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PROPERTIES OF SCHWARZIAN DIFFERENCE EQUATIONS

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ABSTRACT. We consider the Schwarzian type difference equation

$$\Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k = R(z),$$

where $R(z)$ is a nonconstant rational function. We study the existence of rational solutions and value distribution of transcendental meromorphic solutions with finite order of the above equation.

1. Introduction and statement of main results

In this article, we use the basic notions of Nevanlinna's theory [6, 12]. In addition, $\sigma(f)$ denotes the order of growth of the meromorphic function $f(z)$; $\lambda(f)$ and $\lambda\left(\frac{1}{f}\right)$ denote the exponents of convergence of zeros and poles of $f(z)$. Let $S(r,w)$ denote any quantity satisfying $S(r,w) = o\big(T(r,w)\big)$ for all r outside of a set with finite logarithmic measure. A meromorphic solution w of a difference (or differential) equation is called *admissible* if the characteristic function of all coefficients of the equation are $S(r,w)$. For every $n \in \mathbb{N}^+$, the forward differences $\Delta^n f(z)$ are defined in the standard way [11] by

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z).$$

The Schwarzian differential equation

$$\Big[\frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2\Big]^k = R(z,f) = \frac{P(z,f)}{Q(z,f)} \tag{1.1}$$

was studied by Ishizaki [7], and obtained some important results. Chen and Li [3] investigated Schwarzian difference equation, and obtained the following theorem.

Theorem 1.1. *Let $f(z)$ be an admissible solution of difference equation*

$$\Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k = R(z,f) = \frac{P(z,f)}{Q(z,f)}$$

such that $\sigma_2(f) < 1$, where $k(\geq 1)$ is an integer, $P(z,f)$ and $Q(z,f)$ are polynomials with $\deg_f P(z,f) = p$, $\deg_f Q(z,f) = q$, $d = \max\{p,q\}$. *Let $\alpha_1, \dots, \alpha_s$ be $s(\geq 2)$*

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distinct complex constants. Then

$$\sum_{j=1}^{s} \delta(\alpha_j, f) \le 4 - \frac{q}{2k}.$$

In particular, if $N(r,f) = S(r,f)$, *then*

$$\sum_{j=1}^{s} \delta(\alpha_j, f) \le 2 - \frac{d}{2k}.$$

Set $\deg_f P(z,f) = \deg_f Q(z,f) = 0$ in equation (1.1), then $R(z,f) \equiv R(z)$ is a small function with respect to $f(z)$. Liao and Ye [10] studied this type of Schwarzian differential equation, and obtained the following result.

Theorem 1.2. *Let* P *and* Q *be polynomials with* $\deg P = p$, $\deg Q = q$, *and let* $R(z) = \frac{P(z)}{Q(z)}$ *and* k *a positive integer. If* $f(z)$ *is a transcendental meromorphic solution of equation*

$$\Big[\frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2\Big]^k = R(z),$$

then $p - q + 2k > 0$ *and the order* $\sigma(f) = \frac{p-q+2k}{2k}$.

In this article, we study a Schwarzian difference equation, and obtain the following result.

Theorem 1.3. *Let* $R(z) = \frac{P(z)}{Q(z)}$ *be an irreducible rational function with* $\deg P(z) = p$, $\deg Q(z) = q$. *Consider the difference equation*

$$\Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k = R(z), \tag{1.2}$$

where k *is a positive integer. Then*

(i) *every transcendental meromorphic solution* $f(z)$ *of* (1.2) *satisfies* $\sigma(f) \ge 1$; *if* $p - q + 2k > 0$, *then* (1.2) *has no rational solutions;*

(ii) *if* $f(z)$ *is a mereomorphic solution of* (1.2) *with finite order, terms* $\frac{\Delta^2 f(z)}{\Delta f(z)}$ *and* $\frac{\Delta^3 f(z)}{\Delta f(z)}$ *in* (1.2) *are nonconstant rational functions;*

(iii) *every transcendental meromorphic solution* $f(z)$ *with finite order has at most one Borel exceptional value unless*

$$f(z) = b + R_0(z)e^{az}, \tag{1.3}$$

where $b \in \mathbb{C}$, $a \in \mathbb{C} \setminus \{0\}$ *and* $R_0(z)$ *is a nonzero rational function.*

(iv) *if* $p - q + 2k > 0, \sigma(f) < \infty$, *then* $\Delta f(z)$ *has at most one Borel exceptional value unless*

$$\Delta f(z) = R_1(z)e^{az}, \tag{1.4}$$

where $a \in \mathbb{C}$, $a \ne i2k_1\pi$ *for any* $k_1 \in \mathbb{Z}$, *and* $R_1(z)$ *is a nonzero rational function.*

Corollary 1.4. *Let* $f(z)$ *be a finite order meromorphic solution of* (1.2), *if* $p-q+2k > 0$, *then* $f(z)$, $\Delta f(z)$, $\Delta^2 f(z)$ *and* $\Delta^3 f(z)$ *cannot be rational functions, and* $\frac{\Delta^2 f(z)}{\Delta f(z)}$ *and* $\frac{\Delta^3 f(z)}{\Delta f(z)}$ *are nonconstant rational functions.*

Remark 1.5. Let $f(z)$ be the function in the form (1.3), then the Schwarzian difference is an irreducible rational function $R(z) = \frac{P(z)}{Q(z)}$ with $\deg P \le \deg Q$.

Proof. Suppose that $f(z)$ has the form (1.3). Since $R_0(z)$ is a rational function, we see $R_0(z)$ satisfies
$$\frac{R_0(z+j)}{R_0(z)} \to 1, \quad z \to \infty,\ j = 1, 2, 3. \tag{1.5}$$
By (1.3), we have
$$\begin{gathered}
\Delta f(z) = e^{az}(e^a R_0(z+1) - R_0(z));\\
\Delta^2 f(z) = e^{az}(e^{2a} R_0(z+2) - 2e^a R_0(z+1) + R_0(z));\\
\Delta^3 f(z) = e^{az}(e^{3a} R_0(z+3) - 3e^{2a} R_0(z+2) + 3e^a R_0(z+1) - R_0(z)).
\end{gathered}$$
Combining these with (1.5), we have
$$\begin{aligned}
\frac{\Delta^3 f(z)}{\Delta f(z)} &= \frac{e^{3a} R_0(z+3) - 3e^{2a} R_0(z+2) + 3e^a R_0(z+1) - R_0(z)}{e^a R_0(z+1) - R_0(z)}\\
&= \frac{e^{3a} \frac{R_0(z+3)}{R_0(z)} - 3e^{2a} \frac{R_0(z+2)}{R_0(z)} + 3e^a \frac{R_0(z+1)}{R_0(z)} - 1}{e^a \frac{R_0(z+1)}{R_0(z)} - 1}\\
&\to \frac{e^{3a} - 3e^{2a} + 3e^a - 1}{e^a - 1} = (e^a - 1)^2, \quad z \to \infty,
\end{aligned} \tag{1.6}$$
and
$$\begin{aligned}
\frac{\Delta^2 f(z)}{\Delta f(z)} &= \frac{e^{2a} R_0(z+2) - 2e^a R_0(z+1) + R_0(z)}{e^a R_0(z+1) - R_0(z)}\\
&= \frac{e^{2a} \frac{R_0(z+2)}{R_0(z)} - 2e^a \frac{R_0(z+1)}{R_0(z)} + 1}{e^a \frac{R_0(z+1)}{R_0(z)} - 1}\\
&\to \frac{e^{2a} - 2e^a + 1}{e^a - 1} = e^a - 1, \quad z \to \infty.
\end{aligned} \tag{1.7}$$
Thus,
$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 \to (e^a - 1)^2 - \frac{3}{2}(e^a - 1)^2 = -\frac{1}{2}(e^a - 1)^2, \quad z \to \infty. \tag{1.8}$$
By (1.6), (1.7) and $R_0(z)$ begin a rational function, we see that
$$R(z) = \Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k$$
is a rational function. Denote $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are prime polynomials. By (1.8), we see
$$R(z) = \frac{P(z)}{Q(z)} = \Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k \to \frac{(-1)^k}{2^k}(e^a - 1)^{2k}, \quad z \to \infty.$$
If $e^a \neq 1$, then $\deg P = \deg Q$; if $e^a = 1$, then $\deg P < \deg Q$. So, $\deg P \leq \deg Q$. $\square$

Remark 1.6. Checking the proof of Theorem 1.3 (iv), we see that for $f(z)$ a function such that $\Delta f(z)$ in the form (1.4), then the Schwarzian difference satisfies
$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 = e^{2a}\frac{R_1(z+2)}{R_1(z)} - \frac{3}{2}e^{2a}\Big(\frac{R_1(z+1)}{R_1(z)}\Big)^2 + e^a\frac{R_1(z+1)}{R_1(z)} - \frac{1}{2}.$$

Examples 1.7 and 1.8 below show that the condition "$p-q+2k > 0$" in Theorem 1.3 (i) cannot be omitted.

Example 1.7. Consider the Schwarzian type difference equation

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 = -\frac{6}{(2z+1)^2},$$

where $k=1$, $p=0$, $q=2$, and $p-q+2k=0$. This equation has a rational solution $f_1(z)=z^2$, and a transcendental meromorphic solution $f_2(z)=e^{i2\pi z}+z^2$.

Example 1.8. Consider the Schwarzian type difference equation

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 = \frac{-6}{(z+3)(z+2)^2},$$

where $k=1$, $p=0$, $q=3$, and $p-q+2k=-1<0$. This equation has a rational solution $f_1(z)=\frac{1}{z}$, and a transcendental meromorphic solution $f_2(z)=e^{i2\pi z}+\frac{1}{z}$.

Example 1.9. The function $f(z)=ze^{(\log 3)z}$ satisfies Schwarzian type difference equation

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 = \frac{-8z^2-48z-108}{(2z+3)^2}.$$

We see $\sigma(f)=1$ and $f(z)$ has finitely many zeros and poles. It shows the result of Theorem 1.3 (iii) is precise.

2. Preliminaries

Lemma 2.1 ([2])**.** *Let $f(z)$ be a meromorphic function of finite order σ and let η be a nonzero complex constant. Then for each $\varepsilon(0<\varepsilon<1)$, we have*

$$m\Big(r,\frac{f(z+\eta)}{f(z)}\Big)+m\Big(r,\frac{f(z)}{f(z+\eta)}\Big)=O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.2 ([2])**.** *Let $f(z)$ be a meromorphic function with order $\sigma=\sigma(f),\sigma<\infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon>0$,*

$$T(r,f(z+\eta))=T(r,f(z))+O\big(r^{\sigma-1+\varepsilon}\big)+O(\log r).$$

Lemma 2.3 ([4, Theorem 1.8.1], [9])**.** *Let $c\in\mathbb{C}\setminus\{0\}$ and $f(z)$ be a finite order meromorphic function with two finite Borel exceptional values a and b. Then for every $n\in\mathbb{N}^+$,*

$$T(r,\Delta^n f)=(n+1)T(r,f)+S(r,f)$$

unless $f(z)$ and c satisfy

$$f(z)=b+\frac{b-a}{pe^{dz}-1},\quad p,d\in\mathbb{C}\setminus\{0\},$$
$$mdc=i2k_1\pi\quad \text{for some } k_1\in\mathbb{Z} \text{ and } m\in\{1,2,\dots,n\}.$$

Remark 2.4. Checking the proof of Lemma 2.3, we point out that when $c\in\mathbb{C}\setminus\{0\}$ and $f(z)$ is a finite order meromorphic function with two finite Borel exceptional values, for every $n\in\mathbb{N}^+$, if $c,2c,\dots,nc$ are not periods of $f(z)$, then

$$T(r,\Delta^n f)=(n+1)T(r,f)+S(r,f).$$

Lemma 2.5 ([1])**.** *Let $f(z)$ be a function transcendental and meromorphic in the plane which satisfies*

$$\liminf_{r\to\infty}\frac{T(r,f)}{r}=0.$$

Then Δf and $\Delta f/f$ are both transcendental.

Lemma 2.6. *Suppose that $f(z) = H(z)e^{az}$, where $a \neq 0$ is a constant, $H(z)$ is a transcendental meromorphic function with $\sigma(H) < 1$. Then $\frac{\Delta f(z)}{f(z)}$ is transcendental.*

Proof. Substituting $f(z) = H(z)e^{az}$ into $\frac{\Delta f(z)}{f(z)}$, we see that

$$\begin{aligned} \frac{\Delta f(z)}{f(z)} &= \frac{f(z+1) - f(z)}{f(z)} = \frac{H(z+1)e^{a(z+1)} - H(z)e^{az}}{H(z)e^{az}} \\ &= e^a \frac{H(z+1)}{H(z)} - 1 = e^a \Big(\frac{H(z+1)}{H(z)} - 1 \Big) + e^a - 1 \\ &= e^a \frac{\Delta H(z)}{H(z)} + e^a - 1. \end{aligned} \tag{2.1}$$

From the fact $\sigma(H) < 1$, we see that

$$\limsup_{r\to\infty} \frac{\log T(r,H)}{\log r} = \sigma(H) < 1.$$

Then for large enough r, choose $\varepsilon = \frac{1-\sigma(H)}{2} > 0$, we have

$$\log T(r,H) < (\sigma(H) + \varepsilon) \log r;$$

that is,

$$T(r,H) < r^{\sigma(H)+\varepsilon}.$$

Thus,

$$\liminf_{r\to\infty} \frac{T(r,H)}{r} \leq \liminf_{r\to\infty} \frac{r^{\sigma(H)+\varepsilon}}{r} = \liminf_{r\to\infty} r^{\sigma(H)+\varepsilon-1} = \liminf_{r\to\infty} r^{-\varepsilon} = 0. \tag{2.2}$$

So, $H(z)$ is a transcendental meromorphic function which satisfies (2.2). From Lemma 2.5, we see $\frac{\Delta H(z)}{H(z)}$ is transcendental. By (2.1), $\frac{\Delta f(z)}{f(z)}$ is transcendental too. □

Lemma 2.7 ([4, Lemma 5.2.2])**.** *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) < 1$, and let $g_1(z)$ and $g_2(z)(\not\equiv 0)$ be polynomials, c_1, c_2 ($c_1 \neq c_2$) be constants. Then*

$$h(z) = g_2(z)f(z+c_2) + g_1(z)f(z+c_1)$$

is transcendental.

Lemma 2.8 ([5, 8])**.** *Let w be a transcendental meromorphic solution with finite order of difference equation*

$$P(z,w) = 0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a) \not\equiv 0$ for a meromorphic function a, where a is a small function with respect to w, then

$$m\big(r, \frac{1}{w-a}\big) = S(r,w).$$

Remark 2.9. Ishizaki [7, Remark 1] pointed out that if $P(z,w)$ and $Q(z,w)$ are mutually prime, there exist polynomials of $w, U(z,w)$ and $V(z,w)$ such that

$$U(z,w)P(z,w) + V(z,w)Q(z,w) = s(z),$$

where $s(z)$ and coefficients of $U(z,w)$ and $V(z,w)$ are small functions with respect to $w(z)$.

Lemma 2.10. *Let $R(z)$ be a nonconstant rational function. Suppose that $f(z)$ is a transcendental meromorphic solution of equation* (1.2) *with finite order, then in* (1.2)*, terms $\frac{\Delta^2 f(z)}{\Delta f(z)}$ and $\frac{\Delta^3 f(z)}{\Delta f(z)}$ are both nonconstant rational functions.*

Proof. Set $G(z) = \frac{\Delta f(z+1)}{\Delta f(z)}$. Then $G(z)$ is a meromorphic function with finite order, and

$$\Delta f(z+1) = G(z)\Delta f(z),$$
$$\Delta f(z+2) = G(z+1)\Delta f(z+1) = G(z+1)G(z)\Delta f(z).$$

Hence,

$$\Delta^2 f(z) = \Delta f(z+1) - \Delta f(z) = (G(z)-1)\Delta f(z), \tag{2.3}$$

and

$$\begin{aligned}\Delta^3 f(z) &= \Delta^2(\Delta f(z)) = \Delta f(z+2) - 2\Delta f(z+1) + \Delta f(z)\\ &= (G(z+1)G(z) - 2G(z) + 1)\Delta f(z).\end{aligned} \tag{2.4}$$

From (1.2),

$$R(z) = \Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k$$

is a nonconstant rational function, then

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2$$

is also a nonconstant rational function. Denote

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 = R_2(z), \tag{2.5}$$

where $R_2(z)$ is a nonconstant rational function.

It follows from (2.3)–(2.5) that

$$G(z+1)G(z) - 2G(z) + 1 - \frac{3}{2}(G(z)-1)^2 = R_2(z); \tag{2.6}$$

that is,

$$G(z+1) = \frac{\frac{3}{2}G^2(z) - G(z) + R_2(z) + \frac{1}{2}}{G(z)}. \tag{2.7}$$

Since $R_2(z)$ is a nonconstant rational function, by (2.6), $G(z)$ cannot be a constant. Suppose that $G(z)$ is transcendental. We see that

$$\frac{3}{2}G^2(z) - G(z) + R_2(z) + \frac{1}{2} + (-\frac{3}{2}G(z)+1)G(z) = R_2(z) + \frac{1}{2}.$$

Together with Remark 2.9, $\frac{3}{2}G^2(z) - G(z) + R_2(z) + \frac{1}{2}$ and $G(z)$ are irreducible. Applying Valiron-Mohon'ko Theorem to (2.7), we have

$$T(r, G(z+1)) = 2T(r, G(z)) + S(r, G),$$

which contradicts Lemma 2.2. So, $G(z)$ is a nonconstant rational function. By (2.3) and (2.4), we see that $\frac{\Delta^2 f(z)}{\Delta f(z)}$ and $\frac{\Delta^3 f(z)}{\Delta f(z)}$ are nonconstant rational functions. □

3. Proofs of theorems

Proof of Theorem 1.3. (i) Suppose that $f(z)$ is a transcendental meromorphic solution of equation (1.2) with $\sigma(f) < 1$. Lemma 2.5 shows $g(z) = \Delta f(z)$ is transcendental with $\sigma(g) < 1$. Again by Lemma 2.5, we see $\frac{\Delta^2 f(z)}{\Delta f(z)} = \frac{\Delta g(z)}{g(z)}$ is also transcendental, which contradicts with Lemma 2.10. Thus, $\sigma(f) \geq 1$.

Next, we prove that if $f(z)$ is a rational solution of equation (1.2), then $p - q + 2k \leq 0$. Set $g(z) = \Delta f(z)$. By (1.2), we see

$$\Big[\frac{\Delta^2 g(z)}{g(z)} - \frac{3}{2}\Big(\frac{\Delta g(z)}{g(z)}\Big)^2\Big]^k = R(z). \tag{3.1}$$

Thus, $g(z)$ is a rational solution of equation

$$\frac{\Delta^2 g(z)}{g(z)} - \frac{3}{2}\Big(\frac{\Delta g(z)}{g(z)}\Big)^2 = R_2(z),$$

or

$$g(z)\Delta^2 g(z) - \frac{3}{2}(\Delta g(z))^2 = R_2(z) g^2(z), \tag{3.2}$$

where $R_2(z)$ is some rational function such that $R_2^k(z) = R(z)$. Since $R(z) = Az^{p-q}(1 + o(1))$, where A is some nonzero constant, then

$$R_2(z) = Bz^{\frac{p-q}{k}}(1 + o(1)), \tag{3.3}$$

where B is some nonzero constant.

Suppose that

$$g(z) = h(z) + \frac{m(z)}{n(z)}, \tag{3.4}$$

where $h(z), m(z)$ and $n(z)$ are polynomials with $\deg h(z) = l(\geq 0)$, $\deg m(z) = m$, $\deg n(z) = n$ with $m < n$. Denote

$$h(z) = c_0 z^l + \dots + c_l, \quad m(z) = a_0 z^m + \dots + a_m, \quad n(z) = b_0 z^n + \dots + b_n, \tag{3.5}$$

where $c_0, \dots, c_l$, $a_0, \dots, a_m$, $b_0, \dots, b_n$ are constants, with $a_0 \neq 0$ and $b_0 \neq 0$.

We divide this proof into the following three cases.

Case 1. $l > 0$. By (3.4) and (3.5), when z is large enough, $g(z)$ can be written as

$$g(z) = c_0 z^l (1 + o(1)). \tag{3.6}$$

Hence,

$$\Delta g(z) = lc_0 z^{l-1}(1 + o(1)), \quad \Delta^2 g(z) = l(l-1)c_0 z^{l-2}(1 + o(1)). \tag{3.7}$$

Substituting (3.3), (3.6), (3.7) in (3.2), we obtain

$$c_0 z^l l(l-1)c_0 z^{l-2}(1 + o(1)) - \frac{3}{2}(lc_0 z^{l-1})^2(1 + o(1)) = Bz^{\frac{p-q}{k}} c_0^2 z^{2l}(1 + o(1));$$

that is,

$$-\big(\frac{l}{2} + 1\big) l c_0^2 z^{2l-2}(1 + o(1)) = Bz^{\frac{p-q}{k}} c_0^2 z^{2l}(1 + o(1)),$$

from which it follows

$$2l - 2 = \frac{p-q}{k} + 2l.$$

So, $p - q + 2k = 0$.

Case 2. $l = 0$, $c_0 \neq 0$. By (3.4) and (3.5), when z is large enough, $g(z)$ can be written as

$$g(z) = c_0 + \frac{m(z)}{n(z)} = c_0 + o(1). \tag{3.8}$$

By calculation and $m < n$, we see that

$$n(z)n(z+1) = b_0^2 z^{2n}(1 + o(1)),$$
$$m(z+1)n(z) - m(z)n(z+1) = (m-n)a_0 b_0 z^{m+n-1}(1 + o(1)).$$

Thus,

$$\Delta g(z) = \frac{m(z+1)n(z) - m(z)n(z+1)}{n(z)n(z+1)} = (m-n)\frac{a_0}{b_0} z^{m-n-1}(1 + o(1)). \tag{3.9}$$

Again by calculations, we have

$$\Delta^2 g(z) = (m-n)(m-n-1)\frac{a_0}{b_0} z^{m-n-2}(1 + o(1)). \tag{3.10}$$

Submitting (3.3), (3.8)–(3.10) in (3.2), since $2(m-n-1) < m-n-2 < 0$, we have

$$\begin{aligned} Bz^{\frac{p-q}{k}}(c_0^2 + o(1)) &= c_0(m-n)(m-n-1)\frac{a_0}{b_0} z^{m-n-2}(1 + o(1)) \\ &\quad - \frac{3}{2}\Big((m-n)\frac{a_0}{b_0} z^{m-n-1}\Big)^2 (1 + o(1)) \\ &= c_0(m-n)(m-n-1)\frac{a_0}{b_0} z^{m-n-2}(1 + o(1)). \end{aligned}$$

Hence, $p - q = k(m-n-2) = k(m-n) - 2k < -2k$. That is, $p - q + 2k < 0$.

Case 3. $l = 0$, $c_0 = 0$. Because $m < n$, we see that

$$g(z) = \frac{m(z)}{n(z)} = \frac{a_0}{b_0} z^{m-n}(1 + o(1)). \tag{3.11}$$

We also obtain (3.9) and (3.10). Substituting (3.3), (3.9)–(3.11) into (3.2), we have

$$\frac{n-m-2}{2}(m-n)\frac{a_0^2}{b_0^2} z^{2m-2n-2}(1 + o(1)) = Bz^{\frac{p-q}{k}} \frac{a_0^2}{b_0^2} z^{2m-2n}(1 + o(1)). \tag{3.12}$$

If $n \neq m + 2$, by (3.12),

$$2m - 2n - 2 = \frac{p-q}{k} + (2m - 2n);$$

thus, $p - q + 2k = 0$.

If $n = m + 2$, by (3.12),

$$2m - 2n - 2 > \frac{p-q}{k} + (2m - 2n),$$

thus, $p - q + 2k < 0$.

By the above Cases 1–3, we see if (1.2) has a rational solution $f(z)$, then $p - q + 2k \leq 0$.

(ii) By Lemma 2.10, we see that Theorem 1.3 (ii) holds.

(iii) Set $G(z) = \frac{\Delta^2 f(z)}{\Delta f(z)}$. Lemma 2.10 shows $G(z)$ is a nonconstant rational function. Then

$$\Delta^2 f(z) = G(z)\Delta f(z), \tag{3.13}$$

By (1.2), we easily see $\Delta f(z) \not\equiv 0$, that is $f(z+1) \not\equiv f(z)$. Assert that $f(z+2) \not\equiv f(z)$. Otherwise,

$$\Delta^2 f(z) = f(z+2) - 2f(z+1) + f(z) = 2f(z) - 2f(z+1) = -2\Delta f(z).$$

Together with (3.13),

$$G(z) = \frac{\Delta^2 f(z)}{\Delta f(z)} \equiv -2,$$

which contradicts with the fact $G(z)$ is a nonconstant rational function.

If $f(z)$ has two finite Borel exceptional values, by $f(z+2) \not\equiv f(z)$, $f(z+1) \not\equiv f(z)$ and Remark 2.4, we have

$$T(r, \Delta^2 f) = 3T(r,f) + S(r,f), \quad T(r, \Delta f) = 2T(r,f) + S(r,f).$$

On the other hand, (3.13) shows that

$$T(r, \Delta^2 f) = T(r, \Delta f) + O(\log r).$$

The last two equalities follows $T(r,f) = S(r,f)$. It is a contradiction. So, $f(z)$ cannot have two finite Borel exceptional values.

Suppose that $f(z)$ has two Borel exceptional values $b \in \mathbb{C}$ and ∞. By Hadamard's factorization theory, $f(z)$ takes the form

$$f(z) = b + R_0(z)e^{h(z)}, \tag{3.14}$$

where $R_0(z)$ is a meromorphic function, and $h(z)$ is a polynomial such that

$$\sigma(R_0) = \max\big\{\lambda(f-b), \lambda\big(\frac{1}{f}\big)\big\} < \deg h.$$

Thus,

$$\Delta f(z) = \Big(R_0(z+1)e^{h(z+1)-h(z)} - R_0(z)\Big)\, e^{h(z)} = R_1(z)e^{h(z)}, \tag{3.15}$$

where $R_1(z) = R_0(z+1)e^{h(z+1)-h(z)} - R_0(z)$. Obviously,

$$\sigma(R_1) = \sigma\Big(R_0(z+1)e^{h(z+1)-h(z)} - R_0(z)\Big) \le \max\{\sigma(R_0), \deg h - 1\} < \deg h. \tag{3.16}$$

From (3.15) and (3.16), we see that $\sigma(\Delta f) = \sigma(f)$, and $\Delta f(z)$ has two Borel exceptional values 0 and ∞. Substituting $\Delta f(z) = R_1(z)e^{h(z)}$ into (3.13), we have

$$R_1(z+1)e^{h(z+1)-h(z)} = R_1(z)(G(z)+1). \tag{3.17}$$

If $\deg h \ge 2$, then $\sigma(e^{h(z+1)-h(z)}) = \deg h - 1 \ge 1$. By (3.17) and Lemma 2.1, for any given $\varepsilon > 0$, we have

$$\begin{aligned} m(r, e^{h(z+1)-h(z)}) &\le m\Big(r, \frac{R_1(z)}{R_1(z+1)}\Big) + m(r, G(z)+1) \\ &= O(r^{\sigma(R_1)-1+\varepsilon}) + O(\log r), \end{aligned}$$

which yields $\deg h - 1 \le \sigma(R_1) - 1 + \varepsilon$. Letting $\varepsilon \to 0$, we have $\deg h \le \sigma(R_1)$, which contradicts with (3.16). Hence, if $\deg h \ge 2$, then $f(z)$ has at most one Borel exceptional value.

If $\deg h = 1$, then $F(z) = \Delta f(z) = R_1(z)e^{az}$, where $a \in \mathbb{C} \setminus \{0\}$. If $R_1(z)$ is transcendental with $\sigma(R_1) < 1$, by Lemma 2.6, we see $G(z) = \frac{\Delta^2 f(z)}{\Delta f(z)} = \frac{\Delta F(z)}{F(z)}$ is also transcendental. This contradicts with the fact $G(z)$ is a rational function.

Therefore, $R_1(z)$ is a rational function. Combining this with (3.14) and (3.15), we have

$$f(z) = b + R_0(z)e^{az} \tag{3.18}$$

and

$$R_1(z) = e^a R_0(z+1) - R_0(z),$$

where $\sigma(R_0) < 1$. If $R_0(z)$ is transcendental, by Lemma 2.7, we see $e^a R_0(z+1) - R_0(z)$ is transcendental, which contradicts with $R_1(z) = e^a R_0(z+1) - R_0(z)$ is a rational function. Hence, $R_0(z)$ is a rational function.

(iv) Suppose that $f(z)$ is a meromorphic solution of equation (1.2), then $g(z) = \Delta f(z)$ is a meromorphic solution of equation (3.1). Checking the proof of (i), we see if $g(z)$ is a rational solution of (3.1), then $p - q + 2k \le 0$. Since $p - q + 2k > 0$, we know $\Delta f(z)$ is transcendental. (3.13) still hold. By (3.13), set

$$P(z, \Delta f) := \Delta^2 f(z) - G(z)\Delta f(z) = 0.$$

Since $G(z)$ is a nonconstant rational function, then for any given $a \in \mathbb{C} \setminus \{0\}$, we have $P(z, a) = -aG(z) \not\equiv 0$. Together with Lemma 2.8 , we have $m\big(r, \frac{1}{\Delta f - a}\big) = S(r, \Delta f)$. Thus, $\delta(a, \Delta f) = 0$. By this and the proof of (iii), we see taht $\Delta f(z)$ has at most one Borel exceptional value 0 or ∞ unless

$$\Delta f(z) = R_1(z)e^{az} \tag{3.19}$$

where $a \in \mathbb{C}\setminus\{0\}, R_1(z)$ is a nonzero rational function. Now we prove that $a \ne i2k_1\pi$ for any $k_1 \in \mathbb{Z}$. We see $R_1(z)$ satisfies

$$\frac{R_1(z+2)}{R_1(z)} \to 1, \quad \frac{R_1(z+1)}{R_1(z)} \to 1, \quad z \to \infty. \tag{3.20}$$

By (3.19), we have

$$\begin{gathered} \Delta^2 f(z) = \Delta(\Delta f(z)) = e^{az}(e^a R_1(z+1) - R_1(z)), \\ \Delta^3 f(z) = \Delta^2(\Delta f(z)) = e^{az}(e^{2a} R_1(z+2) - 2e^a R_1(z+1) + R_1(z)). \end{gathered} \tag{3.21}$$

From (3.19)–(3.21), we deduce that

$$\begin{aligned} &\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2 \\ &= e^{2a}\frac{R_1(z+2)}{R_1(z)} - \frac{3}{2}e^{2a}\Big(\frac{R_1(z+1)}{R_1(z)}\Big)^2 + e^a\frac{R_1(z+1)}{R_1(z)} - \frac{1}{2} \\ &\to e^{2a} - \frac{3}{2}e^{2a} + e^a - \frac{1}{2} = -\frac{1}{2}(e^a - 1)^2, \quad z \to \infty. \end{aligned}$$

Combining this with (1.2), we have

$$\Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k = R(z) \to \frac{(-1)^k}{2^k}(e^a - 1)^{2k}, \quad z \to \infty.$$

If $e^a = 1$, by (3.21), we have

$$\Delta^2 f(z) = e^{az}\Delta R_1(z), \quad \Delta^3 f(z) = e^{az}\Delta^2 R_1(z).$$

Combining this with (1.2) and (3.19), we obtain

$$\Big[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\Big(\frac{\Delta^2 f(z)}{\Delta f(z)}\Big)^2\Big]^k = \Big[\frac{\Delta^2 R_1(z)}{R_1(z)} - \frac{3}{2}\Big(\frac{\Delta R_1(z)}{R_1(z)}\Big)^2\Big]^k = R(z).$$

Hence, $R_1(z)$ is a rational solution of the equation

$$\Big[\frac{\Delta^2 g(z)}{g(z)}-\frac{3}{2}\Big(\frac{\Delta g(z)}{g(z)}\Big)^2\Big]^k=R(z). \tag{3.22}$$

By the conclusion of (i), we see if $p-q+2k>0$, equation (3.22) has no rational solutions. It is a contradiction. Thus, $e^a\neq 1$. So, $a\neq i2k_1\pi$ for any $k_1\in\mathbb{Z}$. $\square$

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