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SPIKE-LAYER SOLUTIONS TO NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS WITH ALMOST OPTIMAL NONLINEARITIES

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Abstract. In this article, we are interested in singularly perturbed nonlinear elliptic problems involving a fractional Laplacian. Under a class of nonlinearity which is believed to be almost optimal, we construct a positive solution which exhibits multiple spikes near any given local minimum components of an exterior potential of the problem.

1. Introduction

Let $N \geq 2$ and $(-\Delta)^s, 0 < s < 1$ be the usual fractional Laplace operator on $\mathbb{R}^N$. We study the singularly perturbed elliptic problem of fractional order

$$\begin{gathered} \varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \\ \lim_{x\to\infty} u(x) = 0, \end{gathered} \tag{1.1}$$

which is derived from the nonlinear fractional Schrödinger equation

$$i\hbar\psi_t - \hbar^2(-\Delta)^s\psi - V(x)\psi + f(\psi) = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

where $\hbar$ is the Plank constant, i is the imaginary unit. Equation (1.2) can be understood as a nonlinear counterpart of the fractional Schrödinger equations formulated by Laskin who defined fractional path integrals over the paths of the Lévy flights and found fractional generalization of the Schrödinger equations in [19, 20]. We refer to [21] for more physical background. We are concerned with standing waves of (1.2), solutions of the form

$$\psi(x,t) = e^{-i\omega t/\hbar}u(x). \tag{1.3}$$

By assuming that $f : \mathbb{C} \to \mathbb{C}$ is continuous such that

$$f(e^{i\theta}u) = e^{i\theta}f(u) \quad \text{for } u \in \mathbb{R}$$

and inserting the ansatz (1.3) to (1.2), we obtain

$$\hbar^2(-\Delta)^s u + (V(x) - \omega)u = f(u) \quad \text{in } x \in \mathbb{R}^N.$$

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Then we set $\hbar = \varepsilon^s$ and redefine $V(x) - \omega$ by $V(x)$ to derive (1.1). Since the plank constant $\hbar > 0$ is a very small physical quantity, we may assume $\varepsilon > 0$ is a small parameter. Throughout this article, we assume

- (F1') $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{t\to 0} f(t)/t = 0$;
- (V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$.

When $s = 1$, the equation (1.1) becomes a local elliptic equation

$$\begin{gathered} -\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N \\ \lim_{x\to\infty} u(x) = 0 \end{gathered} \tag{1.4}$$

to which a great deal of work has been devoted during the last three decades. The main concern is to construct a family of positive solution which exhibits spikes near some critical points of V. Floer and Weinstein proved in their pioneering paper [16] that if $N = 1$ and $f(u) = u^3$ then for small $\varepsilon > 0$, there exists a positive solution of (1.4) which develops a spike near any given non-degenerate critical point of V. In [25, 26], Oh generalized this result by constructing a positive solution which develops multiple spikes near any given finite set of non-degenerate critical points of V when $N \geq 1$ and $f(u) = u^p$, where $p \in (1, \infty)$ if $N = 1, 2$ and $p \in (1, (N + 2)/(N - 2))$ if $N \geq 3$. We also refer to [18] in which Kang and Wei proved that for small $\varepsilon > 0$, there exists a positive solution which is clustered near a strict local maximum point of V, i.e., a solution which develops k-spikes near a strict local maximum point of V for any given $k \in \mathbb{N}$.

The results mentioned above make use of Lyapunov-Schmidt reduction method, which is a very powerful tool especially when we construct highly unstable solutions. This method essentially requires uniqueness and non-degeneracy of a positive radial solution of the autonomous equation

$$\begin{gathered} -\Delta v + V(x_0)v = f(v) \quad \text{in } \mathbb{R}^N,\ x_0 \text{ is fixed} \\ \lim_{x\to\infty} v(x) = 0\,. \end{gathered} \tag{1.5}$$

Equation (1.5) is called the limit equation of (1.4) because if u is a solution of (1.5), by defining $v(x) := u(\varepsilon x + x_0)$, v satisfies

$$-\Delta v + V(\varepsilon x + x_0)v = f(v),$$

which approaches as $\varepsilon \to 0$ to (1.5). Then, one can apply the Lyapunov-Schmidt reduction method to search a solution of (1.4) near the set of positive solutions of (1.5) with x_0 which is a critical point of V.

Unfortunately, the uniqueness and non-degeneracy of a positive radial solution of (1.5) are known for a very restrictive class of f, for example, $f(u) = u^p$ while the existence of a radial positive solution of (1.5) can be obtained for a quite general class of nonlinearity f. Using variational approaches, Berestycki and Lions proved in [1] that there exists a positive radial least energy solution of (1.5) if f satisfies the conditions (F1') and

- (F2') $\limsup_{t\to\infty} |f(t)/t^p| < C$ for some $C > 0$ and $p \in (1, (n + 2)/(n - 2))$,
- (F3') there is $T > 0$ such that $F(T) > V(x_0)T^2/2$ where $F(t) = \int_0^t f(s)\,ds$,

which are believed to be almost optimal for the existence of solutions of (1.5). Therefore one can ask the following question:

Question 1. Under conditions (F1')–(F3'), does (1.4) admit spike-layer solutions described in [16, 25, 26, 18]?

In this regard, we refer to a series of works [11, 12, 13] in which Del pino and Felmer had developed interesting variational techniques to construct spike layer solutions of (1.4), concentrating near any given topologically nontrivial critical point of V for a wide class of nonlinearity f. For example, it is assumed in [13] that f satisfies

(1) $f \in C^1$ and $f'(t)t$ is locally Lipschitz on $[0, \infty)$,
(2) $\limsup_{t\to\infty} f'(t)/s^{p-1} < \infty$ for some $p \in (1, (n+2)/(n-2))$,
(3) $f'(t)t \le Cf(t)$ for some $C > 0$ and all $s \in (0, 1)$,
(4) $0 < qf(t) \le f'(t)t$ for some $q > 1$ and all $s > 0$.

Note that neither uniqueness nor non-degeneracy of a positive solution is assumed in (1)–(4) although these are still more restrictive than the optimal conditions (F1')–(F3').

After the works [11, 12, 13], much effort has been made to positively answer Question 1. Byeon and Jeanjean proved in [3, 4] that under the conditions (F1), (F2') and (F3'), if $\varepsilon > 0$ is small, there exists a positive solution which exhibits multiple spikes near any given finite family of local minimum components of V. Later, Byeon and Tanaka proved in [5] the existence of a spike solution near a given structurally stable critical point of V by further assuming $f \in C^1$. Very recently, Byeon and Tanaka proved in [6] that for small $\varepsilon > 0$, there is a solution clustered near a given local maximum component of V by assuming (F2'), (F3'), $f \in C^1$ and $f = o(t^q)$ as $t \to 0$ for some $q > 1$.

Now, we turn our attention to the case $s \in (0, 1)$. For the power type nonlinearity $f(u) = u^p$, Dávila, Del pino and Wei [10] obtained the existence of positive solutions of (1.1), exhibiting multiple spikes near given topologically nontrivial critical points of V or clustered near a given local maximum point of V by applying the Lyapunov-Schmidt reduction method. In this paper, we are interested in extending the result by Byeon and Jeanjean to the nonlocal equation (1.1). This is a first step to answer Question 1 for nonlocal equation (1.1). The main idea is adopted from [4] but we are not going to introduce the penalization term Q_ε in [3, 4] since it causes some technical complications. Instead, we show a kind of intersection lemma using degree theory as in [7, 22]. This makes the point of whole proof much clearer.

We define a concept of classical solution to (1.1) by following [15]. Recall that $(-\Delta)^s$ is defined as

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u)), \tag{1.6}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transform respectively. If u is sufficiently smooth, it is known that (see [14]) definition (1.6) is equivalent to

$$(-\Delta)^s u := -\frac{1}{2}C(N, s)\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\,dy, \tag{1.7}$$

where

$$C(N, s) = \Big(\int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}}\,d\xi\Big)^{-1}.$$

By a classical solution of (1.1), we mean a continuous function that (1.7) is well defined for all $x \in \mathbb{R}^N$ and satisfies (1.1) in pointwise sense.

Next, we give precise descriptions for assumptions about V and f. We assume that

(V2) there are k local minimum components of V for some $k \geq 1$, i.e., there are k bounded open sets $O_1, \dots, O_k \subset \mathbb{R}^N$ such that $m_i := \inf_{x \in O_i} V(x) < \min_{x \in \partial O_i} V(x), i = 1, \dots, k$.

Denote $\mathcal{M}_i := \{x \in O_i : V(x) = m_i\}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

(F1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(t) = o(t)$ as $t \to 0$;

(F2) $\limsup_{t \to \infty} |f(t)/t^p| < C$ for some $C > 0$ and $p \in (1, (n+2s)/(n-2s))$;

(F3) there is $T > 0$ such that $F(T) > mT^2/2$ where $F(t) = \int_0^t f(s)\, ds$ and $m = \max_{i=1,\dots,k} m_i$.

Now, we state the main theorem.

Theorem 1.1. *Let $N \geq 2$. Fix arbitrary $s \in (0,1)$. We assume $V : \mathbb{R}^N \to \mathbb{R}$ is a C^1 function satisfying* (V1)–(V2) *and f satisfies* (F1)–(F3). *Then, for any small $\varepsilon > 0$, there exists a positive classical solution u_ε of* (1.1) *which exhibits k spikes near each $\mathcal{M}_i$. More precisely, u_ε develops k local maximum points $y_{\varepsilon,i} \in O_i$ satisfying*

(i) $\operatorname{dist}(y_{\varepsilon,i}, \mathcal{M}_i) \to 0$ *up to a subsequence;*

(ii) $u_\varepsilon(y_{\varepsilon,i}) > c$ *for some constant $c > 0$ independent of $\varepsilon > 0$;*

(iii)
$$u_\varepsilon(x) \leq C \sum_{i=1}^{k} \frac{\varepsilon^{N+2s}}{(\varepsilon^2 + |x - y_{\varepsilon,i}|^2)^{\frac{N+2s}{2}}}.$$

In this article, we only pay attention to the singularly perturbed setting. For readers interested in non-perturbed setting, for example, equations of the form
$$(-\Delta)^s u + V(x)u = \lambda f(x,u) \quad \text{in } \mathbb{R}^n,$$
we refer to [24] in which at least two nontrivial solutions are constructed under a class of exterior potential V and nonlinearity f with sublinear growth.

We close this section by introducing some notation:

- $B_r^N(x)$: N-dim Euclidean ball with center x and radius r.
- $C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$: the set of infinitely differentiable functions with compact support in $\overline{\mathbb{R}_+^{N+1}}$.
- $C^\sigma(\mathbb{R}^N), n \in \mathbb{N}, \sigma \geq 0$: the set of $[\sigma]$ times differentiable functions whose $[\sigma]$th derivatives are Hölder continuous with exponent $\sigma - [\sigma]$.
- C : positive generic constant which can vary from line to line.
- $L^p(\Omega)$: the set of p-th integrable functions on Ω.
- $L^p_W(\Omega)$: the set of p-th weighted integrable functions on Ω with weight W.

2. Preliminaries

By the change of variable $x \to x/\varepsilon$, (1.1) is equivalent to
$$\begin{gathered} (-\Delta)^s u + V_\varepsilon(x)u = f(u) \quad \text{in } \mathbb{R}^N \\ \lim_{x \to \infty} u(x) = 0, \end{gathered} \tag{2.1}$$
where $V_\varepsilon(x) := V(\varepsilon x)$.

Let $D^s(\mathbb{R}^N)$ denote the homogeneous fractional Sobolev space, defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm
$$\|u\|_{D^s} := \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\, dx \Big)^{1/2}$$

and $H^s(\mathbb{R}^N)$ denote the standard fractional Sobolev space, defined as the set of $u \in D^s(\mathbb{R}^N)$ satisfying $u \in L^2(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s} := \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + u^2\,dx\Big)^{1/2}.$$

Let $H^s_{V_\varepsilon}(\mathbb{R}^N)$ be the set of $u \in D^s(\mathbb{R}^N)$ satisfying

$$\|u\|_{H^s_{V_\varepsilon}} := \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V_\varepsilon(x)u^2\,dx\Big)^{1/2} < \infty.$$

Then we have an obvious embedding $H^s_{V_\varepsilon}(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$. For any $u \in H^s(\mathbb{R}^N)$, it is well known that the following fractional version of Sobolev inequality holds

$$\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \le C\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}. \tag{2.2}$$

We say $u \in H^s_{V_\varepsilon}(\mathbb{R}^N)$ is a weak solution to (2.1) if u satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\,dx + \int_{\mathbb{R}^N} V_\varepsilon(x)uv\,dx = \int_{\mathbb{R}^N} f(u)v\,dx$$

for all $v \in H^s_{V_\varepsilon}(\mathbb{R}^N)$. Making use of the definition of $H^s_{V_\varepsilon}$, Sobolev inequality (2.2) and the conditions (F1)–(F2), one can see every integral in the above weak formulation is well defined.

2.1. **Extended problems.** Now, we introduce a local extended problem

$$\begin{gathered}
\operatorname{div}(t^{1-2s}\nabla U) = 0 \quad \text{in } \mathbb{R}^{N+1}_+\\
\lim_{t\to 0} -t^{1-2s}\partial_t U = -V_\varepsilon(x)U(x,0) + f(U(x,0)) \quad \text{in } \in \mathbb{R}^N\\
\lim_{(x,t)\to\infty} U(x,t) = 0.
\end{gathered} \tag{2.3}$$

It is shown by Caffarelli and Silvestre [8] that, up to a normalization constant, the equation (2.3) is equivalent to (2.1). Let $D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ denote the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|U\|_{D^1(t^{1-2s},\mathbb{R}^{N+1}_+)} = \int_{\mathbb{R}^{N+1}_+} |\nabla U(x,t)|^2 t^{1-2s}\,dx\,dt.$$

It is known that (see [17]) for any $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$, its trace $U(x,0)$ belongs to $D^s(\mathbb{R}^N)$ and the trace map is continuous as follows:

$$\|U(\cdot,0)\|_{D^s(\mathbb{R}^N)} \le C\|U\|_{D^1(t^{1-2s},\mathbb{R}^{N+1}_+)}. \tag{2.4}$$

Then the trace Sobolev inequality

$$\|U(x,0)\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \le C\|U\|_{D^1(t^{1-2s},\mathbb{R}^{N+1}_+)} \tag{2.5}$$

is derived from (2.4) and (2.2). We define function spaces H_0 by the set of all $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ satisfying

$$\|U\|_0^2 := \int_{\mathbb{R}^{N+1}_+} |\nabla U(x,t)|^2 t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} U(x,0)^2\,dx < \infty$$

and H_ε by the set of all $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ satisfying

$$\|U\|_\varepsilon^2 := \int_{\mathbb{R}^{N+1}_+} |\nabla U(x,t)|^2 t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} V_\varepsilon(x)U(x,0)^2\,dx < \infty.$$

We say $U \in H_\varepsilon$ is a weak solution of (2.3) if U satisfies

$$\int_{\mathbb{R}^{N+1}_+} \nabla U \cdot \nabla V t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} V(\varepsilon x)U(x,0)V(x,0) - f(U(x,0))V(x,0)\,dx$$

for every $V \in H_\varepsilon$. It is well known that (See [8] and [17]) if $W_\varepsilon \in H_\varepsilon$ is a weak solution of (2.3), then $W_\varepsilon(\cdot,0) \in H^s_{V_\varepsilon}(\mathbb{R}^N)$ is a weak solution of (2.1). Also, it is standard to show that $U \in H_\varepsilon$ is a weak solution of (2.3) if and only if it is a critical point of the following C^1 functional on H_ε,

$$\Gamma_\varepsilon(U) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} \frac{1}{2}V_\varepsilon(x)U(x,0)^2 - F(U(x,0))\,dx,$$

where $F(s) = \int_0^s f(\sigma)\,d\sigma$.

2.2. **Elliptic estimates.** Let $\mathcal{D}_r := B^N_r(0) \times (0,r)$. Consider the nonlinear Neumann boundary value problem

$$\begin{gathered} -\operatorname{div}(t^{1-2s}\nabla U) = 0 \quad \text{in } \mathcal{D}_1 \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t U(x,0) = a(x)U(x,0) + g(x) \quad \text{in } B^N_1(0) \end{gathered} \tag{2.6}$$

Let $L^2(t^{1-2s},\mathcal{D}_r)$ be a weighted L^2 space on $\mathcal{D}_r$ with the weight t^{1-2s}. Also, $H^1(t^{1-2s},\mathcal{D}_r)$ denotes the corresponding weighted Sobolev space. Here we collect some standard elliptic estimates for (2.6).

Proposition 2.1 (De Giorgi-Nash-Moser type estimate, [17, Proposition 2.4])**.** *Suppose $a, g \in L^p(B^N_1(0))$ for some $p > N/(2s)$.*

(i) *Let $U \in H^1(t^{1-2s},\mathcal{D}_1)$ be a weak solution to* (2.6)*. Then, $U \in L^\infty(\mathcal{D}_{1/2})$ and there is a constant $C > 0$ depending only on N, s, p and $\|a\|_{L^p(B^N_1(0))}$ such that*
$$\sup_{\mathcal{D}_{1/2}} U \le C\Big(\|U\|_{L^2(t^{1-2s},\mathcal{D}_1)} + \|g\|_{L^p(B^N_1(0))}\Big);$$

(ii) *Let $U \in H^1(t^{1-2s},\mathcal{D}_1)$ be a weak solution to* (2.6)*. Then, there is $\alpha \in (0,1)$ depending only on N, s, p such that $U \in C^\alpha(\overline{\mathcal{D}_{1/2}})$ and there is a constant $C > 0$ depending only on N, s, p and $\|a^+\|_{L^p(B^N_1(0))}$ such that*
$$\|U\|_{C^\alpha(\overline{\mathcal{D}_{1/2}})} \le C\Big(\|U^+\|_{L^\infty(\mathcal{D}_1)} + \|g\|_{L^p(B^N_1(0))}\Big).$$

Proposition 2.2 (Schauder estimate [17, Theorem 2.3])**.** *Suppose a, g belong to $C^\sigma(B^N_1(0))$ for some $\sigma \notin \mathbb{N}$. Let $U \in H^1(t^{1-2s},\mathcal{D}_1)$ be a weak solution to* (2.6)*. If $2s+\sigma$ is not an integer, then $U(\cdot,0) \in C^{2s+\sigma}(B^N_{1/2}(0))$ and there is a constant $C > 0$ depending only on N, s, σ and $\|a\|_{C^\sigma(B^N_1(0))}$ such that*
$$\|U(\cdot,0)\|_{C^{2s+\alpha}(B^N_{1/2}(0))} \le C(\|U\|_{L^\infty(\mathcal{D}_1)} + \|g\|_{C^\sigma(B^N_1(0))})$$

2.3. **Regularity of weak solutions.** We assume $f(t) = 0$ for any $t \le 0$ by redefining f if necessary. Then, we see that every weak solution W of (2.3) is nonnegative by testing W^-, the negative part of W to the equation (2.1). Actually we can prove W is classical and positive everywhere.

Proposition 2.3. *Let W_ε be a weak solution of* (2.3)*. Then its trace $W_\varepsilon(\cdot,0)$ is a classical solution of* (2.1) *and positive.*

Proof. Since $W_\varepsilon \in H_\varepsilon \subset H_0$ solves

$$\begin{gathered}\operatorname{div}(t^{1-2s}\nabla U) = 0 \quad \text{in } \mathbb{R}^{N+1}_+ \\ \lim_{t\to 0} U(x,t) = W_\varepsilon(x,0) \quad \text{in } \in \mathbb{R}^N,\end{gathered}$$

it is given by

$$W_\varepsilon(x,t) = \beta(N,s)\int_{\mathbb{R}^N} \frac{t^{2s}}{(|x-y|^2+t^2)^{\frac{N+2s}{2}}} W_\varepsilon(x,0)\,dx,$$

and consequently $W_\varepsilon \in L^2(t^{1-2s},\mathcal{D}_1)$(see [17]). Thus the De Giorgi-Nash-Moser type estimate (Proposition 2.1) applies to deduce $W_\varepsilon(\cdot,0)$ is Hölder continuous. Then, applying the Schauder estimate (Proposition 2.2) iteratively, we see that $W_\varepsilon(\cdot,0) \in C^{2+\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ if $s > 1/2$ and $W_\varepsilon(\cdot,0) \in C^{1+\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ if $s \le 1/2$. We refer to [15] to inform that these are enough regularities to makes weak solutions classical. Suppose that there is $x_0 \in \mathbb{R}^N$ such that $W_\varepsilon(x_0,0) = 0$. Since $W_\varepsilon(x_0,0) \ge 0$, x_0 is a global minimum of W_ε. But this contradicts with the expression

$$\begin{aligned}&-\frac{1}{2}C(N,s)\int_{\mathbb{R}^N} \frac{W_\varepsilon(x_0+y,0)+W_\varepsilon(x_0-y,0)-2W_\varepsilon(x_0,0)}{|y|^{N+2s}}\,dy \\ &= -W_\varepsilon(x_0,0) + f(W_\varepsilon(x_0,0))\end{aligned}$$

because the left-hand side is positive but the right-hand side vanishes. ☐

2.4. **limit equations.** As $\varepsilon \to 0$, we obtain the following limit equation of (2.3)

$$\begin{gathered}\operatorname{div}(t^{1-2s}\nabla U) = 0 \quad \text{in } \mathbb{R}^{N+1}_+ \\ \lim_{t\to 0} -t^{1-2s}\partial_t U = -aU(x,0) + f(U(x,0)) \quad \text{in } \in \mathbb{R}^N \\ \lim_{(x,t)\to\infty} U(x,t) = 0,\end{gathered} \tag{2.7}$$

where a is a real constant.

As before, a weak solution of (2.7) is defined by a function $W \in H_0$ satisfying

$$\int_{\mathbb{R}^{N+1}_+} \nabla W\cdot\nabla V t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} aW(x,0)V(x,0) - f(W(x,0))V(x,0)\,dx$$

for every $V \in H_0$, and it is a critical point of the functional

$$\Gamma_a(U) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} \frac{1}{2}aU(x,0)^2 - F(U(x,0))\,dx.$$

It is also true that if $W \in H_0$ is a weak solution of (2.7), then $W(\cdot,0) \in H^s(\mathbb{R}^N)$ is a weak solution of

$$\begin{gathered}(-\Delta)^s u + au = f(u) \quad \text{in } \mathbb{R}^N \\ \lim_{x\to\infty} u(x) = 0.\end{gathered} \tag{2.8}$$

In [9], the following Pohozaev identity is proved for $a > 0$. We note that the argument employed in [9], originally developed in [1], does not require the positiveness of a. Thus we have the following result.

Proposition 2.4. *For any $a \in \mathbb{R}$, if $W \in H_0$ is a weak solution of* (2.7)*, it satisfies the following integral identity, which is called the Pohozaev identity*

$$\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla W|^2 t^{1-2s}\,dx\,dt + N\int_{\mathbb{R}^N} \frac{a}{2}W(x,0)^2 - F(W(x,0))\,dx = 0. \quad (2.9)$$

We say $L \in H_0$ is a least energy solution of (2.7) if L is a weak solution of (2.7) and satisfies $\Gamma_a(L) \le \Gamma_a(W)$ whenever W is a nontrivial weak solution of (2.7).

Proposition 2.5. *Let $s \in (0,1)$ and f satisfies $(F1)$–$(F3)$. Then, for any $a > 0$ there exists a positive least energy solution to* (2.7)*. Moreover, every least energy solution L satisfies the following properties:*

(i) *$L(\cdot,0)$ is a classical solution of* (2.8) *and positive.*
(ii) *$\Gamma_a(L) \le \Gamma_a(V)$ whenever $V \in H_0$ is nontrivial and satisfies the Pohozaev identity* (2.9)*.*
(iii) *there is a constant C depending only on N, s, f and $\|L\|_{H_0}$ such that*

$$|L(x,0)| \le C\frac{1}{|x|^{N+2s}} \quad \textit{for } |x| > 1;$$

Proof. The existence of a solution W of (2.7) satisfying

$$\Gamma_a(W) = c_a := \min_{\gamma\in T}\max_{\sigma\in[0,1]} \Gamma_a(\gamma(\sigma)), \quad (2.10)$$

where $T = \{\gamma \in C([0,1], H_0 : \gamma(0) = 0, \gamma(1) < 0\}$, is obtained in [9] but it is not clear at this point whether or not it is of least energy and it satisfies (ii). Here, we reconstruct a solution by following the approach employed in [2] that finding a minimizer of a minimization problem below

$$\min_{U\in H_0\setminus\{0\}} \Big\{\Gamma_a(U) : \frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s}\,dx\,dt + N\int_{\mathbb{R}^N} \frac{a}{2}U(x,0)^2 - F(U(x,0))\,dx = 0\Big\}.$$

Using the Pohozaev identity (2.9), it is easy to check that there is a minimizer $L \in H_0$, which is a solution of (2.7). For detail, we refer to [2]. From the definition it is clear that *(ii)* holds. Since every weak solution satisfies (2.9), we see L is a least energy solution. Applying the same argument in the proof of Proposition 2.3, we also obtain (i).

Now, we prove (iii). Applying (i) of Proposition 2.1, we see that $\|L(\cdot,0)\|_{L^\infty(\mathbb{R}^N)}$ is bounded and $\|L(\cdot,0)\|_{L^\infty(B^N_1(x))} \to 0$ as $|x| \to \infty$ so that $L(\cdot,0)$ satisfies

$$(-\Delta)^s L(\cdot,0) + \frac{a}{2}L(\cdot,0) \le 0 \quad \text{on } \mathbb{R}^N \setminus\ B^N_R(0)$$

by taking large $R > 0$. It is proved in [15] that there is a positive function $K \in H^s(\mathbb{R}^N)$ satisfying

$$(-\Delta)^s K(x) + \frac{a}{2}K(x) = 0 \quad \text{in } \mathbb{R}^N \setminus B^N_1(0),$$
$$K(x) \le C\frac{1}{|x|^{N+2s}} \quad \text{on } \mathbb{R}^N \setminus B^N_1(0).$$

Then, the comparison principle (we refer to [15]) applies to see

$$L(x,0) \le CK(x) \le C\frac{1}{|x|^{N+2s}} \quad \text{on } \mathbb{R}^N \setminus B^N_R(0)$$

for some $C > 0$. $\square$

Let S_a denote the set of least energy solutions L of (2.7) such that $L(x, 0)$ attains its maximum at $0 \in \mathbb{R}^N$. The compactness of S_a can be proved by following the argument in the proof of [3, Proposition 1] with no changes. We omit the details.

Proposition 2.6. *For any $a > 0$, the set S_a is compact in H_0.*

We denote $E_a := \Gamma_a(L)$, where $L \in S_a$.

Proposition 2.7. *Suppose that $0 < a \le b$. Then $E_a \le E_b$.*

Proof. Recall $c_a := \min_{\gamma\in T} \max_{\sigma\in[0,1]} \Gamma_a(\gamma(\sigma))$. We claim that $E_a = c_a$. Then the proposition follows from the definition of c_a. Let L be a least energy solution of (2.7). Using the Pohozaev identity (2.9), we see that

$$\Gamma_a(L(\cdot/\sigma, \cdot/\sigma)) = \Big(\frac{\sigma^{N-2s}}{2} - \frac{N-2s}{2N}\sigma^N\Big) \int_{\mathbb{R}^{N+1}_+} |\nabla L|^2 t^{1-2s}\, dx\, dt \tag{2.11}$$

so that there is $\sigma_0 > 1$ such that $\Gamma_a(L(\cdot/\sigma_0, \cdot/\sigma_0)) < 0$. Define a path $\gamma_0(\sigma)$: $[0, 1] \to H_0$ by $\gamma_0(\sigma) = L(\cdot/(\sigma_0\sigma), \cdot/(\sigma_0\sigma))$. Then γ_0 is continuous such that $\gamma_0(0) = 0$ and $\gamma_0(1) < 0$. We also see from (2.11) that

$$\Gamma_a(L) = \max_{\sigma\in[0,1]} \Gamma_a(\gamma_0(\sigma)),$$

which shows $c_a \le E_a$.

Conversely, take arbitrary path $\gamma \in T$. It is standard to see from the mountain pass geometry of Γ_a that we may assume $\gamma(1) = \gamma_0(1)$. For any $U \in H_0 \setminus \{0\}$, define a map

$$\mathcal{P}_a(U) := \sqrt{\Big(\frac{N \int_{\mathbb{R}^N} F(U(x, 0)) - \frac{a}{2} U(x, 0)^2\, dx}{\frac{N-2s}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s}\, dx\, dt}\Big)_+}.$$

Observe $\mathcal{P}_a(\gamma_0(\sigma)) = \sigma_0\sigma$. For proving $E_a \le c_a$, it is sufficient to show there is $\sigma_1 > 0$ such that $\mathcal{P}_a(\gamma(\sigma_1)) = 1$, due to the (ii) of Proposition 2.5. Define a homotopy $H_a : [0, 1] \times [0, 1]$ by

$$H_a(\sigma, t) = (1 - t)\gamma_0(\sigma) + t\gamma(\sigma).$$

Since $\deg(\mathcal{P}_a(\gamma_0(\sigma)), [0, 1], 1) = \deg(\sigma_0\mathrm{Id}, [0, 1], 1) \neq 0$ and $\gamma_0(0), \gamma_0(1) \neq 1$, we are done because the homotopy fixes the boundary. Later, we shall apply a similar argument when we prove Proposition 4.2. $\square$

3. Local concentration compactness result

Let $10\delta := \min\{\min_i \operatorname{dist}(\mathcal{M}_i, \partial O_i), \min_{i\neq j} \operatorname{dist}(O_i, O_j)\}$ and $\varphi : \mathbb{R}^N \to [0, 1]$ be a smooth cut-off function satisfying

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \le \delta \\ 0 & \text{for } |x| \ge 2\delta. \end{cases}$$

We denote $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$. For any set $A \subset \mathbb{R}^N$, we mean by A^δ the δ-neighbor hood of A, i.e., $\{x \in \mathbb{R}^N : \operatorname{dist}(x, A) \le \delta\}$. Then one can define a set of approximation solutions by

$$N_\varepsilon(\rho) = \Big\{ \sum_{i=1}^k \varphi_\varepsilon(\cdot - x_i/\varepsilon) L_i(\cdot - x_i/\varepsilon, t) + \omega :$$

$$x_i \in \mathcal{M}_i^\delta,\ L_i \in S_{m_i},\ \omega \in H_\varepsilon,\ \|\omega\|_\varepsilon \le \rho\Big\}$$

In this section, we prove a concentration compactness type result which gives a decomposition of so-called ε-PS sequences when they belong to $N_\varepsilon(\rho)$. For a sequence $\{\varepsilon_n\} > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$, we say a sequence $\{U_n\} \in H_{\varepsilon_n}$ is an ε-PS sequence of Γ_ε at level $c \in \mathbb{R}$ if

$$\lim_{n\to\infty} \Gamma'_{\varepsilon_n}(U_n) = 0 \text{ in } (H_{\varepsilon_n})^{-1} \quad \text{and} \quad \lim_{n\to\infty} \Gamma_{\varepsilon_n}(U_n) \to c.$$

Proposition 3.1. *Let $\{U_n\} \in N_{\varepsilon_n}(\rho)$ be an ε-PS sequence of Γ_ε at level $c \le \sum_{i=1}^k E_i$. Then for sufficiently small $\rho > 0$, there exist k functions $L_i \in S_{m_i}$ and k sequences $\{x_n^i\} \in \mathbb{R}^N$ such that*

$$\varepsilon_n x_n^i \to x_i \text{ for some } x_i \in \mathcal{M}_i \text{ and } \Big\|U_n - \sum_{i=1}^k \varphi_{\varepsilon_n}(\cdot - x_n^i) L_i(\cdot - x_n^i, \cdot)\Big\|_{\varepsilon_n} \to 0$$

as $n \to \infty$ up to a subsequence.

Proof. Since $U_n \in N_{\varepsilon_n}(\rho)$, there exist k sequences of functions $\{L_n^i\} \in S_{m_i}$ and k sequences of points $\{y_n^i\} \in \mathcal{M}_i^\delta$ such that

$$U_n(x,t) = \sum_{i=1}^k \varphi_{\varepsilon_n}(x - y_n^i/\varepsilon_n) L_n^i(x - y_n^i/\varepsilon_n, t) + \omega_n, \text{with } \|\omega_n\|_{\varepsilon_n} \le \rho. \tag{3.1}$$

Then, we see by the compactness of S_{m_i} and $\mathcal{M}_i^\delta$, as $n \to \infty$,

$$L_n^i \to L_i \text{ in } H_0, \quad y_n^i \to y_i \text{ in } \mathbb{R}^N$$

for some $L_i \in S_{m_i}$ and $y_i \in \mathcal{M}_i^\delta$ up to a subsequence. Define $U_n^1 := \sum_{i=1}^k \varphi_{\varepsilon_n}(\cdot - y_n^i/\varepsilon_n) U_n$ and $U_n^2 = U_n - U_n^1$. We denote $U_n^{1,i} = \varphi_{\varepsilon_n}(\cdot - y_n^i/\varepsilon_n) U_n$ so that $U_n^1 = \sum_{i=1}^k U_n^{1,i}$. Now, we divide the remaining proof into several steps.

Step 1. Fix an arbitrary i. From the definition, it is clear that W_n^i is bounded in H_0 so that we may assume it weakly converges to some $\underline{W_i \in H_0}$ by taking a subsequence. Choose an arbitrary test function $\Psi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$. We denote $u_n := U_n(\cdot, 0)$ and $\psi := \Psi(\cdot, 0)$. Also, for $z \in \mathbb{R}^N$, we denote $\Psi_z := \Psi(\cdot - z, t)$ and $\psi_z := \Psi(\cdot - z, 0)$. Since U_n is an ε-(PS) sequence, we have as $n \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}_+^{N+1}} \nabla U_n \cdot \nabla \Psi_{y_n^i/\varepsilon_n} t^{1-2s}\, dx\, dt + \int_{\mathbb{R}^N} V_{\varepsilon_n} u_n \psi_{y_n^i/\varepsilon_n} - F(u_n) \psi_{y_n^i/\varepsilon_n}\, dx \\
&= o(1) \|\psi_{y_n^i/\varepsilon}\|_{\varepsilon_n} = o(1)
\end{aligned}$$

For n as large as $\operatorname{supp}(\Psi) \subset B_{\delta/\varepsilon_n}^N(0) \times [0, \infty)$, the LHS of the above equation becomes by a change of variable $x \mapsto x + y_n^i$,

$$\int_{\mathbb{R}_+^{N+1}} \nabla W_n^i \cdot \nabla \Psi t^{1-2s}\, dx\, dt + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n^i) w_n^i \psi - F(w_n^i) \psi\, dx,$$

which converges to

$$\int_{\mathbb{R}_+^{N+1}} \nabla W_i \cdot \nabla \Psi t^{1-2s}\, dx\, dt + \int_{\mathbb{R}^N} V(y_i) w_i \psi - F(w_i) \psi\, dx \quad \text{as } n \to \infty$$

due to the weak convergence of W_n^i, conditions $(F1)$–$(F2)$ of the nonlinearity f and compact Sobolev embedding. This proves Step 1.

Step 2. We define $\tilde{O}_{\varepsilon_n} := \big(\mathbb{R}^N \setminus \cup_{i=1}^k B^N_{\delta/\varepsilon_n}(y_n^i/\varepsilon_n)\big) \times [0,\infty)$ so that $\operatorname{supp}(U_n^2) \subset \tilde{O}$ and $\tilde{\varphi}_{\varepsilon_n} := \sum_{i=1}^k \varphi_{\varepsilon_n}(\cdot - y_n^i/\varepsilon_n)$ so that $U_n^2 = (1-\tilde{\varphi}_{\varepsilon_n})U_n$. Then, using the fact that $\|U_n\|_{\varepsilon_n}$ is bounded, we can see if n is sufficiently large,

$$
\begin{aligned}
\|U_n^2\|_{\varepsilon_n} &= \|U_n^2\|_{H_{\varepsilon_n}(\tilde{O})} = \|(1-\tilde{\varphi}_{\varepsilon_n})U_n\|_{H_{\varepsilon_n}(\tilde{O})} \\
&\le \|U_n\|_{H_{\varepsilon_n}(\tilde{O})} + o(1) \\
&\le \|U_n - \sum_{i=1}^k \varphi_{\varepsilon_n}(\cdot - y_n^i/\varepsilon_n) L_n^i(\cdot - y_n^i/\varepsilon_n, \cdot)\|_{H_{\varepsilon_n}(\tilde{O})} \\
&\quad + \|\sum_{i=1}^k \varphi_{\varepsilon_n}(\cdot - y_n^i/\varepsilon_n) L_n^i(\cdot - y_n^i/\varepsilon_n, \cdot)\|_{H_{\varepsilon_n}(\tilde{O})} + o(1) \\
&\le \rho + o(1) \le 2\rho
\end{aligned}
\tag{3.2}
$$

Then, we have from the conditions (F1)–(F2) and Sobolev trace inequality that

$$
\begin{aligned}
\Gamma_{\varepsilon_n}(U_n^2) &\ge \frac{1}{2}\|U_n^2\|_{\varepsilon_n}^2 - \frac{1}{4}V_0\|U^2(\cdot,0)\|_{L^2(\mathbb{R}^N)}^2 - C\|U^2(\cdot,0)\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\
&\ge \frac{1}{4}\|U_n^2\|_{\varepsilon_n}^2 - C\|U_n^2\|_{\varepsilon_n}^{p+1} \\
&= \|U_n^2\|_{\varepsilon_n}^2\big(\frac{1}{4} - C\|U_n^2\|_{\varepsilon_n}^{p-1}\big).
\end{aligned}
\tag{3.3}
$$

Thus, we can deduce $\Gamma_{\varepsilon_n}(U_n^2) \ge 0$ by taking $\rho > 0$ small.

Next, we prove the second assertion. Define

$$
A(n,i) := B^N_{2\delta/\varepsilon_n}(y_n^i/\varepsilon_n) \setminus B^N_{\delta/\varepsilon_n}(y_n^i/\varepsilon_n).
$$

Let $u_n = U_n(\cdot,0)$. We claim that for all $i \in 1,\dots,k$

$$
\liminf_{n\to\infty} \int_{A(n,i)} |u_n|^{p+1}\,dx = 0,
$$

from which we can deduce that

$$
\liminf_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) - F(u_n^1) - F(u_n^2)\,dx = 0
$$

by using the conditions (F1)–(F2). Then our assertion follows because

$$
\begin{aligned}
\Gamma_{\varepsilon_n}(U_n) = \Gamma_{\varepsilon_n}(U_n^1) &+ \Gamma_{\varepsilon_n}(U_n^2) + \int_{\mathbb{R}^{N+1}_+} \tilde{\varphi}_{\varepsilon_n}(1-\tilde{\varphi}_{\varepsilon_n})|\nabla U_n|^2 t^{1-2s}\,dx\,dt \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)\tilde{\varphi}_{\varepsilon_n}(1-\tilde{\varphi}_{\varepsilon_n})(u_n)^2\,dx \\
&+ \int_{\mathbb{R}^N} F(u_n) - F(u_n^1) - F(u_n^2)\,dx + o(1).
\end{aligned}
$$

Arguing indirectly, suppose that $\liminf_{n\to\infty}\int_{A(n,i)} |u_n|^{p+1}\,dx \ne 0$ for some i. We let $g_n := |u_n|\chi_{A(n,i)}$. Then, as is proved in [23], there exist a positive $R > 0$ and a sequence $\{z_n\} \in \mathbb{R}^N$ such that

$$
\liminf_{n\to\infty} \int_{B_R^N(z_n)} g_n^2\,dx > 0.
$$

Since the support of g_n is contained in $A(n,i)$, we may assume that

$$z_n \in B^N_{\frac{2\delta}{\varepsilon_n}+R}(y_n^i/\varepsilon_n) \setminus B^N_{\frac{\delta}{\varepsilon_n}-R}(y_n^i/\varepsilon_n).$$

Define $\tilde{W}_n := U_n(\cdot + z_n, \cdot)$. Then, As in Step 1, we see that $\tilde{W}_n$ weakly converges in H_0 to some $\tilde{W} \in H_0$, which is a solution of (2.7) with $a = V(z_0)$ for some $z_0 \in B^N_{2\delta}(y^i) \setminus B^N_{\delta}(y^i)$. Also, $\tilde{W}$ is nontrivial because

$$\int_{B_R(0)} \tilde{w}^2\,dx = \liminf_{n\to\infty} \int_{B_R(0)} \tilde{w}_n^2\,dx \ge \liminf_{n\to\infty} \int_{B^N_R(z_n)} g_n^2\,dx > 0.$$

Then we see from Proposition 2.7, the definition of $E_{V(z_0)}$ and the Pohozaev identity (2.9) that

$$E_{m_i} \le E_{V(z_0)} \le \Gamma_{V(z_0)}(\tilde{W}) = \frac{s}{N}\int_{\mathbb{R}^N} |\nabla \tilde{W}|^2 t^{1-2s}\,dx\,dt.$$

Thus, by taking large R, we have

$$\frac{E_{m_i}}{2} \le \frac{s}{N}\int_{B_R(0)} |\nabla \tilde{W}|^2 t^{1-2s}\,dx\,dt \le \liminf_{n\to\infty} \frac{s}{N}\int_{B_R(0)} |\nabla \tilde{W}_n|^2 t^{1-2s}\,dx\,dt$$

But, by taking small $\rho < E_{m_i}/2$, this contradicts with (3.1) and proves the claim.

Step 3. As before, we set $w_n^i := W_n^i(\cdot, 0)$. We first claim that w_n^i strongly converges to w_i in $L^{p+1}(\mathbb{R}^N)$ for p in the condition $(F2)$. Fix an i. Arguing indirectly, suppose that $w_n^i \not\to w_i$ in $L^{p+1}(\mathbb{R}^N)$. Then, again using the argument in [23], there are an unbounded sequence $\{z_n\} \in \mathbb{R}^N$ and a positive number $R > 0$ such that $\liminf_{n\to\infty} \int_{B_R(z_n)} (w_n^i)^2\,dx > 0$. Since each w_n^i is supported in $B_{2\delta/\varepsilon_n}(0)$, we may assume that $z_n \in B_{\frac{2\delta}{\varepsilon_n}+R}(0)$ by choosing a subsequence. Define $\tilde{W}_n^i := W_n^i(\cdot + z_n, \cdot)$. As before, we can deduce by taking a subsequence, $\tilde{W}_n^i$ weakly converges in H_0 to some $\tilde{W}_i \in H_0$, which is a nontrivial solution to (2.7) with $a = V(z_0)$, where z_0 is a point in O_i. Then, as in Step 2, this contradicts with (3.1) and the claim follows. Now, the conditions (F1)–(F3) imply

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(w_n^i)\,dx = \int_{\mathbb{R}^N} F(w_i)\,dx,$$

which shows

$$\begin{aligned}
&\liminf_{n\to\infty} \Gamma_{\varepsilon_n}(U_n^{1,i}) \\
&= \liminf_{n\to\infty} \Big(\frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla W_n^i| t^{12-s}\,dx\,dt \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n^i)(w_n^i)^2\,dx - \int_{\mathbb{R}^N} F(w_n^i)\,dx\Big) \\
&\ge \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla W_i| t^{12-s}\,dx\,dt + \frac{1}{2}\int_{\mathbb{R}^N} V(y_i)(w_i)^2\,dx - \int_{\mathbb{R}^N} F(w_i)\,dx \ge E_i.
\end{aligned} \tag{3.4}$$

Also, we see from Step 2

$$\begin{aligned}
\limsup_{n\to\infty} \sum_{i=1}^k \Gamma_{\varepsilon_n}(U_n^{1,i}) &\le \limsup_{n\to\infty} \Big(\sum_{i=1}^k \Gamma_{\varepsilon_n}(U_n^{1,i}) + \Gamma_{\varepsilon_n}(U_n^2)\Big) \\
&\le \limsup_{n\to\infty} \Gamma_{\varepsilon_n}(U_n) \le \sum_{i=1}^k E_i.
\end{aligned} \tag{3.5}$$

The inequalities (3.4) and (3.5) imply that $\lim_{n\to\infty}\Gamma_{\varepsilon_n}(U_n^{1,i}) = E_i$. We again use (3.4) and (3.5) to see that $\Gamma_{V(y_i)}(W_i) = E_i$, from which and Proposition 2.7 we deduce $y_i \in \mathcal{M}_i$. Therefore we see also there exist $L_i \in S_{m_i}$ and $z_i \in \mathbb{R}^N$ such that $W_i = L_i(\cdot - z_i)$ from Proposition 2.5. Now observe

$$\begin{aligned}\int_{\mathbb{R}^N} V(y_i)(w_n^i)^2\,dx &= \int_{\mathbb{R}^N} V(y_i)(u_n^{1,i})^2\,dx\\ &\le \int_{\mathbb{R}^N} V(\varepsilon_n x)(u_n^{1,i})^2\,dx\\ &= \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n^i)(w_n^i)^2\,dx,\end{aligned}$$

where we used the fact that $V(y^i) \le V(\varepsilon_n x)$ on the support of $u_n^{1,i}$ so that

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\int_{\mathbb{R}_+^{N+1}} |\nabla W_n^i|^2 t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} V(y_i)(w_n^i)^2\,dx\Big)\\ &\le 2(E_i + \int_{\mathbb{R}^N} F(w_i)\,dx)\\ &= \int_{\mathbb{R}_+^{N+1}} |\nabla W_i|^2 t^{1-2s}\,dx\,dt + \int_{\mathbb{R}^N} V(y_i)(w_i)^2\,dx.\end{aligned}$$

This shows that $W_n^i \to W_i$ in H_0 as $n\to\infty$.

Step 4. *Completion of the proof.* Combining (3.4) and (3.5), we also deduce $\lim_{n\to\infty}\Gamma_{\varepsilon_n}(U_n^2) = 0$. Then, using (3.3), it holds that

$$\lim_{n\to\infty}\|U_n^2\|_{\varepsilon_n} = 0. \tag{3.6}$$

By setting $x_n^i = y_n^i/\varepsilon_n + z_i$, the whole proof of Proposition 3.1 follows from combining (3.6) and Steps 1–3. $\square$

Now by Proposition 3.1, we obtain a sufficient condition for proving Theorem 1.1.

Proposition 3.2. *For sufficiently small $\rho > 0$, every family of critical points $U_\varepsilon \in N_\varepsilon(\rho)$ of Γ_ε with $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(U_\varepsilon) \le \sum_{i=1}^k E_i$ satisfies the conclusion of Theorem 1.1.*

Proof. From Proposition 2.3, $u_\varepsilon := U_\varepsilon(\cdot,0)$ is a classical solution of (2.1). We denote $O_{\varepsilon,i} := \{x \in \mathbb{R}^N : \varepsilon x \in O_i\}$. Let $x_{\varepsilon,i}$ be a point satisfying $u(x_{\varepsilon,i}) = \max_{x\in\overline{O_{\varepsilon,i}}} u_\varepsilon(x)$. We first note that U_ε fulfills the hypothesis of Proposition 3.1. Then using Proposition 3.1, we may assume that if $\varepsilon > 0$ is sufficiently small, $u(x_{\varepsilon,i})$ is bounded below uniformly for $\varepsilon > 0$ and $\mathrm{dist}(\varepsilon x_{\varepsilon,i}, \mathcal{M}_i) \to 0$ as $\varepsilon \to 0$. This implies that $\{x_{\varepsilon,i}\}, i = 1,\dots,k$ are k-local maximums of u_ε. By scaling back $x \to x/\varepsilon$, this proves all the assertion in Theorem 1.1 except (iii).

Now, we prove (iii). Define a set $A_\varepsilon := \mathbb{R}^N \setminus \cup_{i=1}^k B_R^N(x_{\varepsilon,i})$. Using Proposition 3.1 again we see that $\|U_\varepsilon\|_\varepsilon$ is bounded and $\|U_\varepsilon\|_{H_\varepsilon(A_\varepsilon\times(0,\infty))}$ is arbitrarily small by taking $\rho > 0$ small and $R > 0$ large. Then are before, applying Applying (i) of Proposition 2.1 we can deduce $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ is bounded and $\|u_\varepsilon\|_{L^\infty(A_\varepsilon)}$ can be controlled as small as u_ε satisfies $(-\Delta)^s u_\varepsilon + \frac12 V_0 u_\varepsilon \le 0$ on $\mathbb{R}^N \setminus \cup_{i=1}^k B_R^N(x_{\varepsilon,i})$ by taking small $\rho > 0$ and large $R > 0$. Let Φ be the fundamental solution of equation $(-\Delta)^s u + \frac12 V_0 u = 0$ in $\mathbb{R}^N$. Then $K_\varepsilon := \sum_{i=1}^k \{\Phi * \chi_{B_R^N(0)}\}(\cdot - x_{\varepsilon,i})$ satisfies the equation $(-\Delta)^s u + \frac12 V_0 u = 0$ on $\mathbb{R}^N \setminus \cup_{i=1}^k B_R^N(x_{\varepsilon,i})$ so that one can use the comparison principle to see $u_\varepsilon \le CK_\varepsilon$ on $\mathbb{R}^N \setminus \cup_{i=1}^k B_R^N(x_{\varepsilon,i})$ for some $C > 0$. It is proved in [15] that $\Phi * \chi_{B_R^N(0)} \le C\frac{1}{|x|^{N+2s}}$, on $\mathbb{R}^N \setminus B_R^N(0)$ which implies that

$$u_\varepsilon(x) \le C \sum_{i=1}^k \frac{C}{(1+|x-x_{\varepsilon,i}|^2)^{\frac{N+2s}{2}}} \quad \text{on } \mathbb{R}^N.$$

Then, scaling back $x \to x/\varepsilon$ again, we see (iii) holds. This completes the proof. $\square$

From now on, we devote the rest of the paper to prove the existence of a family of critical points $U_\varepsilon \in N_\varepsilon(\rho_0)$ of Γ_ε with $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(U_\varepsilon) \le \sum_{i=1}^k E_i$.

4. Initial surface and intersection lemma

Choose and fix $x_i \in \mathcal{M}_i$ and $L_i \in S_{m_i}$ for each i. Define a k-dimensional surface $\gamma_\varepsilon : [0,\infty)^k \to H_\varepsilon$ as

$$\gamma_\varepsilon(\sigma_1, \cdots, \sigma_k) := \sum_{i=1}^k \varphi_\varepsilon(\cdot - x_i/\varepsilon) L_i\Big(\frac{\cdot - x_i/\varepsilon}{\sigma_i}, \frac{\cdot}{\sigma_i}\Big).$$

We call γ_ε the initial surface because it will be deformed by a deformation flow later. We denote by $[a,b]^k$ and $\partial([a,b]^k)$ a k-dimensional cube $[a,b]\times\cdots\times[a,b]$ and its boundary respectively. We may choose small $\rho_0 > 0$ which makes the conclusion of Proposition 3.2 holds, $\sigma_- \in (0,1)$ and $\sigma_+ \in (1,\infty)$ such that if $\varepsilon > 0$ is small, $\gamma_\varepsilon(\sigma) \notin N_\varepsilon(2\rho_0)$ on $\sigma = (\sigma_1, \cdots, \sigma_k) \in \partial([\sigma_-,\sigma_+]^k)$. Let $D_\varepsilon := \max_{\sigma\in[\sigma_-,\sigma_+]^k} \Gamma_\varepsilon(\gamma_\varepsilon(\sigma))$.

Proposition 4.1. *It holds that*

(i) $\lim_{\varepsilon\to 0} D_\varepsilon = \sum_{i=1}^k E_i$.
(ii) *there exists $\alpha > 0$ such that for all small $\varepsilon > 0$,*

$$\Gamma_\varepsilon(\gamma_\varepsilon(\sigma)) \ge D_\varepsilon - \alpha \Rightarrow \gamma_\varepsilon(\sigma) \in N_\varepsilon(\rho_0/2).$$

Proof. Let us denote by $o_\sigma(1)$ a quantity going to zero uniformly for $\sigma \in [\sigma_-,\sigma_+]^k$ as $\varepsilon \to 0$. We compute by setting $l_i = L_i(\cdot, 0)$,

$$\begin{aligned}
&\Gamma_\varepsilon(\gamma_\varepsilon(\sigma)) \\
&= \sum_{i=1}^k \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} |\nabla(\phi_\varepsilon L_i(\cdot/\sigma_i, \cdot/\sigma_i))|^2 t^{1-2s}\,dx\,dt \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + x_i)\varphi_\varepsilon(x) l_i(x/\sigma_i)^2\,dx - \int_{\mathbb{R}^N} F(\varphi_\varepsilon(x) l_i(x/\sigma_i))\,dx \\
&= \sum_{i=1}^k \frac{\sigma_i^{N-2s}}{2} \int_{\mathbb{R}_+^{N+1}} |\nabla L_i|^2 t^{1-2s}\,dx\,dt + \sigma_i^N \Big(\int_{\mathbb{R}^N} \frac{m_i}{2} l_i^2 - F(l_i)\,dx\Big) + o_\sigma(1) \\
&= \sum_{i=1}^k \Big(\frac{\sigma_i^{N-2s}}{2} - \frac{N-2s}{2N}\sigma_i^N\Big) \int_{\mathbb{R}_+^{N+1}} |\nabla L_i|^2 t^{1-2s}\,dx\,dt + o_\sigma(1),
\end{aligned}$$

where we used the decay property of l_i and the Pohozaev identity (2.9). Therefore we see

$$\begin{aligned} D_\varepsilon &= \max_{\sigma\in[\sigma_-,\sigma_+]^k} \Gamma_\varepsilon(\gamma_\varepsilon(\sigma)) \\ &= \sum_{i=1}^k (\frac{1}{2} - \frac{N-2s}{2N}) \int_{\mathbb{R}_+^{N+1}} |\nabla L_i|^2 t^{1-2s}\, dx\, dt + o_\sigma(1) \\ &= \sum_i^k E_i + o_\sigma(1) \end{aligned} \tag{4.1}$$

since the function $g(t) := t^{N-2s}/2 - \frac{N-2s}{2N} t^N$ attains it maximum at $t = 1$. This proves (i).

Next, we prove (ii). Suppose that (ii) does not hold. Then by (i), there exist $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\{\sigma_n\} \in [\sigma_-, \sigma_+]^k$ such that $\lim_{n\to\infty} \Gamma_{\varepsilon_n}(\gamma_{\varepsilon_n}(\sigma_n)) = \sum_{i=1}^k E_i$ and $\gamma_{\varepsilon_n}(\sigma_n) \notin N_{\varepsilon_n}(\rho_0/2)$ for all n. Since $[\sigma_-, \sigma_+]^k$ is compact, we may assume $\sigma_n \to \sigma_0$ to some $\sigma_0 \in [\sigma_-, \sigma_+]^k$ as $n \to \infty$. Then, we use (4.1) to deduce

$$\sum_{i=1}^k \Big(\frac{(\sigma_0)_i^{N-2s}}{2} - \frac{N-2s}{2N}(\sigma_0)_i^N\Big) \int_{\mathbb{R}_+^{N+1}} |\nabla L_i|^2 t^{1-2s}\, dx\, dt = \sum_{i=1}^k E_i.$$

By explicitly computing $g'(t)$, we can easily see $t = 1$ is a unique solution of $g'(t) = 0$. This implies that $\sigma_0 = (1, \cdots 1)$ so that σ_n is arbitrarily close to $(1, \cdots, 1)$ as $n \to \infty$. This however contradicts with $\gamma_{\varepsilon_n}(\sigma_n) \notin N_{\varepsilon_n}(\rho_0/2)$. This completes the proof. □

We say a k-dimensional continuous map $\tilde\gamma_\varepsilon : [\sigma_-, \sigma_+]^k \to H_\varepsilon$ is a boundary fixing deformation of γ_ε if there exists a continuous map $K_\varepsilon : [\sigma_-, \sigma_+] \times [0, 1] \to H_\varepsilon$ such that $K_\varepsilon(\sigma, t) = \gamma_\varepsilon(\sigma)$ for any $(\sigma, t) \in \partial([\sigma_-, \sigma_+]^k) \times [0, 1]$, $K_\varepsilon(\cdot, 0) = \gamma_\varepsilon$ and $K_\varepsilon(\cdot, 1) = \tilde\gamma_\varepsilon$. For each $i = 1, \dots, k$, let $\zeta_i : \mathbb{R}^N \to \mathbb{R}_+$ be a cut-off function satisfying $\zeta_i(x) = 1$ if $x \in O_i^\delta$, $\zeta_i(x) = 0$ if $x \notin O_i^{2\delta}$ and $|\nabla\zeta_i| \le 2/\delta$. For any $U \in H_0 \setminus \{0\}$, we define functionals

$$\mathcal{P}_i(U) := \sqrt{\Big(\frac{N\int_{\mathbb{R}^N} F(U(x,0)) - \frac{m_i}{2} U(x,0)^2\, dx}{\frac{N-2s}{2}\int_{\mathbb{R}_+^{N+1}} |\nabla U|^2 t^{1-2s}\, dx\, dt}\Big)_+}$$

and a set

$$P_\varepsilon := \{U \in H_\varepsilon : \mathcal{P}_i(\zeta_i(\varepsilon\cdot)U) = 1 \text{ for every } i = 1, \dots, k\}.$$

Proposition 4.2 (Intersection lemma)**.** *Let $\tilde\gamma_\varepsilon$ be a boundary fixing deformation of γ_ε. Then for any small $\varepsilon > 0$*

$$\tilde\gamma_\varepsilon([\sigma_-, \sigma_+]^k) \cap P_\varepsilon \neq \emptyset.$$

Proof. Since $\tilde\gamma_\varepsilon$ is a boundary fixing deformation of γ_ε, there is a continuous map $K_\varepsilon : [\sigma_-, \sigma_+]^k \times [0, 1] \to H_\varepsilon$ satisfying the definition above. We define a continuous map $\Xi_\varepsilon : [\sigma_-, \sigma_+]^k \times [0, 1] \to \mathbb{R}^k$ by

$$\Xi_\varepsilon(\sigma, t) := (\mathcal{P}_1(\zeta_1(\varepsilon\cdot)K_\varepsilon(\sigma, t)), \dots, \mathcal{P}_k(\zeta_k(\varepsilon\cdot)K_\varepsilon(\sigma, t))).$$

We have to prove that for any small $\varepsilon > 0$, there exists some $\sigma_\varepsilon \in [\sigma_-, \sigma_+]^k$ such that

$$\Xi_\varepsilon(\sigma_\varepsilon, 1) = (1, \dots, 1).$$

This is the case when $\deg(\Xi_\varepsilon(\cdot,1),[\sigma_-,\sigma_+]^k,(1,\dots,1))\neq 0$. Using elementary computation, we can see

$$\Xi_\varepsilon(\sigma,0)=(\mathcal{P}_1(\varphi_\varepsilon L_1(\cdot/\sigma_1)),\dots,\mathcal{P}_k(\varphi_\varepsilon L_k(\cdot/\sigma_1)))=(\sigma_1,\dots,\sigma_k)+o_\sigma(1), \quad (4.2)$$

which shows for sufficiently small $\varepsilon>0$,

$$\deg(\Xi_\varepsilon(\cdot,1),[\sigma_-,\sigma_+]^k,(1,\dots,1))=\deg(\Xi_\varepsilon(\cdot,0),[\sigma_-,\sigma_+]^k,(1,\dots,1))$$

because Ξ_ε fixes the boundary and $\Xi_\varepsilon(\cdot,0)$ does not touch $(1,\dots,1)$ at the boundary. Next, we define another homotopy map $\chi:[\sigma_-,\sigma_+]^k\times[0,1]$ by

$$\chi(\sigma,\varepsilon)=(\mathcal{P}_1(\varphi_\varepsilon L_1(\cdot/\sigma_1)),\dots,\mathcal{P}_k(\varphi_\varepsilon L_k(\cdot/\sigma_1)))$$

Then we again use (4.2) to see χ connects $\Xi_\varepsilon(\cdot,0)$ with the identity map I and for sufficiently small $\varepsilon>0$,

$$\deg(\Xi_\varepsilon(\cdot,0),[\sigma_-,\sigma_+]^k,(1,\dots,1))=\deg(I,[\sigma_-,\sigma_+]^k,(1,\dots,1))=1.$$

This completes the proof. □

Now, the following corollary immediately follows from the above intersection lemma.

Corollary 4.3. *For any boundary fixing deformation $\tilde\gamma_\varepsilon$ of γ_ε, it holds that*

$$\liminf_{\varepsilon\to\infty}\max_{\sigma\in[\sigma_-,\sigma_+]^k}\Gamma_\varepsilon(\tilde\gamma_\varepsilon(\sigma))\geq\sum_{i=1}^k E_i.$$

Proof. Define $\tilde\gamma_\varepsilon^1=\sum_{i=1}^k\zeta_i(\varepsilon\cdot)\tilde\gamma_\varepsilon$ and $\tilde\gamma_\varepsilon^2=\tilde\gamma_\varepsilon-\tilde\gamma_\varepsilon^1$. Arguing similarly in the proof of Proposition 3.1, we can check

$$\Gamma_{\varepsilon_n}(\tilde\gamma_\varepsilon^2)\geq 0 \quad\text{and}\quad \Gamma_{\varepsilon_n}(\tilde\gamma_\varepsilon)\geq\Gamma_{\varepsilon_n}(\tilde\gamma_\varepsilon^1)+\Gamma_{\varepsilon_n}(\tilde\gamma_\varepsilon^2)+o_\sigma(1).$$

Let $\sigma_\varepsilon\in[\sigma_-,\sigma_+]^k$ be such that $\tilde\gamma_\varepsilon(\sigma_\varepsilon)\in P_\varepsilon$. Then we have from Proposition 2.5 that

$$\begin{aligned}\Gamma_\varepsilon(\tilde\gamma_\varepsilon(\sigma_\varepsilon))&\geq\Gamma_\varepsilon(\tilde\gamma_\varepsilon^1(\sigma_\varepsilon))+o_\sigma(1)=\sum_{i=1}^k\Gamma_\varepsilon(\zeta_i(\varepsilon\cdot)\tilde\gamma_\varepsilon(\sigma_\varepsilon))+o_\sigma(1)\\&=\sum_{i=1}^k\Gamma_{m_i}(\zeta_i(\varepsilon\cdot)\tilde\gamma_\varepsilon(\sigma_\varepsilon))+o_\sigma(1)\\&\geq\sum_{i=1}^k\Gamma_{m_i}(L_i)+o_\sigma(1)=\sum_{i=1}^k E_i+o_\sigma(1),\end{aligned}$$

which shows the result. □

5. Deformation argument and Completion of the proof

In this section, we first show the existence of a (PS) sequence $\{u_n\}\in N_\varepsilon(\rho_0)$ of Γ_ε with $\Gamma_\varepsilon(u_n)\leq D_\varepsilon$ by applying a deformation argument. Then the existence of a critical point follows.

Proposition 5.1. *For each small $\varepsilon>0$ there exists a sequence $\{U_n\}\in N_\varepsilon(\rho_0)$ of Γ_ε satisfying $\Gamma_\varepsilon(U_n)\leq D_\varepsilon$ for all n and $\lim_{n\to\infty}\Gamma_\varepsilon'(U_n)\to 0$ in H_ε^{-1}.*

Proof. Arguing indirectly, suppose that there exists no such a sequence. We define a set $X_\varepsilon(\rho)$ by

$$X_\varepsilon(\rho) := \{U \in N_\varepsilon(\rho) : \Gamma_\varepsilon(U) \leq D_\varepsilon\}.$$

Then there is a real number $\beta_\varepsilon > 0$ such that $\|\Gamma'_\varepsilon\|_{H_\varepsilon^{-1}} \geq \beta_\varepsilon$ on $X_\varepsilon(\rho_0)$. Also, Proposition 3.1 implies that there is a real number $\eta > 0$ independent of small $\varepsilon > 0$ such that $\|\Gamma'_\varepsilon\|_{H_\varepsilon^{-1}} \geq \eta$ on $X_\varepsilon(\rho_0) \setminus X_\varepsilon(\rho_0/2)$. Then, by combining this facts with Proposition 4.1, we can apply the standard deformation theory(see for example [27]) to show the existence of k-dimensional surface $\tilde{\gamma}_\varepsilon(\sigma)$ homotopic to initial surface $\gamma_\varepsilon(\sigma)$ satisfying $\tilde{\gamma}_\varepsilon(\sigma) = \gamma_\varepsilon(\sigma)$ on $\sigma \in \partial([\sigma_-, \sigma_+]^k)$ and

$$\max_{\sigma \in [\sigma_-, \sigma_+]^k} \Gamma_\varepsilon(\tilde{\gamma}_\varepsilon(\sigma)) \leq D_\varepsilon - \kappa$$

for some $\kappa > 0$ independent of $\varepsilon > 0$. But, taking the limit $\varepsilon \to 0$, this contradicts with Corollary 4.3. This completes the proof. □

Now we are ready to complete the proof of Theorem 1.1.

Proposition 5.2. *For sufficiently small $\varepsilon > 0$, there exists a critical point $U_\varepsilon \in N_\varepsilon(\rho_0)$ of Γ_ε with $\Gamma_\varepsilon(U_\varepsilon) \leq D_\varepsilon$.*

Proof. We fix small $\varepsilon > 0$. Let $\{U_n\}$ be the (PS) sequence obtained in Proposition 5.1. Then there are V_n and W_n such that $U_n = V_n + W_n$, $V_n \in N_\varepsilon(0)$ and $\|W_n\|_\varepsilon \leq \rho_0$. Since $\{U_n\}$ is a (PS) sequence and bounded in H_ε, it weakly converges, up to a subsequence, to a critical point $U \in H_\varepsilon$ of Γ_ε. It is easily checked from the compactness of S_{m_i} in H_0 that $N_\varepsilon(0)$ is compact in H_ε. Thus, there is $V \in N_\varepsilon(0)$ such that $V_n \to V$ in H_ε, up to a subsequence, as $n \to \infty$. Also there is $W \in H_\varepsilon$ which weakly converges, up to a subsequence, to some $W \in H_\varepsilon$ so that $U = V + W$. Then we have

$$\|U - V\|_\varepsilon = \|W\|_\varepsilon \leq \liminf_{n \to \infty} \|W_n\|_\varepsilon \leq \rho_0,$$

which shows $U \in N_\varepsilon(\rho_0)$.

Next, we claim that $\Gamma_\varepsilon(U) \leq \limsup_{n\to\infty} \Gamma_\varepsilon(U_n)$ to show $\Gamma_\varepsilon(U) \leq D_\varepsilon$. By using the same argument for proving well known Brezis-Lieb lemma (for example see [27]), it can be seen

$$\int_{\mathbb{R}^N} F(U_n(x,0))\,dx = \int_{\mathbb{R}^N} F(U(x,0))\,dx + \int_{\mathbb{R}^N} F(U_n(x,0) - U(x,0))\,dx + o(1),$$

which shows

$$\Gamma_\varepsilon(U_n) = \Gamma_\varepsilon(U) + \Gamma_\varepsilon(U_n - U) + o(1).$$

Observe

$$\|U_n - U\|_\varepsilon \leq \|V_n - V\|_\varepsilon + \|W_n - W\|_\varepsilon \leq 2\rho_0 + o(1).$$

Then, arguing similarly as in the proof of Proposition 3.1, it holds that $\Gamma_\varepsilon(U_n - U) \geq 0$, which proves the claim. □

Theorem 1.1 follows from Propositions 5.2, 3.2 and the fact that $\lim_{\varepsilon\to 0} D_\varepsilon = \sum_{i=1}^k E_i$.

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