Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 197, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONLINEAR ELLIPTIC PROBLEMS INVOLVING THE n-LAPLACIAN WITH MAXIMAL GROWTH

ABDERRAHIM EL ATTAR, SAID EL MANOUNI, OMAR SIDKI

Abstract. In this article, we study the limit case of some elliptic problems involving nonlinearities having the maximal growth with Dirichlet boundary conditions. We apply a result by Ricceri [12] to prove the existence of multiple nontrivial solutions using Trudinger-Moser estimates.

1. Introduction

In this article, we study the limit case of the following two nonlinear problems with Dirichlet boundary condition:

$$\begin{gathered} -\operatorname{div}(m(|\nabla u|^2)\nabla u) = \lambda\beta(x)(u^q e^{\frac{\eta u^2}{\ln(|u|+3)}} + \gamma|u|u) + \mu|u|^{s-2}u \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} -\operatorname{div}(a(|\nabla u|^n)|\nabla u|^{n-2}\nabla u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) \quad \text{in } \Omega \subset \mathbb{R}^n \\ -\operatorname{div}(a(|\nabla v|^n)|\nabla v|^{n-2}\nabla v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) \quad \text{in } \Omega \subset \mathbb{R}^n \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where Ω is a bounded domain with C^1-boundary $\partial\Omega, \lambda, \mu \in \mathbb{R}$ are parameters, $0 < \gamma \leq \eta$, $q > 1$, $s > 1$, $\beta : \Omega \to \mathbb{R}$ be a bounded measurable function and $m, a : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions satisfying some assumptions to be specified in next sections. F_u, F_v are continuous nonlinearities having a maximal growth and G_u, G_v are Carathéodory functions satisfying polynomial growth conditions.

Note that the maximal growth is motivated by the Trudinger-Moser inequality [11, 18] which will be introduced in sections 3 and 4. Recall that Trudinger-Moser type inequalities have a wide variety of applications to the study of nonlinear elliptic partial differential equations involving the limiting case of Sobolev inequalities.

Since the appearance of the abstract result proved by Ricceri in [15] and its revisited note established in [13] dealing with variational equations with both Dirichlet and Neumann conditions, they have extensively been investigated and have widely been applied for the study of the existence of multiple nontrivial solutions and in recent years a lot of papers has been appeared in the scalar case and systems of

2010 *Mathematics Subject Classification.* 35J35, 35J60, 35J66.
Key words and phrases. Limit case; maximal growth; Trudinger-Moser's inequality; Ricceri's principle.

Submitted July 6, 2015. Published July 30, 2015.

elliptic equations. We can cite, among others, the articles [1, 2, 3, 4, 5] and the references therein. In [12], Ricceri obtained a general three critical points theorem, that has been applied for a class of elliptic operators involving nonlinearities of polynomial growth. We will not mention such applications here since the reader can easily access such works.

Concerning the limit case of Dirichlet problems involving n-Laplacian with nonlinearities having exponential growth in bounded domains in $\mathbb{R}^n, n \geq 2$, let us mention here that several studies have been devoted to the investigation of related problems in the scalar case and a lot of papers have appeared in the last years; see for example [6, 8] and the references therein. The solvability of nonlinear boundary value problems in the presence of an exponential nonlinearity has been considered by several authors with the purpose to generalize to a wider class of nonlinearities, classical results from the critical point theory.

Let us point out that El Manouni and Faraci [4] applied the result given in [12] and obtained three weak solutions for a class of variational perturbed problems involving n-Laplacian with nonlinearities having maximal growths, while in [6, 8], by using standard variational approach, the authors prove existence results.

The purpose in this article is to investigate the limiting case for some weighted differential operators, consider nonlinearities with exponential growth and prove multiple nontrivial solutions to Dirichlet boundary value problems. Precisely, we are interested in extending some results to a more general class of elliptic equations and systems by making also use of the variational principle of Ricceri [12].

This article is organized as follows. In section 2 we recall some important definitions and the crucial result of Ricceri [12], as the basis for the study of the existence of at least three weak solutions for the given problems. In section 3 we treat the limit case of a weighted elliptic equation involving n-Laplacian ($n = 2$) and we will prove multiple results by applying Ricceri's principle in [12]. Finally, in section 4, we extend the result of the last section to general elliptic systems of two second order nonlinear partial differential equations governed essentially by the n-Laplacian operator with $n > 2$ and use again Ricceri's principle to prove the multiplicity of solutions.

2. Preliminaries

Let us denote by $\mathcal{A}$ the class of the Carathéodory functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

- for every $M > 0, \sup_{|t|\leq M} |h(x,t)| \in L^\infty(\Omega)$;
- for every $\delta > 0$,
$$\lim_{|u|\to\infty} \sup_{x\in\Omega} \frac{|h(x,u)|}{e^{\delta|u|^2}} = 0. \tag{2.1}$$

Before recalling the result proved in [12], which will be the key for the study of the existence of at least three weak solutions for problems (1.1) and (1.2) when the nonlinearities have a maximal growth, let us first recall that if X is a real Banach space, we denote by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$, possessing the following property: if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Theorem 2.1. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$, a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$,* *bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0) = J(x_0) = 0$. Finally, set*

$$\alpha = \max\big\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\big\},$$
$$\beta = \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(x)}{\Phi(x)},$$

and assume that $\alpha < \beta$. Then, for each compact interval $[a, b] \subset]1/\beta, 1/\alpha[$ (with the conventions $1/0 = +\infty$, $1/\infty = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$, and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu\Psi'(x)$$

has at least three solutions in X whose norms are less than r.

3. Weighted Laplace equations

Consider the Dirichlet problem

$$\begin{gathered} -\operatorname{div}(m(|\nabla u|^2)\nabla u) = \lambda\beta(x)(u^q e^{\frac{\eta u^2}{\ln(|u|+3)}} + \gamma|u|u) + \mu|u|^{s-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with C^1-boundary $\partial\Omega, q > 1, s > 1, \beta : \Omega \to \mathbb{R}$ be a bounded measurable function and $m : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying the following assumption.

There exist positive constants $p \in]1, 2]$, b_1, b_2, c_1, c_2 such that

$$c_1 + b_1|u|^{2-p} \le |u|^{2-p}m(u^2) \le c_2 + b_2|u|^{2-p} \quad \forall u \in \mathbb{R}. \tag{3.2}$$

Remark 3.1. (1) The function $f(x, t) = t^q e^{\frac{\eta t^2}{\ln(|t|+3)}} + \gamma|t|t$, with $q > 1$ and $0 < \gamma \le \eta$, and the function $g(x, t) = |t|^{s-2}t$ with $s > 1$ belong to the class $\mathcal{A}$.
(2) The operator considered here has been studied by Hirano [7] and by Ubilla [19] with nonlinearities having a polynomial growth.
(3) If $m(u) = 1 + u^{\frac{p-2}{2}}, p \le 2$, then (3.2) holds and $-\operatorname{div}(m(|\nabla u|^2)\nabla u)$ in (3.1) becomes $-\Delta u - \Delta_p u$, where $\Delta_p \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator.

To study the Dirichlet problem (3.1), we use the space $W = W_0^{1,2}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}.$$

Motivated by the following result due to Trudinger-Moser (cf. [11, 18]), problem (3.1) can be treated variationally in W since W is embedded in the class of Orlicz-Lebesgue space $L_\phi(\Omega)$ generated by the function $\phi(t) = \exp\big(t^2\big) - 1$, i.e.,

$$L_\phi(\Omega) = \{u : \Omega \to \mathbb{R},\ \text{measurable} : \int_\Omega |\phi(u)|\,dx < \infty\},$$

equipped with the norm

$$\|u\|_\phi = \inf\Big\{\lambda > 0 : \int_\Omega \phi(\frac{u(x)}{\lambda})dx \le 1\Big\}.$$

Moreover,

$$\exp\big(\delta|u|^2\big) \in L^1(\Omega), \quad \forall u \in W, \forall \delta > 0,$$
$$\sup_{\|u\|\le 1} \int_\Omega \exp\big(\delta|u|^2\big)\,dx \le C \quad \text{if } \delta \le \alpha_2,$$

where $\alpha_2 = 2\omega_1$ and ω_1 is the 1-dimensional surface of the unit disk in $\mathbb{R}^2$.

Now we state the main result of this section.

Theorem 3.2. *Assume that $m : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying (3.2) such that the function $k(u) = m(|u|^2)u$ is strictly uniformly monotone and $k(u) \to 0$ as $u \to 0^+$. Furthermore, assume that there exists a constant $C' > 0$ such that $\big(\int_\Omega e^{\delta u^2}\,dx\big)^{1/2} \le C'\|u\|$ for all $u \in W$ and all $\delta > 0$, and that there exists $u_0 \in W$ such that*

$$\int_\Omega \beta(x) \int_0^{u_0(x)} (\xi^q e^{\frac{\eta\xi^2}{\ln(|\xi|+3)}} + \gamma|\xi|\xi)\,d\xi\,dx > 0.$$

Then, if we set

$$\begin{aligned} \omega = \frac{1}{2}\inf\Big\{&\frac{\int_\Omega M(|\nabla u|^2)\,dx}{\int_\Omega \beta(x)\int_0^u (\xi^q e^{\frac{\eta\xi^2}{\ln(|\xi|+3)}} + \gamma|\xi|\xi)\,d\xi\,dx},\\ &u \in W,\ \int_\Omega \beta(x)\int_0^u (\xi^q e^{\frac{\eta\xi^2}{\ln(|\xi|+3)}} + \gamma|\xi|\xi)\,d\xi\,dx > 0\Big\}, \end{aligned} \tag{3.3}$$

with $M(\xi) = \int_0^\xi m(t)\,dt$, for each compact interval $[a,b] \subset]\omega, +\infty[$, there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\sigma > 0$ such that for each $\mu \in [0,\sigma]$, problem (3.1) has at least three solutions in W whose norms are less than r.

Remark 3.3. The Trudinger-Moser inequality [11, 18] implies that $\exp(\delta|u|^2) \in L^1(\Omega)$ for all $u \in W$ and all $\delta > 0$, so that the condition $\big(\int_\Omega e^{\delta u^2}\,dx\big)^{1/2} \le C'\|u\|$, given in Theorem 3.2, makes sense since the quantity $\int_\Omega e^{\delta u^2}\,dx$ depends on u.

Recall that a weak solution of the problem (3.1) is any $u \in W$ such that

$$\int_\Omega m(|\nabla u|^2)\nabla u \nabla v\,dx = \int_\Omega (\lambda\beta(x)(u^q e^{\frac{\eta u^2}{\ln(|u|+3)}} + \gamma|u|u) + \mu|u|^{s-2}u)v\,dx \quad \forall v \in W.$$

Remark 3.4.
(1) The maximal growth (2.1) guarantees that integrals given in the right side are well defined.
(2) The exponential growth condition (2.1) covers a general class of functions where particularly polynomials are taken.

Now we state and prove the following lemma which will be needed later.

Lemma 3.5. *If $h \in \mathcal{A}$, then the functional $N : W \to \mathbb{R}$ defined by $N(u) = \int_\Omega H(x, u(x))\,dx$, where $H(x,\xi) = \int_0^\xi h(x,t)\,dt$, is continuously differentiable with compact derivative.*

Proof. Since h belongs to $\mathcal{A}$, for $\delta > 0$, there exists $C_1 > 0$ such that

$$|h(x,u)| \le C_1 e^{\delta |u|^2}, \quad \forall (x,u) \in \Omega \times \mathbb{R}.$$

Hence, it follows that for every $u \in W$, and almost every $x \in \Omega$,

$$|H(x,u(x))| \le C_2 |u(x)| \exp(\delta |u(x)|^2).$$

Then, using Trudinger-Moser inequality [11, 18] and Hölder inequality, we see that N is well defined on W. Further, Lebesgue Theorem implies that

$$N'(u)v = \lim_{t \to 0^+} \frac{N(u+tv) - N(u)}{t} = \int_\Omega h(x,u(x))v(x)\,dx.$$

Hence, N is Gâteaux differentiable with derivative given by

$$N'(u)v = \int_\Omega h(x,u)v\,dx$$

for all $u, v \in W$. Let us now show that N' is continuous from W to its dual W^*. Indeed, let $\{u_k\}$ be a sequence converging to some u in W.

On one hand, there exists a subsequence, denoted again by $\{u_k\}$ such that

$$u_k \to u \quad \text{strongly in } L^{p_1}(\Omega),$$

as $k \to \infty$ for all $p_1 > 1$. Hence $u_k \to u$ a.e. in Ω.

On the other hand, we have

$$\begin{aligned} \int_\Omega |h(x,u_k)|^{p_1}\,dx &\le C_3 \int_\Omega \exp(p_1 \delta |u_k|^2)\,dx \\ &\le C_3 \int_\Omega \exp\Big(p_1 \delta \|u_k\|^2 \Big(\frac{|u_k|}{\|u_k\|}\Big)^2\Big)\,dx, \end{aligned}$$

for some constant $C_3 > 0$. Since $\{u_k\}$ is a bounded sequence, we may choose δ sufficiently small such that $p_1 \delta \|u_k\|^2 < \alpha_2$. Then one has

$$\sup_k \int_\Omega |h(x,u_k)|^{p_1}\,dx \le C_4,$$

for some constant $C_4 > 0$. Similarly there exists $C_5 > 0$ such that

$$\int_\Omega |h(x,u)|^{p_1}\,dx \le C_5.$$

Let E be a measurable subset of Ω and let $\varepsilon > 0$, we have

$$\begin{aligned} \int_E |h(x,u_k) - h(x,u)|^{p_1}\,dx &\le \big(\operatorname{meas}(E)\big)^{1/\zeta'} \Big(\int_\Omega |h(x,u_k) - h(x,u)|^{p_1 \zeta}\,dx\Big)^{1/\zeta} \\ &\le K\big(\operatorname{meas}(E)\big)^{1/\zeta'}, \end{aligned}$$

where K is a positive constant which is independent of k and $\frac{1}{\zeta} + \frac{1}{\zeta'} = 1$. Then, for $\operatorname{meas}(E)$ sufficiently small, we obtain

$$\int_E |h(x,u_k) - h(x,u)|^{p_1}\,dx \le \varepsilon.$$

Since $u_k \to u$ a.e. in Ω, we have $h(x,u_k) \to h(x,u)$ a.e. in Ω. Then in view of Vitali's convergence theorem,

$$h(x,u_k) \to h(x,u) \quad \text{in } L^{p_1}(\Omega).$$

Hence, using Hölder inequality, we obtain

$$\int_\Omega (h(x,u_k)-h(x,u))v\,dx \le \Big[\int_\Omega |h(x,u_k)-h(x,u)|^{p_1}\,dx\Big]^{1/p_1}\Big[\int_\Omega |v|^{p'_1}\,dx\Big]^{1/p'_1},$$

where p'_1 is the conjugate of p_1. This shows that N' is continuous. Similarly, we prove that N' is a compact map from W to its dual W^*. □

Proof of Theorem 3.2. It follows from (3.2) that for all $u \in \mathbb{R}$,

$$\frac{1}{2}M(|u|^2) \ge \frac{b_1}{2}|u|^2 + \frac{c_1}{p}|u|^p$$
$$\frac{1}{2}M(|u|^2) \le \frac{b_2}{2}|u|^2 + \frac{c_2}{p}|u|^p,$$

with $p>1$. Furthermore the function $h(u)=M(|u|^2)$ is strictly convex. Consequently, the functional $\Phi : W \to \mathbb{R}$ defined as

$$\Phi(u) = \frac{1}{2}\int_\Omega M(|\nabla u|^2)\,dx$$

is well defined, coercive, weakly lower semicontinuous, Gâteaux differentiable and belongs to $C^1(W,\mathbb{R})$. Moreover Φ is bounded on each bounded subset of W. By using [20, Theorem 26.A], we deduce that Φ' admits a continuous inverse on W^* since the operator $I : W \to W^*$ defined by

$$I(u)v = \int_\Omega m(|\nabla u|^2)\nabla u \nabla v\,dx$$

is uniformly monotone. This implies that $\Phi \in \mathcal{W}_W$. Let us prove that

$$\limsup_{u\to 0}\frac{J(u)}{\Phi(u)} = 0.$$

Set

$$f(x,u) = \beta(x)(u^q e^{\frac{\eta u^2}{\ln(|u|+3)}} + \gamma|u|u)$$

for all $(x,u)\in\Omega\times\mathbb{R}$ and

$$J(u) = \int_\Omega F(x,u)\,dx$$

for all $u \in W$, where

$$F(x,u) = \beta(x)\int_0^u (t^q e^{\frac{\eta t^2}{\ln(|t|+3)}} + \gamma|t|t)\,dt.$$

It is easy to see that

$$\begin{aligned} F(x,u) &\le \beta(x)(|u|^{q+1}e^{\frac{\eta u^2}{\ln(|u|+3)}} + \gamma\frac{|u|^3}{3}) \\ &\le \beta(x)(|u|^{q+1}e^{\eta u^2} + \gamma\frac{|u|^3}{3}) \end{aligned}$$

for all $(x,u)\in\Omega\times W$. Then for some positive constants C_6 and C_7, we obtain

$$\begin{aligned} \frac{J(u)}{\Phi(u)} &\le C_6 \sup_{x\in\Omega}\beta(x)\frac{\int_\Omega \big(|u|^{q+1}e^{\eta u^2} + \gamma\frac{|u|^3}{3}\big)\,dx}{C_7\int_\Omega |\nabla u|^2 + |\nabla u|^p\,dx}, \\ &\le C_6 \sup_{x\in\Omega}\beta(x)\frac{\Big(\int_\Omega e^{z'\|u\|^2\eta(\frac{|u|}{\|u\|})^2}\Big)^{1/z'}\Big(\int_\Omega |u|^{z(q+1)}\,dx\Big)^{1/z} + \int_\Omega \gamma\frac{|u|^3}{3}\,dx}{C_7\int_\Omega(|\nabla u|^2 + |\nabla u|^p)\,dx}, \end{aligned}$$

with $\frac{1}{z}+\frac{1}{z'}=1$. Then, in view of Trudinger-Moser's inequality and since W is continuously embedded in $L^{z(q+1)}(\Omega)$, there exists a constant $C_8>0$ such that

$$\frac{J(u)}{\Phi(u)} \le C_8 \sup_{x\in\Omega}\beta(x)\frac{\|u\|^{q+1}+\|u\|^3}{\|u\|^2}, \tag{3.4}$$

for $\|u\|$ small enough such that $z'\|u\|^2\eta<4\pi$. Hence, since $q>1$, (3.4) implies that

$$\limsup_{u\to 0}\frac{J(u)}{\Phi(u)} \le 0. \tag{3.5}$$

Using again the fact that $F(x,u)\le \beta(x)(|u|^{q+1}e^{\eta u^2}+\gamma\frac{u^3}{3})$ for all $(x,u)\in\Omega\times W$. Then, applying again Trudinger-Moser inequality and by the same argument as in the proof of Lemma 3.5, we deduce that

$$J(u)=\int_\Omega F(x,u)\,dx$$

is well defined and continuously Gâteaux differentiable, with compact derivative,

$$J'(u)v=\int_\Omega f(x,u)v\,dx$$

for all $u,v\in W$. Notice that the compactness of J' holds since $f\in\mathcal{A}$. Let us prove that

$$\limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)}\le 0.$$

On one hand, we have for all $\delta>0$,

$$\limsup_{|t|\to\infty}\frac{\sup_{x\in\Omega}F(x,t)}{e^{\delta t^2}}=0,$$

and so for every $\varepsilon>0$, there exists some positive ρ such that, for every $x\in\Omega$ and $|t|>\rho$,

$$F(x,t)\le \varepsilon e^{\delta t^2}.$$

On the other hand, we can easily see that for every $M>0$, $\sup_{|t|\le M}|f(x,t)|\in L^\infty(\Omega)$. Hence, there exists some constant $C_9>0$ such that, for every $x\in\Omega$,

$$\sup_{|t|\le\rho}|f(x,t)|\le C_9.$$

Then, for every $x\in\Omega$ and $t\in\mathbb{R}$,

$$F(x,t)\le C_9\rho+\varepsilon e^{\delta t^2}$$

and so

$$J(u)\le C_9\rho\,\mathrm{meas}(\Omega)+\varepsilon\int_\Omega e^{\delta u^2}dx.$$

Since $\left(\int_\Omega e^{\delta u^2}\,dx\right)^{\frac12}\le C'\|u\|$ for all $\delta>0$ and all $u\in W$, we obtain

$$\frac{J(u)}{\Phi(u)}\le 2\frac{C_9\rho\,\mathrm{meas}(\Omega)}{\|u\|^2}+2\varepsilon\frac{C'^2\|u\|^2}{\|u\|^2}.$$

Therefore,

$$\limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)}\le 2\varepsilon C'^2. \tag{3.6}$$

Finally, for an arbitrary ε and in view of (3.5) and (3.6), we obtain

$$\max\Big\{\limsup_{\|x\|\to+\infty}\frac{J(x)}{\Phi(x)},\limsup_{x\to 0}\frac{J(x)}{\Phi(x)}\Big\}\leq 0.$$

Hence, all the assumptions of Theorem 2.1 are satisfied (with $x_0=0$). Moreover, the functional

$$\Psi(u)=\frac{1}{s}\|u\|^s_{L^s(\Omega)}$$

is continuously Gâteaux differentiable on W, with compact derivative. Consequently the result follows and the proof is complete. □

Remark 3.6. Consider the following general case of problems (3.1):

$$\begin{gathered}-\operatorname{div}(m(|\nabla u|^n)|\nabla|^{n-2}\nabla u)=\lambda f(x,u)+\mu g(x,u)\quad\text{in }\Omega\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{3.7}$$

where Ω is a bounded domain of $\mathbb{R}^n$ and the nonlinearity $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function having an exponential growth on Ω: i.e.,

(H0) For all $\delta>0$,

$$\lim_{|u|\to\infty}\frac{|f(x,u)|}{e^{\delta(|u|^{\frac{n}{n-1}})}}=0\quad\text{uniformly in }\Omega,$$

with $p=n\neq 2$. $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$|g(x,u)|\leq a|u|^{s-1}+b,$$

for all $(x,u)\in\Omega\times\mathbb{R}$ with $s>1$, $a,b>0$. Regarding the function m, it is assumed to satisfy the following conditions:

(M1) $m:\mathbb{R}^+\to\mathbb{R}$ is continuous;

(M2) there exist positive constants $p\in]1,n]$, b_1,b_2,c_1,c_2 such that

$$c_1+b_1u^{n-p}\leq u^{n-p}m(u^n)\leq c_2+b_2u^{n-p}\quad\forall u\in\mathbb{R}^+;$$

(M3) the function $k:\mathbb{R}\to\mathbb{R}$, $k(u)=m(|u|^n)|u|^{n-2}u$ is strictly increasing and $k(u)\to 0$ as $u\to 0^+$.

The assertion in Theorem 3.2 concerning problem (3.7) should be a generalization of problem (3.1) and the proof is substantively similar to the previous where we adopt the variational principle of Ricceri [12].

4. Nonlinear elliptic systems with $n>2$

Let $\Omega\subset\mathbb{R}^n$, with $n>2$, be a bounded domain with smooth boundary $\partial\Omega$. In this section we shall be concerned with existence of solutions for the problem

$$\begin{gathered}-\operatorname{div}\big(a(|\nabla u|^n)|\nabla u|^{n-2}\nabla u\big)=\lambda F_u(x,u,v)+\mu G_u(x,u,v)\quad\text{in }\Omega\\ -\operatorname{div}\big(a(|\nabla v|^n)|\nabla v|^{n-2}\nabla v\big)=\lambda F_v(x,u,v)+\mu G_v(x,u,v)\quad\text{in }\Omega\\ u=v=0\quad\text{on }\partial\Omega,\end{gathered}\tag{4.1}$$

where the nonlinearity $F:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is assumed to be measurable in Ω and C^1 in $\mathbb{R}\times\mathbb{R}$ satisfying

(H1) For every $M,\sup_{|(t,s)|\leq M}(|F_t(x,t,s)|+|F_s(x,t,s)|)\in L^\infty(\Omega)$,

and having an exponential growth on Ω; i.e.,

(H2) For all $\delta > 0$
$$\lim_{|(t,s)|\to\infty} \frac{|F_t(x,t,s)| + |F_s(x,t,s)|}{e^{\delta(|t|^n+|s|^n)^{1/(n-1)}}} = 0 \quad \text{Uniformly in } \Omega,$$
with $F(.,0,0) \in L^1(\Omega)$.

Regarding the function G, it is assumed to be a measurable function with respect to x in Ω for every (s,t) in $\mathbb{R}\times\mathbb{R}$, and is a C^1-function with respect to (s,t) in $\mathbb{R}\times\mathbb{R}$ for almost every x in Ω and satisfies

(G1)
$$\sup_{|(t,s)|\le k} (|G_t(x,t,s)| + |G_s(x,t,s)|) \le h_k(x),$$
for all $k > 0$ and some $h_k \in L^1(\Omega)$ with $G(.,0,0) \in L^1(\Omega)$.

Finally, we make the following assumptions on the function a.

(A1) $a : \mathbb{R}^+ \to \mathbb{R}$ is continuous;

(A2) There exist positive constants $p \in]1,n]$, b_1, b_2, c_1, c_2 such that
$$c_1 + b_1 u^{n-p} \le u^{n-p} a(u^n) \le c_2 + b_2 u^{n-p} \quad \forall u \in \mathbb{R}^+;$$

(A3) The function $k : \mathbb{R} \to \mathbb{R}$, $k(u) = a(|u|^n)|u|^{n-2}u$ is strictly increasing and $k(u) \to 0$ as $u \to 0^+$.

We shall look for a weak-solution of (4.1) in the space $W = W_0^{1,n}(\Omega) \times W_0^{1,n}(\Omega)$ which is endowed with the norm
$$\|U\|_W^n = \int_\Omega |\nabla U|^n \, dx = \int_\Omega (|\nabla u|^n + |\nabla v|^n) \, dx$$
where $U = (u,v) \in W$. Motivated by the following result due to Trudinger and Moser (cf. [11], [18]) in the case where $n \ne 2$, we remark that the space W is embeded in the class of Orlicz-Lebesgue space
$$L_\phi = \{U : \Omega \to \mathbb{R}^2, \text{ measurable} : \int_\Omega \phi(U) < \infty\},$$
where $\phi(s,t) = \exp\big(s^{\frac{n}{n-1}} + t^{\frac{n}{n-1}}\big)$. Moreover, there exists a constant C_n depending on n and on the measure of Ω such that
$$\sup_{\|(u,v)\|_W \le 1} \int_\Omega \exp\big(\delta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}\big) \, dx \le C_n \quad \text{for every } 0 < \delta \le \alpha_n,$$
where $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, being ω_{n-1} the measure of the $(n-1)$ dimensional surface of the unit sphere in $\mathbb{R}^n$.

Remark 4.1. The operator considered here has been studied by Hirano [7] and Ubilla [19] with nonlinearities having polynomial growth.

We shall denote by λ_1 the smallest eigenvalue [17] for the problem
$$\begin{gathered} -\Delta_n u = \lambda |u|^{\alpha-1} u |v|^{\beta+1} \quad \text{in } \Omega \subset \mathbb{R}^n \\ -\Delta_n v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v \quad \text{in } \Omega \subset \mathbb{R}^n \\ u = v = 0 \quad \text{on } \partial\Omega; \end{gathered}$$
i.e.,
$$\lambda_1 = \inf\Big\{ \frac{\alpha+1}{n} \int_\Omega |\nabla u|^n \, dx + \frac{\beta+1}{n} \int_\Omega |\nabla v|^n \, dx :$$

$$(u,v)\in W,\ \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx=1\Big\}$$

where $\alpha+\beta=n-2$ and $\alpha,\beta>-1$.

Remark 4.2. Since $\alpha+\beta=n-2$ implies that $\frac{\alpha+1}{n}+\frac{\beta+1}{n}=1$, one has for all $(u,v)\in W\backslash\{(0,0)\}$

$$\lambda_1\le \frac{\alpha+1}{n}\int_\Omega |\nabla u_1|^n\,dx+\frac{\beta+1}{n}\int_\Omega |\nabla v_1|^n\,dx \quad\text{and}\quad \int_\Omega |u_1|^{\alpha+1}|v_1|^{\beta+1}\,dx=1$$

with

$$u_1=\frac{u}{(\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx)^{1/n}},\quad v_1=\frac{v}{(\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx)^{1/n}}.$$

So that

$$\lambda_1\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx\le \frac{\alpha+1}{n}\int_\Omega |\nabla u|^n\,dx+\frac{\beta+1}{n}\int_\Omega |\nabla v|^n\,dx$$

for all $(u,v)\in W\backslash\{(0,0)\}$.

Definition 4.3. We say that a pair $(u,v)\in W$ is a weak solution of (4.1) if for all $(\varphi,\psi)\in W$,

$$\begin{gathered}\int_\Omega a(|\nabla u|^n)|\nabla u|^{n-2}\nabla u\nabla\varphi\,dx=\int_\Omega(\lambda F_u(x,u,v)+\mu G_u(x,u,v))\varphi\,dx\\ \int_\Omega a(|\nabla v|^n)|\nabla v|^{n-2}\nabla v\nabla\psi\,dx=\int_\Omega(\lambda F_v(x,u,v)+\mu G_v(x,u,v))\psi\,dx\end{gathered}\tag{4.2}$$

Now we state our second main result.

Theorem 4.4. *Suppose that F_u and F_v are continuous functions satisfying* (H1)-(H2) *and that a satisfies* (A1)-(A3)*. Furthermore, assume that*

$$\lim_{|U|\to0}\sup\frac{pF(x,U)}{|u|^{\alpha+1}|v|^{\beta+1}}<\lambda_1,\tag{4.3}$$

$$\limsup_{|U|\to\infty}\frac{\sup_{x\in\Omega}F(x,U)}{|U|^n}\le0,\tag{4.4}$$

uniformly on $x\in\Omega$, and

$$\sup_{U\in W}\int_\Omega F(x,U)\,dx>0,$$

with $U=(u,v)$. Then, if we set

$$\theta=\frac1n\inf\Big\{\frac{\int_\Omega A(|\nabla u|^n)+A(|\nabla v|^n)\,dx}{\int_\Omega F(x,u(x),v(x))\,dx}:(u,v)\in W,\int_\Omega F(x,u(x),v(x))\,dx>0\Big\},\tag{4.5}$$

with $A(t)=\int_0^t a(\tau)\,d\tau$, for each compact interval $[a,b]\subset]\theta,+\infty[$, there exists $r>0$ with the following property: for every $\lambda\in[a,b]$ and for every function G satisfying (G1) there exists $\sigma>0$ such that for each $\mu\in[0,\sigma]$, problem (4.1) *has at least three weak solutions in W whose norms are less than r.*

Remark 4.5. If $a(u)=1+u^{\frac{p-n}{n}}$ and conditions (A2)–(A3) hold, then problem (4.1) can be formulated as follows

$$\begin{gathered}-\Delta_n u-\Delta_p u=\lambda F_u(x,u,v)+\mu G_u(x,u,v)\\ -\Delta_n v-\Delta_p v=\lambda F_v(x,u,v)+\mu G_v(x,u,v),\end{gathered}$$

where $\Delta_p \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator.

The maximal growth of $F_u(x,u,v)$ and $F_v(x,u,v)$ will allow us to treat variationally system (4.1) in the product Sobolev space W. This exponential growth is relatively motivated by Trudinger-Moser inequality [11, 18]. Now, it follows from the assumptions on the function a that for all $t \in \mathbb{R}$,

$$\frac{1}{n}A(|t|^n) \geq \frac{b_1}{n}|t|^n + \frac{c_1}{p}|t|^p$$
$$\frac{1}{n}A(|t|^n) \leq \frac{b_2}{n}|t|^n + \frac{c_2}{p}|t|^p,$$

where $A(t) = \int_0^t a(\tau)\,d\tau$. Furthermore, the function $l(t) = A(|t|^n)$ is strictly convex. Consequently, the functional $\Phi : W \to \mathbb{R}$ defined as

$$\Phi(u,v) = \frac{1}{n}\int_\Omega A(|\nabla u|^n) + A(|\nabla v|^n)\,dx$$

is well defined, weakly lower semicontinuous, Frêchet differentiable and belongs to $C^1(W,\mathbb{R})$.

Proposition 4.6. *Let $I : W \to W^*$ be the operator defined by*

$$I(u,v)(\varphi,\psi) = \int_\Omega a(|\nabla u|^n)|\nabla u|^{n-2}\nabla u\nabla\varphi + a(|\nabla v|^n)|\nabla v|^{n-2}\nabla v\nabla\psi\,dx$$

for all $(u,v),(\varphi,\psi) \in W$. Then I admits a continuous inverse on W^*.

Proof. Denoting by $\langle\cdot,\cdot\rangle$ the usual inner product in $\mathbb{R}^n$, for $\kappa \geq 2$ there exists a positive constants c_κ such that the following inequality (see [16])

$$\langle |x|^{\kappa-2}x - |y|^{\kappa-2}y, x-y\rangle \geq c_\kappa|x-y|^\kappa$$

holds for all $x, y \in \mathbb{R}^n$. Thus, it is easy to see, using (A2), that

$$\langle I(u_1,v_1) - I(u_2,v_2), (u_1-u_2, v_1-v_2)\rangle \geq c_n(\|u_1-u_2\|_1^n + \|v_1-v_2\|_2^n),$$

for every (u_1,v_1) and (u_2,v_2) belonging to W. This means that I is an uniformly monotone operator in W. Moreover I is coercive and hemicontinuous in W. Therefore, the conclusion follows directly from [20, Theorem 26.A]. Moreover, Φ is coercive, weakly lower semicontinuous, bounded on bounded subsets of W and it belongs to $\mathcal{W}_W$. $\square$

Lemma 4.7. *Assume that F satisfies* (H1)–(H2)*. Then the functional $J : W \to \mathbb{R}$ defined by*

$$J(u,v) = \int_\Omega F(x,u,v)\,dx$$

is continuously differentiable with compact derivative.

Proof. On one hand, if the functions F_u and F_v are continuous and have an exponential growth, then there exists a positive constant C_{10} such that

$$|F_u(x,u,v)| + |F_v(x,u,v)| \leq C_{10}\exp\big(\delta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}})\big), \tag{4.6}$$

for all $(x,u,v) \in \Omega \times \mathbb{R}^2$. Consequently the functional $J : W \to \mathbb{R}$ defined by

$$J(u,v) = \int_\Omega F(x,u,v)\,dx$$

is well defined. On the other hand, as in Lemma 3.5, we conclude that J is continuously Gâteau differentiable with

$$J'(u,v)(\varphi,\psi)=\int_\Omega F_u(x,u,v)\varphi+F_v(x,u,v)\psi\,dx$$

for all $(u,v),(\psi,\varphi)\in W$. Let us now show that J' is continuous from W to its dual W^*. Indeed, let $\{(u_k,v_k)\})$ be a sequence converging to a some $\{(u,v)\}$ in W. Thus, there exists a subsequence, denoted again by $\{(u_k,v_k)\}$ such that

$$\begin{gathered}u_k\to u\quad\text{in } L^{q_1}(\Omega),\\ v_k\to v\quad\text{in } L^{q_2}(\Omega),\end{gathered}$$

as $n\to\infty$ and for all $q_1,q_2>1$. On one hand, we have

$$\begin{aligned}\int_\Omega |F_u(x,u_k,v_k)|^{q_1}\,dx &\le C_{11}\int_\Omega \exp(q_1\delta(|u_k|^{\frac{n}{n-1}}+|v_k|^{\frac{n}{n-1}}))\,dx\\ &\le C_{11}(\int_\Omega \exp(\zeta q_1\delta|u_k|^{\frac{n}{n-1}}))^{\frac{1}{\zeta}}(\int_\Omega \exp(\zeta' q_1\delta|v_k|^{\frac{n}{n-1}}))^{\frac{1}{\zeta'}}\\ &\le C_{11}\Big(\int_\Omega \exp(\zeta q_1\delta\|u_k\|^{\frac{n}{n-1}}_{W_0^{1,n}(\Omega)}(\frac{|u_k|^{\frac{n}{n-1}}}{\|u_k\|_{W_0^{1,n}(\Omega)}}))\Big)^{1/\zeta}\\ &\quad\times\Big(\int_\Omega \exp(\zeta' q_1\delta\|v_k\|^{\frac{n}{n-1}}_{W_0^{1,n}(\Omega)}(\frac{|v_k|^{\frac{n}{n-1}}}{\|v_k\|_{W_0^{1,n}(\Omega)}}))\Big)^{1/\zeta'},\end{aligned}$$

for some positive constant C_{11} and $\frac{1}{\zeta}+\frac{1}{\zeta'}=1$. Since $\{(u_k,v_k)\}$ is a bounded sequence, we may choose δ sufficiently small such that

$$\zeta q_1\delta\|u_k\|^{\frac{n}{n-1}}_{W_0^{1,n}(\Omega)}<\alpha_n\quad\text{and}\quad \zeta' q_1\delta\|v_k\|^{\frac{n}{n-1}}_{W_0^{1,n}(\Omega)}<\alpha_n.$$

Then

$$\int_\Omega |F_u(x,u_k,v_k)|^{q_1}\,dx\le C_{12}$$

for k large and some constant $C_{12}>0$. By the same argument, we also have

$$\int_\Omega |F_v(x,u_k,v_k)|^{q_2}\,dx\le C_{13}$$

for k large and some constant $C_{13}>0$. The proof can now be completed following the same steps as in the proof of Lemma 3.5. $\square$

Proof of Theorem 4.4. Recall that the functionals Φ and J are defined on W as follows

$$\Phi(u,v)=\frac{1}{n}\int_\Omega A(|\nabla u|^n)+A(|\nabla v|^n)\,dx,\quad\text{and}\quad J(u,v)=\int_\Omega F(x,u,v)\,dx,$$

for every $(u,v)\in W$. The goal is to apply Theorem 2.1 to the functionals Φ and J to obtain multiple weak solutions of (4.1). It is well known that Φ is coercive, weakly lower semicontinuous, bounded on bounded subsets of W and it belongs to $\mathcal{W}_W$. Moreover Φ is continuously Gâteaux differentiable in X with derivative

$$\Phi'(u,v)(\varphi,\psi)=\int_\Omega a(|\nabla u|^n)|\nabla u|^{n-2}\nabla u\nabla\varphi+a(|\nabla v|^n)|\nabla v|^{n-2}\nabla v\nabla\psi\,dx$$

for all $(u,v),(\varphi,\psi)\in W$ and Φ' admits a continuous inverse on W^* (see [20, Theorem 26.A]).

In view of Lemma 4.7, J is well defined and continuously Gâteaux differentiable with compact derivative J' given by

$$J'(u,v)(\varphi,\psi) = \int_\Omega F_u(x,u,v)\varphi + F_v(x,u,v)\psi\,dx$$

for all $(u,v),(\varphi,\psi) \in W$. Let us prove now that

$$\limsup_{(u,v)\to(0,0)} \frac{J(u,v)}{\Phi(u,v)} \le 0. \tag{4.7}$$

By (4.3),

$$\limsup_{|(t,s)|\to(0,0)} \frac{\sup_{x\in\Omega} pF(x,t,s)}{|t|^{\alpha+1}|s|^{\beta+1}} \le \lambda_1,$$

and so for every $\varepsilon > 0$ there exists some positive ρ such that, for every $x \in \Omega$ and $|(u,v)| < \rho$,

$$F(x,u,v) < \frac{\varepsilon}{p}\lambda_1 |u|^{\alpha+1}|v|^{\beta+1}.$$

In view of (H2), for fixed $\delta > 0$ and $z > n$ there exists $C_{14} > 0$ such that, for every $x \in \Omega$ and $|(u,v)| \ge \rho$

$$F(x,u,v) \le C_{14}(|u|^z|v|^z e^{\delta(|u|^{\frac{n}{n-1}}+|v|^{\frac{n}{n-1}})}).$$

Then, for every $x \in \Omega$ and $(u,v) \in W$, one has

$$F(x,u,v) \le \frac{\varepsilon}{p}\lambda_1|u|^{\alpha+1}|v|^{\beta+1} + C_{14}(|u|^z|v|^z e^{\delta(|u|^{\frac{n}{n-1}}+|v|^{\frac{n}{n-1}})}).$$

Let $\theta_1,\theta_2,\theta_3,\theta_4 > 1$ with $\sum_{i=1}^4 \frac{1}{\theta_i} = 1$. Then in view of Remark 4.2 and by applying Hölder's inequality, we obtain

$$\begin{aligned} J(u,v) &\le C_{15}\varepsilon\|(u,v)\|^n + C_{14}\Big[\int_\Omega e^{\left(\theta_1\delta\|u\|^{\frac{n}{n-1}}(\frac{|u|}{\|u\|})^{\frac{n}{n-1}}\right)}dx\Big]^{1/\theta_1}\Big(\int_\Omega |u|^{\theta_2 z}\,dx\Big)^{1/\theta_2} \\ &\quad\times\Big[\int_\Omega e^{\left(\theta_3\delta\|v\|^{\frac{n}{n-1}}(\frac{|v|}{\|v\|})^{\frac{n}{n-1}}\right)}dx\Big]^{1/\theta_3}\Big(\int_\Omega |v|^{\theta_4 z}\,dx\Big)^{1/\theta_4}, \end{aligned}$$

for some constant $C_{15} > 0$. By choosing δ sufficiently small such that

$$\theta_1\delta\|u\|^{\frac{n}{n-1}}_{W_0^{1,n}(\Omega)} < \alpha_n \quad\text{and}\quad \theta_3\delta\|v\|^{\frac{n}{n-1}}_{W_0^{1,n}(\Omega)} < \alpha_n,$$

and taking into account that $W_0^{1,n}(\Omega)$ is continuously embedded in $L^\zeta(\Omega)$ for every $\zeta \ge 1$, one has

$$J(u,v) \le C_{16}(\varepsilon\|(u,v)\|^n + \|(u,v)\|^{2z}),$$

for some constant $C_{16} > 0$. Therefore, in view of (A2), we obtain

$$\frac{J(u,v)}{\Phi(u,v)} \le C_{16}\frac{\varepsilon\|(u,v)\|^n + \|(u,v)\|^{2z}}{\|(u,v\|^n}.$$

Since $z > n$, claim (4.7) immediately follows.

Let us prove now that

$$\limsup_{\|(u,v)\|\to\infty} \frac{J(u,v)}{\Phi(u,v)} \le 0. \tag{4.8}$$

By (4.4),

$$\limsup_{|(t,s)|\to\infty} \sup_{x\in\Omega} \frac{F(x,t,s)}{|t|^n + |s|^n} \le 0,$$

and so for every $\varepsilon > 0$, there exists some positive ρ such that, for every $x \in \Omega$ and $|(t,s)| > \rho$,
$$F(x,t,s) \le \varepsilon(|t|^n + |s|^n).$$
From condition (H1), there exists some constant $C_{17} > 0$ such that, for every $x \in \Omega$,
$$\sup_{|(t,s)|\le\rho} (|F_u(x,t,s)|, |F_v(x,t,s)|) \le C_{17}.$$
Then, for every $x \in \Omega$ and $t, s \in \mathbb{R}$,
$$F(x,t,s) \le C_{17}\rho + \varepsilon(|t|^n + |s|^n)$$
and so
$$J(u,v) \le C_{17}\rho \operatorname{meas}(\Omega) + \varepsilon(\int_\Omega |u|^n + |v|^n \, dx).$$
Since $W_0^{1,n}(\Omega)$ is continuously embedded into $L^n(\Omega)$, we obtain
$$\frac{J(u,v)}{\Phi(u,v)} \le n\frac{C_{17}\rho \operatorname{meas}(\Omega)}{\|u\|^n} + \varepsilon n C_{18},$$
for some constant $C_{18} > 0$. Hence, claim (4.8) follows at once. In view of (4.7) and (4.8), we obtain
$$\max\Big\{\limsup_{\|(u,v)\|\to+\infty} \frac{J(u,v)}{\Phi(u,v)}, \limsup_{(u,v)\to(0,0)} \frac{J(u,v)}{\Phi(u,v)}\Big\} \le 0.$$
Now all the assumptions of Theorem 2.1 are satisfied with $\alpha = 0$ and $\beta = \frac{1}{\theta}$, where θ is as in (4.5) and choose $[a,b] \subseteq]\theta, +\infty[$. Moreover, since the function $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a measurable in Ω and C^1 in $\mathbb{R} \times \mathbb{R}$ satisfying the condition (G1), then by Lemma 4.7 the functional $\Psi(u,v) = \int_\Omega G(x,u,v)\,dx$ is well defined and continuously Gâteaux differentiable in W, with compact derivative, and one has
$$\Psi'(u,v)(\varphi,\psi) = \int_\Omega G_u(x,u,v)\varphi + G_v(x,u,v)\psi \, dx$$
for all $(u,v), (\varphi,\psi) \in W$. Then, in view of Proposition 4.6 and Theorem 2.1 there exists $r > 0$ such that for every $\lambda \in [a,b]$, it is possible to find $\sigma > 0$ verifying the following condition: for each $\mu \in [0,\sigma]$, the functional $\Phi - \lambda J - \mu\Psi$ has at least three critical points, which are precisely weak solutions of problem (4.1) whose norms are less than r. The proof is complete. □

Example. Let
$$F(x,u,v) = \frac{\lambda}{p}|u|^{\alpha+1}|v|^{\beta+1} + (1 - \chi(u,v)) \exp\Big(\frac{\sigma(|u|^n + |v|^n)^{\frac{1}{n-1}}}{Log(|u| + |v| + 2)}\Big)$$
where $\chi \in C^1(\mathbb{R}^2, [0,1]), \chi \equiv 1$ on some ball $B(0,r) \subset \mathbb{R}^2$ with $r > 0$, and $\chi \equiv 0$ on $\mathbb{R}^2 \backslash B(0, r+1)$. Thus, it follows immediately that (H1), (H2) and (4.3) are satisfied. Then problem (4.1) has a three nontrivial weak solutions provided that $\lambda < \lambda_1$.

Acknowledgements. The second author gratefully acknowledges the financial support of Deutscher Akademischer Austausch Dienst.

References

[1] G. Anello, G. Cordaro; An existence theorem for the Neumann problem involving the p-Laplacian. *J. Convex. Anal.*, (10)1: 185–198, 2003.

[2] G. Anello G. Cordaro; Existence of solutions of the Neumann problem involving the p-Laplacian via variational principle of Ricceri. *Arch. Math. (Basel)*, 79: 274–287, 2002.

[3] S. El Manouni; A study of nonlinear problems for the p-Laplacian in $\mathbb{R}^n$ via Ricceri's principle. *Nonlinear Anal.* 74, No. 5-6: 4496–4502, 2011.

[4] S. El Manouni, F. Faraci; Multiplicity results for some elliptic problems of n-Laplace type. *Taiwanese J. Math.* 16, No. 3: 901–911, 2012.

[5] S. El Manouni, M. Kbiri Alaoui; A result on elliptic systems with Neumann conditions via Ricceri's three critical points theorem. *Nonlinear Anal.* 71, No. 5-6: 2343–2348, 2009.

[6] D. G. de Figueiredo, O. H. Miyagaki, B. Ruf; Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. *Calc. Var. PDE.*, 3: 139–153, 1995.

[7] N. Hirano; Multiple solutions for quasilinear elliptic equations. *Nonlinear Anal.*, 15, 625–638, 1990.

[8] Marcos B. do Ó, João; Semilinear Dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range. *Differential and Integral Equations*, 9, 5: 967–979, 1996.

[9] G. Molica Bisci, D. Repovš; Nonlinear Algebraic Systems with discontinuous terms. *J. Math. Anal. Appl.*, 398, 846–856, 2013.

[10] G. Molica Bisci; Sequence of weak solutions for fractional equations. *Math. Res. Lett.*, 21, 241–253, 2014.

[11] J. Moser; A sharp form of an equality by N. Trudinger. *Ind. Univ. Math. J.*, 20, 1077–1092, 1971.

[12] B. Ricceri; A further three critical points theorem. *Nonlinear Anal.*, 71, 9: 4151–4157, 2009.

[13] B. Ricceri; A three critical points theorem revisited. *Nonlinear Anal.*, 70, 9: 3084–3089, 2009.

[14] B. Ricceri; A note on the Neumann problem. *Complex Var. Elliptic Equ.*, 55, 5-6: 593–599, 2010.

[15] B. Ricceri; On a three critical points theorem. *Arch. Math. (Basel)*, 75: 220–226, 2000.

[16] J. Simon; Régularité de la solution d'une equation non linéaire dans $\mathbb{R}^N$. LMN 665, P. Benilan ed., Berlin-Heidelberg-New York, 1978.

[17] F. De Thelin; Première valeur propre d'un système elliptique non linéaire. *C. R. Acad. Sci. Paris*, 311, Sèrie 1: 603–606, 1990.

[18] N. S. Trudinger; On imbedding into Orlicz spaces and some applications. *J. Math. Mech.*, 17, 473–483, 1967.

[19] P. Ubilla; Alguns resultados de multiplicidade de solucoes pam equacoes eliticas quasilineares. *Doctoral dissertation*, (Unicamp). 1992.

[20] E. Zeidler; Nonlinear functional analysis and its applications. *Vol. II/B. Berlin, Heidelberg, New York*, 1978.

Abderrahim El Attar
University of Fez, FST, Department of Mathematics, Fez, Morocco
E-mail address: abderrahimelattar@hotmail.com

Said El Manouni
Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Faculty of Sciences, Department of Mathematics, P.O. Box 90950 Riyadh 11623, Saudi Arabia
E-mail address: samanouni@imamu.edu.sa

Omar Sidki
University of Fez, FST, Department of Mathematics, Fez, Morocco
E-mail address: osidki@hotmail.com