Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 203, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

RAPID CONVERGENCE OF APPROXIMATE SOLUTIONS FOR SINGULAR DIFFERENTIAL SYSTEMS

PEIGUANG WANG, XIANG LIU

Abstract. In this article we show the existence and approximation of solutions for a class of singular differential system with initial value condition. We present a generalized quasilinearization method for obtain monotone iterative sequences of approximate solutions converging uniformly to a solution at a rate higher than quadratic.

1. Introduction

In 1974, Rosenbrock [17] introduced the concept of singular systems, which is more complicated than the ordinary ones, and its qualitative analysis involve greater difficulty than those of the ordinary systems. A systematic development of the basic theory of the linear singular systems has been provided by Campbell [5, 6]. The results of qualitative properties for nonlinear singular differential systems can be found in [9, 12, 18, 19, 20, 22, 23]. However, we noticed that most of the previous results focused on stability problems. In fact, the convergence of the solution has an important function in the development of qualitative theory, and the rapid convergence of solutions is also very important in practical applications.

Generalized quasilinearization is an efficient method for constructing approximate solutions of nonlinear problems. Bellman and Kalaba [2], Lakshmikantham and Vatsala [10] gave a systematic development of the method to ordinary differential equations. Up till now, only some rapid convergence results have been found on ordinary differential equations, for initial value problems [3, 11, 13, 14], for boundary value problem [4, 15, 21]. The applications of the method of quasilinearization in singular differential systems are rare [1, 7, 8, 16]. For example, in [1], the authors investigated the uniform and quadratic convergence of singular differential systems. We do not find any results on the rapid convergence of solutions for singular systems.

In this paper, we discuss the existence and uniqueness of solutions and give rapid convergence result of solutions for singular differential systems by using the method of generalized quasilinearization and the order relation of lower and upper solutions.

2010 *Mathematics Subject Classification.* 34A09, 34A45.
Key words and phrases. Singular system; generalized quasilinearization; rapid convergence.

Submitted February 12, 2015. Published August 10, 2015.

2. Preliminaries and Lemmas

We consider the following initial value problem for singular differential system

$$\begin{gathered} Ax' = f(t,x), \quad t \in J, \\ x(0) = x_0. \end{gathered} \tag{2.1}$$

where A is a singular $n \times n$ matrix, $x \in R^n$, $f \in C(J \times R^n, R^n)$, $J = [0,a]$, $a > 0$ is a fixed constant.

Definition 2.1. The function $\alpha_0 \in C^1(J, R^n)$ is called a lower solution of (2.1), if the following inequalities are satisfied:

$$\begin{gathered} A\alpha_0' \le f(t,\alpha_0), \quad t \in J, \\ \alpha_0(0) \le x_0. \end{gathered} \tag{2.2}$$

Definition 2.2. The function $\beta_0 \in C^1(J, R^n)$ is called an upper solution of IVP (2.1), if the following inequalities are satisfied:

$$\begin{gathered} A\beta_0' \ge f(t,\beta_0), \quad t \in J, \\ \beta_0(0) \ge x_0. \end{gathered} \tag{2.3}$$

In our further investigations we will need some results on linear singular differential inequalities and linear singular differential systems. Consider the singular differential inequality

$$Ax' + M(t)x \le 0, \quad x(0) \le 0, \quad t \in J, \tag{2.4}$$

where A, $M(t)$ are $n \times n$ matrices, A is singular and $M(t)$ is continuous for $t \in J$.

Lemma 2.3 ([1]). *Assume that*

- (A1) *There exists a constant λ such that, $L(t) = [\lambda A + M(t)]^{-1}$ exists and $\hat{A} = AL(t)$ is a constant matrix.*
- (A2) *There exists a nonsingular matrix T such that T^{-1}, $(LT)^{-1}$ exist and T^{-1}, (LT), $(LT)^{-1} \ge 0$, satisfying*
$$T^{-1}\hat{A}T = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \quad T^{-1}[I - \lambda\hat{A}]T = \begin{pmatrix} I_1 - \lambda C & 0 \\ 0 & I_2 \end{pmatrix},$$
where C is a diagonal matrix with $C^{-1} \ge 0$.

Then $x(0) \le 0$ implies $x(t) \le 0$ for $t \in J$.

For the singular linear initial value problem

$$Ax' + M(t)x = g(t), \quad x(0) = y_0, \tag{2.5}$$

we have the following result.

Lemma 2.4 ([6]). *Assume that condition* (A1) *holds,* $\operatorname{index}(A) = 1$ *and*

- (A3) *y_0 satisfies $(I - \hat{A}\hat{A}^D)(y_0 - w(0)) = 0$, where $w(t) = \hat{M}^D g(t)$, $\hat{M} = M(t)L(t)$.*

Then the unique solution $y(t)$ of

$$\hat{A}y' + \hat{M}y = g(t), \quad y(0) = y_0. \tag{2.6}$$

is given by

$$y(t) = e^{-\hat{A}^D\hat{M}t}\hat{A}\hat{A}^D y_0 + e^{-\hat{A}^D\hat{M}t}\int_0^t e^{\hat{A}^D\hat{M}s}\hat{A}^D g(s)ds + (I - \hat{A}\hat{A}^D)\hat{M}^D g(t),$$

where the notation $\hat{A}^D, \hat{M}^D$ mean the Drazin inverse of the matrices $\hat{A}, \hat{M}$ respectively. Furthermore, we note that $x(t) = L(t)y(t)$, then we obtain the solution $x(t)$ of (2.5).

To prove our main result, we need the following comparison result.

Lemma 2.5. *Assume that the conditions* (A1), (A2) *hold, and*

- (A4) *The functions α_0, $\beta_0 \in C^1(J, R^n)$ are lower and upper solutions of* (2.1), *f_x exists and are continuous.*

Then $\alpha_0(0) \le \beta_0(0)$ implies that $\alpha_0(t) \le \beta_0(t)$ on J.

Proof. It can be noted from the condition (A4) that

$$\begin{aligned} A\alpha_0' - A\beta_0' &\le f(t, \alpha_0) - f(t, \beta_0) \\ &= \Big(\int_0^1 f_x(t, \sigma\alpha_0 + (1-\sigma)\beta_0) d\sigma \Big)(\alpha_0 - \beta_0). \end{aligned}$$

Taking

$$M(t) = -\Big(\int_0^1 f_x(t, \sigma\alpha_0 + (1-\sigma)\beta_0) d\sigma \Big),$$

we have

$$A(\alpha_0 - \beta_0)' + M(t)(\alpha_0 - \beta_0) \le 0.$$

Noting that $\alpha_0(0) \le \beta_0(0)$, therefore, by Lemma 2.3, we have $\alpha_0(t) \le \beta_0(t)$ on J. $\square$

The following existence result is needed for our main result. For convenience we define the set

$$S(\alpha_0, \beta_0) = \{u \in C(J, R^n) : \alpha_0(t) \le u(t) \le \beta_0(t),\ t \in J\}.$$

Lemma 2.6. *Assume that the conditions* (A1)–(A3) *hold, and*

- (A5) *The functions α_0, $\beta_0 \in C^1(J, R^n)$ are lower and upper solutions of* (2.1) *with $\alpha_0 \le \beta_0$ on J.*
- (A6) *The function $f \in C(S(\alpha_0, \beta_0), R^n)$ satisfies the inequality*

$$f(t, x) - f(t, y) \ge -M_0(x - y)$$

for $x \ge y$, $M(t_0) = M_0$, and $t_0 \in J$.

Then (2.1) *has a solution $x(t)$ that satisfies $\alpha_0(t) \le x(t) \le \beta_0(t)$ on J.*

Proof. Let α_{n+1} and β_{n+1} be the solutions of the singular linear systems

$$\begin{gathered} A\alpha_{n+1}' = f(t, \alpha_n) - M_0(\alpha_{n+1} - \alpha_n), \quad t \in J, \\ \alpha_{n+1}(0) = x_0, \end{gathered} \tag{2.7}$$

and

$$\begin{gathered} A\beta_{n+1}' = f(t, \beta_n) - M_0(\beta_{n+1} - \beta_n), \quad t \in J, \\ \beta_{n+1}(0) = x_0, \end{gathered} \tag{2.8}$$

where α_{n+1} and β_{n+1} exist because of Lemma 2.4. According to the iterative schemes (2.7) and (2.8), we obtain the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ which were generated by the initial conditions $\alpha_0(t)$ and $\beta_0(t)$, respectively.

Let $n = 0$, we first show that $\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t)$ on J.

For this purpose, setting $p(t) = \alpha_0(t) - \alpha_1(t)$. Using the condition (A5), we obtain

$$\begin{aligned} Ap' &= A(\alpha_0 - \alpha_1)' \\ &\le f(t,\alpha_0) - [f(t,\alpha_0) - M_0(\alpha_1 - \alpha_0)] \\ &= -M_0 p, \quad t \in J. \end{aligned}$$

Noting that $p(0) \le 0$, by Lemma 2.3, we have $p(t) \le 0$, that is, $\alpha_0(t) \le \alpha_1(t)$ on J. similarly, letting $p(t) = \beta_1(t) - \beta_0(t)$, we can show that $\beta_1(t) \le \beta_0(t)$ on J.

To prove $\alpha_1(t) \le \beta_1(t)$. Setting $p(t) = \alpha_1(t) - \beta_1(t)$ so that $p(0) = 0$. From the condition (A6), we have

$$\begin{aligned} Ap' &= A(\alpha_1 - \beta_1)' \\ &= f(t,\alpha_0) - M_0(\alpha_1 - \alpha_0) - [f(t,\beta_0) - M_0(\beta_1 - \beta_0)] \\ &\le M_0(\beta_0 - \alpha_0) - M_0(\alpha_1 - \alpha_0) + M_0(\beta_1 - \beta_0) \\ &= -M_0 p, \quad t \in J. \end{aligned}$$

As before, this implies that $\alpha_1(t) \le \beta_1(t)$ on J. Thus, we conclude that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \quad t \in J.$$

The process can be continued successively to obtain that

$$\alpha_0(t) \le \alpha_1(t) \le \dots \alpha_n(t) \le \beta_n(t) \le \dots \le \beta_1(t) \le \beta_0(t), \quad t \in J.$$

It is easy to see that the sequence $\{\alpha_n(t)\}$ is uniformly bounded and equicontinuous, employing the Ascoli-Arzela Theorem, the nondecreasing sequence $\{\alpha_n(t)\}$ has a pointwise limit $x(t)$ that satisfies $\alpha_0(t) \le x(t) \le \beta_0(t)$. A passage to the limit based on the Dominated Convergence Theorem shows that $x(t)$ is a solution of

$$\begin{gathered} Ax' = f(t,x) - M_0(x - x), \quad t \in J, \\ x(0) = x_0; \end{gathered}$$

that is, $x(t)$ is a solution of (2.1). Thus, we conclude that there exists a solution $x(t)$ of IVP (2.1) satisfies $\alpha_0(t) \le x(t) \le \beta_0(t)$ on J. The proof is complete. □

3. Main results

In this section we prove the convergence of the sequence of successive approximations is of order $k \ge 2$.

Theorem 3.1. *Assume that the conditions* (A1)–(A3), (A6) *hold, and*

- (A7) *The functions* α_0, $\beta_0 \in C^1(J, R^n)$ *are lower and upper solutions of* (2.1) *with* $\alpha_0 \le \beta_0$ *on* J.
- (A8) *The Frechet derivatives* $\frac{\partial^i f(t,x)}{\partial x^i}$ $(i = 0, 1, 2, \dots, k)$ *exist and are continuous satisfying* $f(t,x) + Mx^k$ *is* $(k-1)$*-hyperconvex and* $f(t,x) - Nx^k$ *is* $(k-1)$*-hyperconcave in* x*; that is,* $\frac{\partial^k (f(t,x) + Mx^k)}{\partial x^k} \ge 0$ *and* $\frac{\partial^k (f(t,x) - Nx^k)}{\partial x^k} \le 0$, *where the* $n \times n$ *matrices* M, $N > 0$, $k \ge 1$.

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *which convergence is of order* k*, that is, there exist constant matrices* λ_1 *and* $\mu_1 > 0$ *such that for the solution* $x(t)$ *of* (2.1) *in* $S(\alpha_0, \beta_0)$*, the inequalities*

$$\max_{t \in J} |x(t) - \alpha_{n+1}(t)| \le \lambda_1 \max_{t \in J} |x(t) - \alpha_n(t)|^k,$$

$$\max_{t\in J}|\beta_{n+1}(t)-x(t)|\le \mu_1\max_{t\in J}|\beta_n(t)-x(t)|^k$$

hold, where

$$\max_{t\in J}|u(t)|=(\max_{t\in J}|u_1(t)|,\dots,\max_{t\in J}|u_n(t)|)^T,$$

$|u|^k=(|u_1|^k,\dots,|u_n|^k)^T$ *for any function* $u\in C(J,R^n)$.

Proof. Firstly, for $t\in J$, applying Taylor mean value theorem yields

$$\begin{aligned} f(t,x)&=\sum_{i=0}^{k-1}\frac{\partial^i f(t,y)}{\partial x^i}\frac{(x-y)^i}{i!}\\ &\quad+\Big(\int_0^1(1-\sigma)^{k-1}\frac{\partial^k f(t,\sigma x+(1-\sigma)y)}{\partial x^k}d\sigma\Big)\frac{(x-y)^k}{(k-1)!},\end{aligned}\tag{3.1}$$

with $\alpha_0\le x,y\le\beta_0$, $\frac{\partial^0 f}{\partial x^0}=f$. Furthermore, for $\alpha_0\le y\le x\le\beta_0$, in view of the condition (A8), we have

$$f(t,x)\ge\sum_{i=0}^{k-1}\frac{\partial^i f(t,y)}{\partial x^i}\frac{(x-y)^i}{i!}-M(x-y)^k\equiv F(t,x,y),\tag{3.2}$$

and $F(t,x,x)=f(t,x)$, where $x^i=(x_1^i,x_2^i,\dots,x_n^i)^T$, $x\in R^n$, $i=0,1,2,\dots,k$. Similarly, for a giving $t\in J$, $\alpha_0\le x\le y\le\beta_0$, we obtain

$$\begin{aligned} f(t,x)&\le\begin{cases}\sum_{i=0}^{k-1}\frac{\partial^i f(t,y)}{\partial x^i}\frac{(x-y)^i}{i!}-M(x-y)^k, & k=2n+1.\\ \sum_{i=0}^{k-1}\frac{\partial^i f(t,y)}{\partial x^i}\frac{(x-y)^i}{i!}+N(x-y)^k, & k=2n.\end{cases}\\ &\equiv G(t,x,y)\end{aligned}\tag{3.3}$$

and $G(t,x,x)=f(t,x)$.

Consider the singular differential system

$$\begin{gathered} Ax'=\sum_{i=0}^{k-1}\frac{\partial^i f(t,\alpha_0)}{\partial x^i}\frac{(x-\alpha_0)^i}{i!}-M(x-\alpha_0)^k\equiv F(t,x,\alpha_0),\quad t\in J,\\ x(0)=x_0.\end{gathered}\tag{3.4}$$

We will show that α_0 and β_0 are lower and upper solutions of (3.4) respectively. The condition (A7) and the inequality (3.2) imply

$$\begin{gathered} A\alpha_0'\le f(t,\alpha_0)=F(t,\alpha_0,\alpha_0),\quad t\in J,\\ \alpha_0(0)\le x_0,\end{gathered}$$

and

$$\begin{gathered} A\beta_0'\ge f(t,\beta_0)\ge F(t,\beta_0,\alpha_0),\quad t\in J,\\ \beta_0(0)\ge x_0.\end{gathered}$$

Hence by Lemma 2.6, there exists a solution $\alpha_1(t)$ of (3.4) such that $\alpha_0(t)\le\alpha_1(t)\le\beta_0(t)$ on J. Furthermore, we can prove that $\alpha_1(t)$ is the unique solution of (3.4). For this purpose, we assume $x_1(t)$ and $x_2(t)$ are two solutions of (3.4) and $\alpha_0\le x_2\le x_1\le\beta_0$ holds. Then, from (3.4), and noting that $x_1(0)-x_2(0)=0$, and

$$A^i-B^i=(A-B)\sum_{j=0}^{i-1}A^{i-1-j}B^j,$$

we have

$$
\begin{aligned}
Ax_1' - Ax_2' &= F(t, x_1, \alpha_0) - F(t, x_2, \alpha_0) \\
&= \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_0)}{\partial x^i} \frac{(x_1-\alpha_0)^i}{i!} - M(x_1-\alpha_0)^k \\
&\quad - \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_0)}{\partial x^i} \frac{(x_2-\alpha_0)^i}{i!} + M(x_2-\alpha_0)^k \\
&= \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\alpha_0)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} (x_1-\alpha_0)^{i-1-j}(x_2-\alpha_0)^j \Big\}(x_1-x_2) \\
&\quad - M \sum_{j=0}^{k-1} (x_1-\alpha_0)^{k-1-j}(x_2-\alpha_0)^j (x_1-x_2) \\
&\le \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\alpha_0)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} (x_1-\alpha_0)^{i-1-j}(x_2-\alpha_0)^j \Big\}(x_1-x_2) \\
&\le L_1(x_1-x_2),
\end{aligned}
$$

where

$$
\sum_{i=1}^{k-1} \frac{\partial^i f(t,\alpha_0)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} (x_1-\alpha_0)^{i-1-j}(x_2-\alpha_0)^j \le L_1,
$$

$M(\bar{t}) = -L_1$, $\bar{t} \in J$. Using Lemma 2.3, we can get $x_1(t) \le x_2(t)$. Then we have $x_1(t) \equiv x_2(t)$, that is, $\alpha_1(t)$ is the unique solution of (3.4).

Furthermore, we consider the singular differential system

$$
\begin{gathered}
\begin{aligned}
Ay' &= \begin{cases} \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_0)}{\partial x^i} \frac{(y-\beta_0)^i}{i!} - M(y-\beta_0)^k, & k = 2n+1, \\ \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_0)}{\partial x^i} \frac{(y-\beta_0)^i}{i!} + N(y-\beta_0)^k, & k = 2n, \end{cases} \\
&\equiv G(t, y, \beta_0), \quad t \in J,
\end{aligned} \\
y(0) = x_0.
\end{gathered} \tag{3.5}
$$

Now we show that α_0 and β_0 are lower and upper solutions of (3.5) respectively. For this purpose, the condition (A7) and the inequality (3.3) yield

$$
\begin{gathered}
A\alpha_0' \le f(t,\alpha_0) \le G(t,\alpha_0,\beta_0), \quad t \in J, \\
\alpha_0(0) \le x_0,
\end{gathered}
$$

and

$$
\begin{gathered}
A\beta_0' \ge f(t,\beta_0) = G(t,\beta_0,\beta_0), \quad t \in J, \\
\beta_0(0) \ge x_0.
\end{gathered}
$$

Thus from Lemma 2.6, we see that there exists a solution $\beta_1(t)$ of (3.5) such that $\alpha_0(t) \le \beta_1(t) \le \beta_0(t)$ on J. Similarly, we can prove that $\beta_1(t)$ is the unique solution of (3.5). To do this, we talk about two cases. If $k = 2n+1$, the uniqueness of the solution of (3.5) is the same as (3.4). If $k = 2n$, taking two solutions x_1, x_2 of (3.5) such that $\alpha_0 \le x_2 \le x_1 \le \beta_0$. Then, we obtain

$$
Ax_1' - Ax_2' = G(t, x_1, \beta_0) - G(t, x_2, \beta_0)
$$

$$
\begin{aligned}
&= \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\beta_0)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} (x_1-\beta_0)^{i-1-j}(x_2-\beta_0)^j \Big\}(x_1-x_2) \\
&\quad + N \sum_{j=0}^{k-1} (x_1-\beta_0)^{k-1-j}(x_2-\beta_0)^j (x_1-x_2) \\
&\le \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\beta_0)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} (x_1-\beta_0)^{i-1-j}(x_2-\beta_0)^j \Big\}(x_1-x_2) \\
&\le L_1(x_1-x_2).
\end{aligned}
$$

Noting that $x_1(0)-x_2(0)=0$, by Lemma 2.3, we obtain $x_1(t)\le x_2(t)$. Thus, (3.5) has a unique solution.

Next, we prove that $\alpha_1(t)\le\beta_1(t)$ on J. From (3.2), we obtain

$$
\begin{gathered}
A\alpha_1' = F(t,\alpha_1,\alpha_0) \le f(t,\alpha_1), \quad t\in J, \\
\alpha_1(0)=x_0.
\end{gathered}
$$

Proceeding as before, one can obtain that $\beta_1(t)$ is an upper solution of (2.1). It then follows from Lemma 2.5 that $\alpha_1(t)\le\beta_1(t)$ on J. Consequently,

$$
\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t), \quad t\in J.
$$

Continuing this process by induction, we obtain two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ satisfying

$$
\alpha_0(t)\le\alpha_1(t)\le\dots\le\alpha_n(t)\le\beta_n(t)\le\dots\le\beta_1(t)\le\beta_0(t), \quad t\in J.
$$

Let α_n, β_n be lower and upper solutions of (2.1) respectively with $\alpha_n\le\beta_n$ on J. Then we consider the singular differential system

$$
\begin{gathered}
Ax' = \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i}\frac{(x-\alpha_n)^i}{i!} - M(x-\alpha_n)^k \equiv F(t,x,\alpha_n), \quad t\in J, \\
x(0)=x_0.
\end{gathered} \tag{3.6}
$$

In this case, we can show easily that α_n and β_n are lower and upper solutions of (3.6). Therefore, by Lemma 2.6, there exists a solution $\alpha_{n+1}(t)$ of (3.6) such that $\alpha_n(t)\le\alpha_{n+1}(t)\le\beta_n(t)$ on J. The uniqueness of $\alpha_{n+1}(t)$ is analogous to $\alpha_1(t)$, we omit the details.

Next, consider the singular differential system

$$
\begin{gathered}
\begin{aligned}
Ay' &= \begin{cases} \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i}\frac{(y-\beta_n)^i}{i!} - M(y-\beta_n)^k, & k=2n+1, \\ \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i}\frac{(y-\beta_n)^i}{i!} + N(y-\beta_n)^k, & k=2n, \end{cases} \\
&\equiv G(t,y,\beta_n), \quad t\in J,
\end{aligned} \\
y(0)=x_0.
\end{gathered} \tag{3.7}
$$

Similarly, we can show that α_n and β_n are lower and upper solutions of (3.7), respectively. Consequently, by Lemma 2.6, we obtain a solution $\beta_{n+1}(t)$ of (3.7) exists such that $\alpha_n(t)\le\beta_{n+1}(t)\le\beta_n(t)$ on J. Furthermore, we can show that $\alpha_{n+1}(t)\le\beta_{n+1}(t)$ on J. By induction, we have that for all n,

$$
\alpha_0(t)\le\alpha_1(t)\le\dots\le\alpha_n(t)\le\beta_n(t)\le\dots\le\beta_1(t)\le\beta_0(t), \quad t\in J.
$$

Employing the Ascoli-Arzela Theorem, it can be shown that they have pointwise limits $\rho(t)$ and $r(t)$. Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \le r(t) = \lim_{n\to\infty} \beta_n(t).$$

We can show easily that $\rho(t)$ and $r(t)$ are solutions of (2.1).

Finally, we show that the order of convergence is $k \ge 2$. For that, let $x(t)$ be a solution of (2.1) in $S(\alpha_0, \beta_0)$, we define

$$e_n(t) = x(t) - \alpha_n(t),\ a_n(t) = \alpha_{n+1}(t) - \alpha_n(t), \quad t \in J,$$

so that, $e_n \ge 0$, $a_n \ge 0$, $e_n(0) = 0$, $a_n(0) = 0$. From the equality (3.1), we have

$$\begin{aligned} Ax' &= \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i} \frac{(x-\alpha_n)^i}{i!} \\ &\quad + \Big(\int_0^1 (1-\sigma)^{k-1} \frac{\partial^k f(t, \sigma x + (1-\sigma)\alpha_n)}{\partial x^k} d\sigma \Big) \frac{(x-\alpha_n)^k}{(k-1)!}. \end{aligned}$$

On the other hand, by (3.6), we have

$$A\alpha_{n+1}' = \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i} \frac{(\alpha_{n+1}-\alpha_n)^i}{i!} - M(\alpha_{n+1}-\alpha_n)^k.$$

Therefore,

$$\begin{aligned} Ae_{n+1}' &= \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i} \frac{(x-\alpha_n)^i}{i!} \\ &\quad + \Big(\int_0^1 (1-\sigma)^{k-1} \frac{\partial^k f(t, \sigma x + (1-\sigma)\alpha_n)}{\partial x^k} d\sigma \Big) \frac{(x-\alpha_n)^k}{(k-1)!} \\ &\quad - \sum_{i=0}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i} \frac{(\alpha_{n+1}-\alpha_n)^i}{i!} + M(\alpha_{n+1}-\alpha_n)^k \\ &\le \sum_{i=1}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i} \frac{(e_n^i - a_n^i)}{i!} + Ne_n^k + Ma_n^k \\ &\le \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\alpha_n)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} e_n^{i-1-j} a_n^j \Big\} e_{n+1} + ce_n^k \\ &\le L_1 e_{n+1} + ce_n^k, \end{aligned}$$

where $c = N + M$, $a_n \le e_n$. Furthermore, we have

$$Ae_{n+1}' \le -M(\bar{t})e_{n+1} + ce_n^k.$$

Lemma 2.3 implies $e_{n+1}(t) \le x(t)$ on J, where $x(t)$ is the solution of

$$Ax' + M(\bar{t})x = ce_n^k, \quad x(0) = 0.$$

Thus, using the expression of $x(t)$ in Lemma 2.4, we obtain

$$x(t) = [\lambda A + M(\bar{t})]^{-1} \big[e^{-\hat{A}^D \hat{M} t} \int_0^t e^{\hat{A}^D \hat{M} s} \hat{A}^D ce_n^k(s) ds + (I - \hat{A}\hat{A}^D)\hat{M}^D ce_n^k(t) \big],$$

we arrive at after taking suitable estimates

$$\max_{t\in J} |x(t) - \alpha_{n+1}(t)| \le \lambda_1 \max_{t\in J} |x(t) - \alpha_n(t)|^k,$$

where λ_1 is an appropriate positive matrix.

Similarly, we define

$$g_n(t) = x(t) - \beta_n(t), \; b_n(t) = \beta_{n+1}(t) - \beta_n(t), \quad t \in J,$$

so that, $e_n(t) \le 0$, $b_n(t) \le 0$, $g_n(0) = 0$, $b_n(0) = 0$. In view of (3.1), we have

$$\begin{aligned} Ax' &= \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(x-\beta_n)^i}{i!} \\ &\quad + \Big(\int_0^1 (1-\sigma)^{k-1} \frac{\partial^k f(t, \sigma x + (1-\sigma)\beta_n)}{\partial x^k} d\sigma \Big) \frac{(x-\beta_n)^k}{(k-1)!}. \end{aligned}$$

Furthermore, by (3.7), we obtain

$$A\beta_{n+1}' = \begin{cases} \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(\beta_{n+1}-\beta_n)^i}{i!} - M(\beta_{n+1}-\beta_n)^k, & k = 2n+1, \\ \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(\beta_{n+1}-\beta_n)^i}{i!} + N(\beta_{n+1}-\beta_n)^k, & k = 2n. \end{cases}$$

Hence, if $k = 2n+1$, we obtain

$$\begin{aligned} & - Ag_{n+1}' \\ &= A\beta_{n+1}' - Ax' \\ &= \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(\beta_{n+1}-\beta_n)^i}{i!} - M(\beta_{n+1}-\beta_n)^k - \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(x-\beta_n)^i}{i!} \\ &\quad - \Big(\int_0^1 (1-\sigma)^{k-1} \frac{\partial^k f(t, \sigma x + (1-\sigma)\beta_n)}{\partial x^k} d\sigma \Big) \frac{(x-\beta_n)^k}{(k-1)!} \\ &\le \sum_{i=1}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{b_n^i - g_n^i}{i!} - Mb_n^k + N(-g_n)^k \\ &\le \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} b_n^{i-1-j} g_n^j \Big\} (-g_{n+1}) + (M+N)(-g_n)^k \\ &\le L_1(-g_{n+1}) + (M+N)(-g_n)^k. \end{aligned}$$

Furthermore, we obtain

$$A(-g_{n+1})' \le -M(\bar{t})(-g_{n+1}) + (M+N)(-g_n)^k.$$

According to Lemma 2.3, we have $-g_{n+1}(t) \le x(t)$ on J, where $x(t)$ is the solution of

$$Ax' + M(\bar{t})x = (M+N)(-g_n)^k, \quad x(0) = 0.$$

Thus, using the expression of $x(t)$ in Lemma 2.4 and taking suitable estimates, we obtain

$$\max_{t\in J} |\beta_{n+1}(t) - x(t)| \le \mu_1 \max_{t \in J} |\beta_n(t) - x(t)|^k,$$

where μ_1 is a positive matrix.

If $k = 2n$, we obtain

$$\begin{aligned} & - Ag_{n+1}' \\ &= \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(\beta_{n+1}-\beta_n)^i}{i!} + N(\beta_{n+1}-\beta_n)^k - \sum_{i=0}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{(x-\beta_n)^i}{i!} \end{aligned}$$

$$
\begin{aligned}
&- \Big(\int_0^1 (1-\sigma)^{k-1} \frac{\partial^k f(t, \sigma x + (1-\sigma)\beta_n)}{\partial x^k} d\sigma \Big) \frac{(x-\beta_n)^k}{(k-1)!} \\
&\le \sum_{i=1}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{b_n^i - g_n^i}{i!} + N b_n^k + M g_n^k \\
&\le \Big\{ \sum_{i=1}^{k-1} \frac{\partial^i f(t,\beta_n)}{\partial x^i} \frac{1}{i!} \sum_{j=0}^{i-1} b_n^{i-1-j} g_n^j \Big\} (-g_{n+1}) + (M+N) g_n^k \\
&\le L_1(-g_{n+1}) + (M+N) g_n^k.
\end{aligned}
$$

Then, we obtain

$$
A(-g_{n+1})' \le -M(\bar{t})(-g_{n+1}) + (M+N) g_n^k.
$$

Now applying Lemma 2.3, we have $-g_{n+1}(t) \le x(t)$ on J, where $x(t)$ is the solution of

$$
Ax' + M(\bar{t})x = (M+N) g_n^k, \quad x(0) = 0.
$$

Analogous to the above discussion, after taking suitable estimates, we obtain

$$
\max_{t\in J} |\beta_{n+1}(t) - x(t)| \le \mu_1 \max_{t\in J} |\beta_n(t) - x(t)|^k.
$$

The proof is complete. □

The following corollaries are immediate results of Theorem 3.1.

Corollary 3.2. *Assume that conditions* (A1)–(A3), (A6), (A7) *hold, and*

(A9) *The Frechet derivatives* $\frac{\partial^i f(t,x)}{\partial x^i}$ $(i = 0,1,2,3)$ *exist and are continuous, also* $f(t,x) + Mx^3$ *is* (2)*-hyperconvex and* $f(t,x) - Nx^3$ *is* (2)*-hyperconcave in* x*; that is,*

$$
\frac{\partial^3 (f(t,x) + Mx^3)}{\partial x^3} \ge 0, \quad \frac{\partial^3 (f(t,x) - Nx^3)}{\partial x^3} \le 0,
$$

where the $n \times n$ *matrices* M, $N > 0$.

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *which converge uniformly to the solution of* (2.1) *and the convergence is cubic.*

Corollary 3.3. *Assume that conditions* (A1)–(A3), (A6), (A7) *hold, and*

(A10) *The Frechet derivatives* $\frac{\partial^i f(t,x)}{\partial x^i}$ $(i = 0,1,2,3,4)$ *exist and are continuous satisfying* $f(t,x) + Mx^4$ *is* (3)*-hyperconvex and* $f(t,x) - Nx^4$ *is* (3)*-hyperconcave in* x*, that is,* $\frac{\partial^4 (f(t,x)+Mx^4)}{\partial x^4} \ge 0$ *and* $\frac{\partial^4 (f(t,x)-Nx^4)}{\partial x^4} \le 0$*, where the* $n \times n$ *matrices* M, $N > 0$.

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *which converge uniformly to the solution of* (2.1) *and the convergence is quartic.*

Remark 3.4. If $f(t,x)$ is $(k-1)$-hyperconvex or $(k-1)$-hyperconcave in x, by the method of quasilinearization, we also can obtain the convergence of the monotone sequences is of order k (k is odd or even).

3.1. **Acknowledgments.** This research is supported by the National Natural Science Foundation of China (11271106) and the Natural Science Foundation of Hebei Province of China (A2013201232). The authors would like to thank the reviewers for their valuable suggestions and comments.

References

[1] Abd-Ellateef Kamar, Ahmed R.; Attia, Gamal M.; Vajravelu, K.; Mosaad, M.; *Generalized quasilinearization for singular system of differential equations*, Appl. Math. Comput., 114, 69-74 (2000).

[2] Bellman, R., Kalaba, R.; *Quasilinearization and Nonlinear Boundary Value Problems*, Elsevier, New York, 1965.

[3] Cabada, A., Nieto, J. J.; *Rapid convergence of the iterative technique for first order initial value problem*, Applied Mathematics and Computation, 87, 217-226 (1997).

[4] Cabada, A.; Nieto, J. J.; Heikkila, S.; *Rapid convergence of approximate solutions for first order nonlinear boundary value problems*, Inter. J. Math. & Math. Sci., 21(3), 499-506 (1998).

[5] Campbell, S. L.; *Singular systems of differential equations*, Pitman Advanced Publishing Program (I), London, 1982.

[6] Campbell, S. L.; *Singular systems of differential equations*, Pitman Advanced Publishing Program (II), London, 1982.

[7] El-Gebeily, M.; O'Regan, D.; *A quasilinearization method for a class of second order singular nonlinear differential equations with nonlinear boundary conditions*, Nonlinear Anal. RWA, 8(1), 174-186 (2007).

[8] El-Gebeily, M.; O'Regan, D.; *Upper and lower solutions and quasilinearization for a class of second order singular nonlinear differential equations with nonlinear boundary conditions*, Nonlinear Anal. RWA, 8(2), 636-645(2007).

[9] Kurepa, A.; Weithers, H.; *Oscillation of the solution to a singular differential equation*, Electron. J. Diff. Equa., 2, 19-27 (1999).

[10] Lakshmikantham, V.; Vatsala, A. S.; *Generalized quasilinearization for nonlinear problems*, Dordrecht: Kluwer Academic Publishers, 1998.

[11] Melton, T. G.; Vatsala, A. S.; *Generalized quasilinearization and higher order of convergence for first order initial value problems*, Dynamic Systems and Applications, 15, 375-394 (2006).

[12] Mirko, M. M., Vladimir, B. B.; *Stability analysis of singular systems*, Circuits Systems Signal Process, 8(3), 267-287 (1989).

[13] Mohapatra, R. N.; Vajravelu, K.; Yin, Y.; *Generalized quasilinearization method and rapid convergence for first order initial value problems*, J. Math. Anal. Appl.,207, 206-219 (1997).

[14] Mohapatra, R., Vajravelu, K., Yin, Y.; *Generalized quasilinearization method for second order boundary value problems*, Nonlinear Analysis, 36, 799-806 (1999).

[15] Mohapatra, R.; Vajravelu, K.; Yin, Y.; *Extension of the method of quasilinearization and rapid convergence*, *Journal of Optimization Theory and Applications*, 96(3), 667-682 (1998).

[16] O'Regan, D.; El-Gebeily, Mohamed; *Existence, upper and lower solutions and quasilinearization for singular differential equations*, IMA J. Appl. Math., 73, 323-344 (2008).

[17] Rosenbrock, H. H.; *Structural properties of linear dynamical systems*, Internat. J. Control, 200, 191-202 (1974).

[18] Uvah, J. A.; Vatsala, A. S.; *Monotone method for first order singular systems with boundary conditions*, J. Appl. Math. Simulation, 2(4), 217-224 (1989).

[19] Vladimir, B. B., Mirko, M. M.; *Extended stability of motion of semi-state systems*, Internat. J. Control, 46(6), 2183-2197 (1987).

[20] Wang, P. G., Zhang, J.; *Stability of solutions for nonlinear singular systems with delay*, Appl. Math. Lett., 25, 1291-1295 (2012).

[21] Wang, P. G.; Wu, H. X.; Wu Y. H.; *Higher even-order convergence and coupled solutions for second-order boundary value problem on time scales*, Computers and Mathematics with Applications, 55(8), 1693-1705(2008).

[22] Yang, C. Y.; Zhang, Q. L.; Zhou, L. N.; *Practial stability of descriptor systems with delays in terms of two measurements*, J. Franklin Institute, 343(6), 635-646 (2006).

[23] Zhao, Y. L., Xing, L., Tao, J.; *Existence of solutions to a generalized system*, Appl. Math., 3(6), 511-516 (2012).

Peiguang Wang

College of Electronic and Information Engineering, Hebei University, Baoding 071002, China

E-mail address: pgwang@hbu.edu.cn

Xiang Liu
College of Mathematics and Information Science, Hebei University, Baoding 071002, China
E-mail address: 1065265114@qq.com