231124991999486811410920350778464a_2{a_1}^3 \\ &- 4852869311179757471922805489901407470548086994648890234025102002 \\ &3568501430203732065339603894449313619014039201735819511942218584 \end{aligned}$$

$$
\begin{aligned}
&\times a_2^2 {a_1}^2 \\
&+ 2215252747679276791540976334249389814784586273507216960403376135382138357119539873166593570355631072430839228186442419287414353 2 \\
&\times a_1^2 a_2 a_4 \\
&+ 434975337903447724813720015538420471541695549645684668496869670264195433354524408423563189743118796384550034676714774717230756 8 \\
&\times {a_2}^3 a_1 \\
&+ 139925156878486203251650275215133948778485624374385690191244381757296147082412525261272412146533467957996575927345012601889662 4 \\
&\times a_1 {a_2}^2 a_4 \\
&+ 341304513647970775868513297116704058551768818382336833946123818374878047494658668008709582475994561655861103494601099434366516 26 \\
&\times a_2^4 \\
&- 217363999665661786745515906533803710384221177409686756105142957062031823888561134016242011964053474562159924634330414500520983 71 \\
&\times a_2^3 a_4.
\end{aligned}
$$

Here V_j is reduced modulo the Gröbner basis of $\{V_s : 1 \le s \le j-1\}$ for $j \ge 2$.

From (2.5) and the computation of focal values, we get the following result on the stability of equilibrium for system (3.1).

Theorem 3.1. *Let $\mathfrak{X}$ be the vector field determined on $\mathbb{R}^4$ by system* (3.1)*. For any center manifold W^c of* (3.1) *at the origin of $\mathbb{R}^4$, with regard to $\mathfrak{X}|W^c$:*

1. *if $V_1 \ne 0$ then the origin is a first order fine focus, whose stability is determined by $\operatorname{sgn} V_1$ (i.e., is asymptotically stable if and only if $V_1 < 0$);*
2. *if $V_1 = 0, V_2 \ne 0$ then the origin is a second order fine focus, whose stability is determined by $\operatorname{sgn} V_2$;*
3. *if $V_1 = V_2 = 0, V_3 \ne 0$ then the origin is a third order fine focus, whose stability is determined by $\operatorname{sgn} V_3$;*
4. *if $V_1 = V_2 = V_3 = 0, V_4 \ne 0$ then the origin is a fourth order fine focus, whose stability is determined by $\operatorname{sgn} V_4$.*

Theorem 3.2. *Suppose that*

$$
\begin{aligned}
a_1 &= \mu\, a_2, \\
a_3 &= \frac{4}{9}\mu\, a_2, \\
a_4 &= \frac{F_1(a_2)}{F_2},
\end{aligned}
\tag{3.2}
$$

where μ is one of four real roots of the polynomial

$$
\begin{aligned}
P(\lambda) = {} & 10799985775088040041585439500413156070699520\lambda^8 \\
& + 6212790925817703957911789339676019539872928\lambda^7 \\
& - 28584647405659759515776062165338760876812544\lambda^6 \\
& - 15713629519470248682246891010001873803911248\lambda^5 \\
& + 23373024697622801653163411830546195419879000\lambda^4 \\
& + 13015318562221586811114336007971951317356918\lambda^3 \\
& - 4091058129972655937988952508114319926195742\lambda^2 \\
& - 3522262300428215202057982497109271042377107\lambda \\
& - 1499503145083805883162951654763171839112800,
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
F_1(a_2) = -6a_2\Big(& 37452693416446860638128224\mu^4 + 3898593469752064419888376\mu^3 \\
& - 86579440853033160575936716\mu^2 - 434809309878323097572558\mu \\
& + 50631053332409931321548829\Big),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
F_2 = {} & 108275873757375918099514632\mu^3 + 30897799320198541896741988\mu^2 \\
& - 107827566965381888322448434\mu - 22649516190250502017478481,
\end{aligned} \tag{3.5}
$$

$$
a_2 \neq 0, \tag{3.6}
$$

then the origin of system (3.1) *is a fourth order fine focus.*

Proof. Let condition (3.2) be satisfied, then the first four focal values of system (3.1) are as follows: $V_1 = V_2 = V_3 = 0$,

$$
\begin{aligned}
V_4 = {} & -{a_2}^8 Q(\mu)/36887588669248862905116889599301849674634285420053378493 \\
& 1286682303008351322973629074374337069105705808904284987022190533816 \\
& 3182367071153612906739577075384387480378163200000,
\end{aligned}
$$

where

$$
\begin{aligned}
Q(\mu) = {} & 5061565304752756281507283445466696498397806982562902502862568707 \\
& 12912529775726964251246968509875875991228610701216698005689607424 \\
& 73087731876897344650076374604360705836087040\mu^7 \\
& - 340812912483222809928339721244205680969631979595064645753790303 \\
& 68058287137415281782406391986081459447638560645736575391889270596 \\
& 577079487175148071589735417325410968915137744\mu^6 \\
& - 824956771772875920524768605310963288557474695808927379648635146 \\
& 89934278817997190177837875064854601374617126333206151559276636649 \\
& 987895973437781566179221118658577582087242584\mu^5 \\
& + 81584483608286871615772633434382546877277513771408741974445245 \\
& 4845368677016347807819376306567104009462092441916792312570408246
\end{aligned}
$$

$$
\begin{aligned}
&023720310120199978156628709706339448756299277640\,\mu^4 \\
&+ 25896513600610435149766065028213468848105899037674273723965973 \\
&85292517796469005762508262417287664215744015726820682298861228047 \\
&9466387807108601473511477377951770327152251930\mu^3 \\
&- 63871971031084648777832890365823291014772224724716750214834580 \\
&98528211526941914434037357772542521425827671755298957937753965 85 \\
&05625608488378226940819951947411084516792539928\mu^2 \\
&+ 63320363852535951596180397565233937233465160852321916775662978 \\
&7014918692824778554364353862099348531370990541173801942371682611 \\
&4200270000631493834542751507656379882694394067\mu \\
&+ 15864826572799943856438557624408837065205007313716175374949797 \\
&77238640796696548974290929643892678874375049156348547097876683 72 \\
&9010303052391702074091140390304947430207112560 0.
\end{aligned}
$$

Computing the resultant $R(P,Q)$ between two polynomials $P(\lambda), Q(\lambda)$, we obtain $R(P,Q) \neq 0$. Thus $V_4 \neq 0$ and the origin is a fourth order fine focus for system (3.1). □

Theorem 3.3. *If condition* (3.2) *holds, then there are perturbations of system* (3.1) *yielding three small-amplitude limit cycles bifurcating from the origin.*

Proof. Under condition (3.2), the Jacobian matrix of focal values V_1, V_2, V_3 of system (3.1) with respect to a_1, a_3, a_4 has full rank, i.e.,

$$
\operatorname{rank}\big[\frac{\partial(V_1, V_2, V_3)}{\partial(a_1, a_3, a_4)}\big]_{(3.2)} = 3,
$$

hence by [3, Theorem 2.3.2] the claim follows. □

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Bo Sang (corresponding author)
School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China
E-mail address: sangbo_76@163.com

Qing-Long Wang
School of Science, Hezhou University, Hezhou 542800, China
E-mail address: wqinlong@163.com

Wen-Tao Huang
School of Science, Hezhou University, Hezhou 542800, China
E-mail address: huangwentao@163.com