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MULTIPLE SOLUTIONS FOR A DISCRETE ANISOTROPIC $(p_1(k), p_2(k))$-LAPLACIAN EQUATIONS

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Abstract. This article concerns the existence and multiplicity solutions for a discrete Dirichlet Laplacian problems. Our technical approach is based on variational methods.

1. Introduction

In this work, we study the existence and multiplicity solutions of the discrete boundary-value problem

$$\begin{gathered} -\Delta(\phi_{p_1(k-1)}(\Delta u(k-1))) - \Delta(\phi_{p_2(k-1)}(\Delta u(k-1))) = \lambda f(k, u(k)), \\ \forall k \in \mathbb{Z}[1, T], \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.1}$$

where, $\phi_{p_i(k)}(t) = |t|^{p_i(k)-2}t$ $(i = 1, 2)$ for all $t \in \mathbb{R}$ and for each $k \in \mathbb{Z}[1, T]$, $T \geq 2$ is a positive integer, $\mathbb{Z}[1, T]$ is a discrete interval $\{1, 2, \dots, T\}$, λ is a positive parameter, $\Delta u(k-1) := u(k) - u(k-1)$ is the forward difference operator, $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $p_1, p_2 : \mathbb{Z}[0, T] \to [2, +\infty)$.

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain nonlinear problems from biological neural networks, economics, optimal control and other areas of study have led to the rapid development of the theory of difference equations; see the monograph of [2, 3, 11, 31] for an overview on this subject.

Equations involving the discrete p-Laplacian operator, subjected to classical or less classical boundary conditions, have been widely studied by many authors using various techniques. Recently, many results have been established by applying variational methods. In this direction we mention the papers [1, 9, 20, 25, 30] and the references therein. However, problems like (1.1) involving anisotropic exponents have only been started, by Mihailescu, Radulescu and Tersian [27], Kone and Ouaro [21], where known tools from the critical point theory are applied in order to get the existence of solutions. Later considered by many methods and authors, see [6, 7, 13, 15, 26, 29] for an extensive survey of such boundary value problems.

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Our aim is to establish the existence and multiplicity results for problem (1.1) through variational methods. First we will exploit a critical point Theorem 2.1 which provides for the existence of a local minima for a parameterized abstract functional. Next, Theorem 2.2 with the classical Ambrosetti-Rabinowitz condition, guarantee that (1.1) has at least two distinct nontrivial weak solutions (Theorem 3.2). Finally, we will get the existence of at least three nontrivial solutions of the problem (1.1) where the nonlinearity $f(x,u)$ does not satisfy Ambrosetti-Rabinowitz condition (Theorem 3.3), by employing a local minimum Theorem 2.3.

2. Preliminaries and basic notation

In this section, we state some basic properties, definitions and theorems to be used in this article. Let $(X, \|\cdot\|)$ be a finite dimensional Banach space. A functional I_λ is said to verify the Palais-Smale condition (in short $(P.S.)$) whenever one has that any sequence $\{u_n\}$ such that

- $\{I_\lambda(u_n)\}$ is bounded;
- $\{I'_\lambda(u_n)\}$ is convergent at 0 in X^*

admits a subsequence which is converging in X.

Our main tool will be the following three abstract critical point theorems, which are a simple extension of the Ricceri's Variational Principle [28] recalled here on the finite dimensional Banach spaces.

Theorem 2.1. *Let X be a finite dimensional Banach space and let Φ, $\Psi : X \to \mathbb{R}$ two functions of class C^1 on X with Φ is coercive. In addition, suppose that there exist $r \in \mathbb{R}$ and $w \in X$, with $0 < \Phi(w) < r$, such that*

$$\frac{\sup_{\Phi^{-1}([0,r])} \Psi}{r} < \frac{\Psi(w)}{\Phi(w)}. \tag{2.1}$$

Then, for each

$$\lambda \in \Lambda_w := \Big]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\Big[,$$

the function $I_\lambda = \Phi - \lambda\Psi$ admits at least one local minimum $\overline{u} \in X$ such that $\overline{u} \neq 0$, $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([0,r])$ and $I'_\lambda(\overline{u}) = 0$.

Theorem 2.2. *Let X be a finite dimensional Banach space and let Φ, $\Psi : X \to \mathbb{R}$ two functions of class C^1 on X with Φ is coercive. Fix $r > 0$. Assume that for each*

$$\lambda \in \Lambda := \Big]0, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\Big[,$$

the function $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and is unbounded from below. Then, for each $\lambda \in \Lambda$, the function I_λ admits at least two distinct critical points.

Theorem 2.3. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, moreover*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that

(i)
$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}<\frac{\Psi(\bar u)}{\Phi(\bar u)}$$

(ii) *for each* $\lambda\in\Lambda$,
$$\Lambda:=\Big]\frac{\Phi(\bar u)}{\Psi(\bar u)},\frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\Big[,$$
the functional $\Phi-\lambda\Psi$ *is coercive.*

Then, for each $\lambda\in\Lambda$, *the functional* $I_\lambda=\Phi-\lambda\Psi$ *has at least three distinct critical points in* X.

Remark 2.4. It is worth noticing that whenever X is a finite dimensional Banach space, for the Theorem 2.3 shows that regarding to the regularity of the derivative of Φ and Ψ, it is enough to require only that Φ' and Ψ' are two continuous functionals on X^*.

For the rest of this article, we use the following notation:
$$p_{\min}(k):=\min_{i=1,2}p_i(k),\quad p_{\max}(k):=\max_{i=1,2}p_i(k),\quad\text{for all } k\in\mathbb{Z}[0,T];$$
$$p^-_{\min}=\min_{k\in[0,T]}p_{\min}(k),\quad p^+_{\max}=\max_{k\in[0,T]}p_{\max}(k);$$
$$p^-_i=\min_{k\in[0,T]}p_i(k),\quad p^+_i=\max_{k\in[0,T]}p_i(k),\quad\text{for } i=1,2.$$
Define the function space,
$$H:=\{u:[0,T+1]\to\mathbb{R}:u(0)=u(T+1)=0\}.$$
Clearly, H is a T-dimensional Hilbert space (see [2]) with the inner product
$$\langle u,v\rangle:=\sum_{k=1}^{T+1}\Delta u(k-1)\Delta v(k-1),\quad\forall\, u,v\in H.$$
The associated norm is defined by
$$\|u\|:=\Big(\sum_{k=1}^{T+1}|\Delta u(k-1)|^2\Big)^{1/2}.$$
On the other hand, it is useful to introduce other norms on H, namely
$$|u|_m=\Big(\sum_{k=1}^{T}|u(k)|^m\Big)^{1/m},\quad\forall u\in H\text{ and } m\geq 2\,.$$
It can be verified [11] that
$$T^{\frac{2-m}{2m}}|u|_2\leq|u|_m\leq T^{\frac{1}{m}}|u|_2,\quad\forall u\in H\text{ and } m\geq 2.\tag{2.2}$$
We start with the following auxiliary result. For (a), (b) and (c) see [27] and for (d) see [30].

Lemma 2.5. *We have the following assertions:*

(a) *For every* $u\in H$ *with* $\|u\|\leq 1$ *one has*
$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}\geq T^{\frac{p^+-2}{2}}\|u\|^{p^+}.$$

(b) *For every* $u \in H$ *with* $\|u\| \geq 1$ *one has*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}} \|u\|^{p^-} - T.$$

(c) *For any* $m \geq 2$ *there exists a positive constant* c_m *such that*

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall\, u \in H.$$

(d) *For every* $u \in H$ *and for any* p, $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\max_{k \in \mathbb{Z}[1,T]} |u(k)| < \Big(T+1\Big)^{1/q} \Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p\Big)^{1/p}.$$

Definition 2.6. We say that $u \in H$ is a weak solution of problem (1.1) if

$$\begin{aligned} &\sum_{k=1}^{T+1} \big(\phi_{p_1(k-1)}(\Delta u(k-1)) + \phi_{p_2(k-1)}(\Delta u(k-1))\big)\, \Delta v(k-1) \\ &- \lambda \sum_{k=1}^{T} f(k, u(k)) v(k) = 0, \end{aligned}$$

for all $v \in H$.

To treat the Dirichlet problem (1.1), we define the following two functions:

$$\begin{gathered} \Phi(u) = \sum_{k=1}^{T+1} \Big(\frac{|\Delta u(k-1)|^{p_1(k-1)}}{p_1(k-1)} + \frac{|\Delta u(k-1)|^{p_2(k-1)}}{p_2(k-1)}\Big), \\ \Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \end{gathered} \tag{2.3}$$

where $F(k,t) = \int_0^t f(k,s)ds$ for all $(k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$. Further, let us denote

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u), \quad \text{for every } u \in H.$$

The functional I_λ is of class $C^1(H, \mathbb{R})$, and

$$\begin{aligned} \langle I'_\lambda(u), v\rangle = &\sum_{k=1}^{T+1} \big(\phi_{p_1(k-1)}(\Delta u(k-1)) + \phi_{p_2(k-1)}(\Delta u(k-1))\big)\, \Delta v(k-1) \\ &- \lambda \sum_{k=1}^{T} f(k, u(k)) v(k), \end{aligned}$$

for any $u, v \in H$. Thus, critical points of I_λ are weak solutions of (1.1).

3. Main results

To introduce our result, for a nonnegative constant γ, put

$$\sigma(\gamma) := \frac{T^{\frac{2-p^+_{\max}}{2}}}{p^+_{\max}} \Big(\Big(\frac{\gamma}{\sqrt{T+1}}\Big)^{p^-_{\min}} - 2T^{\frac{p^+_{\max}}{2}}\Big).$$

Theorem 3.1. *Assume that there exist two real constants γ and $\delta \geq 1$, with*

$$\gamma \geq \sqrt{T+1}\Big(T^{\frac{p_{\max}^{+}+p_{\min}^{-}-4}{2}} + 2T^{\frac{p_{\max}^{+}}{2}}\Big)^{1/p_{\min}^{-}}, \tag{3.1}$$

$$4\delta^{p_{\max}^{+}} < p_{\min}^{-}\sigma(\gamma) \tag{3.2}$$

such that

(A1)
$$\frac{\sum_{k=1}^{T}\max_{|t|\leq\gamma}F(k,t)}{\sigma(\gamma)} < \frac{p_{\min}^{-}\sum_{k=1}^{T}F(k,\delta)}{4\delta^{p_{\max}^{+}}}.$$

(A2) $F(k,\delta) \geq 0$ *for each* $k \in \mathbb{Z}[1,T]$.

Then, for each

$$\lambda \in \Lambda_w := \Big]\frac{4\delta^{p_{\max}^{+}}}{p_{\min}^{-}\sum_{k=1}^{T}F(k,\delta)}, \frac{\sigma(\gamma)}{\sum_{k=1}^{T}\max_{|t|\leq\gamma}F(k,t)}\Big[, \tag{3.3}$$

problem (1.1) *admits at least one nontrivial solution* $\overline{u} \in H$*, such that* $|\overline{u}| < \gamma$.

Proof. Take the real Banach space H as defined in Section 2, and put Φ, Ψ, as in (2.3). Our aim is to apply Theorem 2.1 to function I_λ. For each $u \in H$ such that $\|u\| \geq 1$, from assertion (b) in Lemma 2.5, we have

$$\begin{aligned} \Phi(u) &\geq \frac{1}{p_{\max}^{+}}\sum_{k=1}^{T+1}\Big(|\Delta u(k-1)|^{p_1(k-1)} + |\Delta u(k-1)|^{p_2(k-1)}\Big) \\ &\geq \frac{1}{p_{\max}^{+}}\Big(T^{\frac{2-p_1^{-}}{2}}\|u\|^{p_1^{-}} + T^{\frac{2-p_2^{-}}{2}}\|u\|^{p_2^{-}} - 2T\Big) \to \infty \quad \text{as } \|u\| \to \infty. \end{aligned}$$

So, Φ is a coercive, and we have the regularity assumptions required on Φ and Ψ. Therefore, it remains to verify assumption (2.1). To this end, we put $r := \sigma(\gamma)$, and pick $w \in H$, defined as

$$w(k) := \begin{cases} \delta, & \text{if } k \in \mathbb{Z}[1,T], \\ 0, & \text{otherwise }. \end{cases} \tag{3.4}$$

Clearly, with $\delta \geq 1$ one has

$$\Phi(w) = \sum_{k=1}^{T+1}\Big(\frac{|\Delta w(k-1)|^{p_1(k-1)}}{p_1(k-1)} + \frac{|\Delta w(k-1)|^{p_2(k-1)}}{p_2(k-1)}\Big) \leq \frac{4\delta^{p_{\max}^{+}}}{p_{\min}^{-}}. \tag{3.5}$$

Hence, it follows from (3.2) that $0 < \Phi(w) < r$. Now, let $u \in H$ such that $u \in \Phi^{-1}([0,r])$, by Lemma 2.5 (a), for any $u \in H$ with $\|u\| < 1$ we obtain

$$\begin{aligned} r \geq \Phi(u) &\geq \frac{1}{p_{\max}^{+}}\Big(T^{\frac{p_1^{+}-2}{2}}\|u\|^{p_1^{+}} + T^{\frac{p_2^{+}-2}{2}}\|u\|^{p_2^{+}}\Big) \\ &\geq \frac{T^{\frac{p_{\min}^{-}-2}{2}}}{p_{\max}^{+}}\|u\|^{p_{\max}^{+}}. \end{aligned} \tag{3.6}$$

Similarly, from Lemma 2.5 (b), for any $u \in H$ with $\|u\| > 1$, we obtain

$$\begin{aligned} r \geq \Phi(u) &\geq \frac{1}{p_{\max}^{+}}\Big(T^{\frac{2-p_1^{-}}{2}}\|u\|^{p_1^{-}} + T^{\frac{2-p_2^{-}}{2}}\|u\|^{p_2^{+}} - 2T\Big) \\ &\geq \frac{1}{p_{\max}^{+}}\Big(T^{\frac{2-p_{\max}^{+}}{2}}\|u\|^{p_{\min}^{-}} - T\Big). \end{aligned} \tag{3.7}$$

Then

$$\|u\| \le \max\Big\{\Big(\frac{rp^+_{\max}}{T^{\frac{p^-_{\min}-2}{2}}}\Big)^{1/p^+_{\max}}, \Big(\frac{rp^+_{\max}}{T^{\frac{2-p^+_{\max}}{2}}}+2T^{\frac{p^+_{\max}}{2}}\Big)^{1/p^-_{\min}}\Big\}.$$

Bearing in mind (3.1), we obtain

$$rp^+_{\max} \ge T^{\frac{p^-_{\min}-2}{2}}.$$

Then, from (3.6) and (3.7) we have

$$\|u\| \le \Big(\frac{rp^+_{\max}}{T^{\frac{2-p^+_{\max}}{2}}}+2T^{\frac{p^+_{\max}}{2}}\Big)^{1/p^-_{\min}}.$$

This together with Lemma 2.5 (d), yields

$$|u(k)| \le \sqrt{T+1}\|u\| \le \sqrt{T+1}\Big(\frac{rp^+_{\max}}{T^{\frac{2-p^+_{\max}}{2}}}+2T^{\frac{p^+_{\max}}{2}}\Big)^{1/p^-_{\min}} = \gamma$$

for all $k \in \mathbb{Z}[1,T]$. Therefore, we have that

$$\sup_{u\in\Phi^{-1}([0,r])}\Psi(u) = \sup_{u\in\Phi^{-1}([0,r])}\sum_{k=1}^{T}F(k,u(k)) \le \sum_{k=1}^{T}\max_{|t|\le\gamma}F(k,t). \tag{3.8}$$

In view of (3.5) and (3.8), taking into account (A1) and (A2), we obtain

$$\begin{aligned}\frac{\sup_{\Phi^{-1}([0,r])}\Psi(u)}{r} &\le \frac{\sum_{k=1}^{T}\max_{|t|\le\gamma}F(k,t)}{\sigma(\gamma)}\\ &< \frac{p^-_{\min}\sum_{k=1}^{T}F(k,\delta)}{4\delta^{p^+_{\max}}} \le \frac{\Psi(w)}{\Phi(w)}.\end{aligned} \tag{3.9}$$

Therefore, condition (2.1) of Theorem 2.1 is verified and all the assumptions of Theorem 2.1 are satisfied. So, for each $\lambda \in \Lambda_w \subset]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])}\Psi(u)}[$, the functional I_λ admits at least one critical point $\overline{u}$ such that $0<\Phi(\overline{u})<r$, and so $\overline{u}$ is a nontrivial weak solution of problem (1.1) such that $|\overline{u}|<\gamma$. □

The following result, in which the global Ambrosetti-Rabinowitz condition is also used, ensures the existence at least two weak solutions.

Theorem 3.2. *We suppose that the assumptions* (3.1) *and* (3.2) *of Theorem 3.1 be satisfied and $f(k,0)\neq 0$ for every $k\in\mathbb{Z}[1,T]$. Assume that there are two positive constants $\mu > p^+_{\max}$ and $R>0$ such that,*

$$0 < \mu F(k,t) \le tf(k,t), \tag{3.10}$$

for all $k\in\mathbb{Z}[1,T]$ and $|t|\ge R$. Then, for each $\lambda\in\Lambda := \Big]0, \frac{\sigma(\gamma)}{\sum_{k=1}^{T}\max_{|t|\le\gamma}F(k,t)}\Big[$, problem (1.1) *admits at least two nontrivial solutions.*

Proof. Let Φ,Ψ be the functionals defined in (2.3) satisfy all regularity assumptions requested in Theorem 2.2. Arguing as in the proof of Theorem 3.1, put $w(k)$ as in (3.4) and $r=\sigma(\gamma)$, for $\lambda\in\Lambda$ we obtain

$$\frac{\sup_{\Phi^{-1}([0,r])}\Psi(u)}{r} \le \frac{\sum_{k=1}^{T}\max_{|t|\le\gamma}F(k,t)}{\sigma(\gamma)} < \frac{1}{\lambda}.$$

Now, From condition (3.10), by standard computations, there is a positive constant c_1 such that

$$F(k,s) \geq c_1|s|^{\mu} \quad \text{for all } k \in \mathbb{Z}[1,T]. \tag{3.11}$$

Hence, for every $\lambda \in \Lambda$, $u \in H\backslash\{0\}$ and $t > 1$, we obtain

$$\begin{aligned}
I_\lambda(tu) &\leq \sum_{k=1}^{T+1}\Big(\frac{|\Delta tu(k-1)|^{p_1(k-1)}}{p_1(k-1)} + \frac{|\Delta tu(k-1)|^{p_2(k-1)}}{p_2(k-1)}\Big) - \lambda c_1 t^{\mu}\sum_{k=1}^{T}|u(k)|^{\mu} \\
&\leq t^{p_{\max}^+}\sum_{k=1}^{T+1}\Big(\frac{|\Delta u(k-1)|^{p_1(k-1)}}{p_1(k-1)} + \frac{|\Delta u(k-1)|^{p_2(k-1)}}{p_2(k-1)}\Big) - \lambda c_1 t^{\mu}\sum_{k=1}^{T}|u(k)|^{\mu}.
\end{aligned}$$

Since $\mu > p_{\max}^+$, $I_\lambda(tu) \to -\infty$ as $t \to \infty$. Then I_λ is unbounded from below. Finally, we verify the (PS)-condition, it is sufficient to prove that any Palais-Smale sequence is bounded. Arguing by contradiction, suppose that there exists a sequence $\{u_n\}$ such that $\{I_\lambda(u_n)\}$ is bounded and $\|I'_\lambda(u_n)\|_{X^*} \to 0$ as $n \to +\infty$ and $\lim_{n\to+\infty}\|u_n\| = +\infty$. Using also (3.10), we deduce that, for all $n \in \mathbb{N}$, it holds

$$\begin{aligned}
&\sum_{k=1}^{T}\Big(\mu F(k,u_n(k)) - u_n(k)f(k,u_n(k))\Big) \\
&\leq \sum_{|u_n(k)|\leq R}\Big(\mu F(k,u_n(k)) - u_n(k)f(k,u_n(k))\Big) \\
&\leq \sum_{k=1}^{T}\max_{|x|\leq R}|\mu F(k,x) - xf(k,x)| =: c_2.
\end{aligned}$$

To this end, taking into account Lemma 2.5 (b) one has

$$\begin{aligned}
M + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle \\
&= \sum_{k=1}^{T+1}\Big(\frac{|\Delta u_n(k-1)|^{p_1(k-1)}}{p_1(k-1)} + \frac{|\Delta u_n(k-1)|^{p_2(k-1)}}{p_2(k-1)}\Big) - \lambda\sum_{k=1}^{T}F(x,u_n(k)) \\
&\quad - \frac{1}{\mu}\sum_{k=1}^{T+1}\Big(|\Delta u_n(k-1)|^{p_1(k-1)} + |\Delta u_n(k-1)|^{p_2(k-1)}\Big) \\
&\quad + \lambda\sum_{k=1}^{T}\frac{1}{\mu}f(x,u_n(k))u_n(k) \\
&\geq \Big(\frac{1}{p_{\max}^+} - \frac{1}{\mu}\Big)\sum_{k=1}^{T+1}\Big(|\Delta u_n(k-1)|^{p_1(k-1)} + |\Delta u_n(k-1)|^{p_2(k-1)}\Big) \\
&\quad - \frac{\lambda}{\mu}\sum_{k=1}^{T}(\mu F(x,u_n(k)) - u_n(k)f(x,u_n(k))) \\
&\geq \Big(\frac{1}{p_{\max}^+} - \frac{1}{\mu}\Big)\Big(T^{\frac{2-p_1^-}{2}}\|u_n\|^{p_1^-} + T^{\frac{2-p_2^-}{2}}\|u_n\|^{p_2^-} - 2T\Big) - \frac{\lambda}{\mu}c_2.
\end{aligned}$$

But, this cannot hold true since p_1^-, $p_2^- > 1$ and $\mu > p_{\max}^+$. Hence, $\{u_n\}$ is bounded. That information combined with the fact that H is a finite dimensional Hilbert space implies that there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in H$

such that u_n converges to u_0 in H. Then, for each $\lambda \in \Lambda$, the function I_λ admits at least two distinct critical points. $\square$

Finally, we give an application of Theorem 2.3.

Theorem 3.3. *Suppose that there exist two constants γ and $\delta \geq 1$ with* (3.1) *and*

$$4\delta^{p_{\min}^-} > p_{\max}^+ \sigma(\gamma) \tag{3.12}$$

such that the assumptions (A1) *and* (A2) *in Theorem 3.1 hold. Assume also*

$$|f(k,t)| \leq a_0(1 + |t|^{\alpha(k)-1}), \tag{3.13}$$

where $a_0 > 0$ and $2 \leq \alpha^- = \min_{k\in[0,T]} \alpha(k) \leq \alpha^+ = \max_{k\in[0,T]} \alpha(k) < p_{\min}^-$. Then, for each $\lambda \in \Lambda_w$, where Λ_w as in (3.3)*, problem* (1.1) *admits at last three weak solutions.*

Proof. Our aim is to verify (i) and (ii) of Theorem 2.3. Arguing as in the proof of Theorem 3.1, put $w(k)$ as in (3.4) and $r = \sigma(\gamma)$, bearing in mind (3.12) we obtain

$$\Phi(w) > r > 0.$$

Therefore, (3.9) holds and the assumption (i) of Theorem 2.3 is satisfied. Now, we prove that the functional I_λ is coercive. For $u \in H$ such that $\|u\| \to +\infty$, in fact by using condition (3.13), we have

$$\begin{aligned} I_\lambda(u) \geq{}& \frac{1}{p_{\max}^+} \sum_{k=1}^{T+1} \Big(|\Delta u(k-1)|^{p_1(k-1)} + |\Delta u(k-1)|^{p_2(k-1)} \Big) \\ & - \lambda a_1 \sum_{k=1}^{T} \frac{|u(k)|^{\alpha(k)}}{\alpha(k)} - a_2, \end{aligned}$$

where a_1, a_2 are positive constants. Now, for $k \in \mathbb{Z}[1,T]$ we point out that

$$|u(k)|^{\alpha(k)} \leq |u(k)|^{\alpha^-} + |u(k)|^{\alpha^+}.$$

Thus, using (2.2) and Lemma 2.5 (c), we obtain

$$\begin{aligned} |u|_{\alpha^\pm}^{\alpha^\pm} &= \sum_{k=1}^{T} |u(k)|^{\alpha^\pm} \leq T|u|_2^{\alpha^\pm} = T\Big(\sum_{k=1}^{T} |u(k)|^2 \Big)^{\alpha^\pm/2} \\ &\leq T\Big(c_2 \sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \Big)^{\alpha^\pm/2} = TC_{\alpha^\pm}\|u\|^{\alpha^\pm}. \end{aligned}$$

Then, for every $\lambda \in \Lambda$ we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p_{\max}^+} \Big(T^{\frac{2-p_1^-}{2}} \|u\|^{p_1^-} + T^{\frac{2-p_2^-}{2}} \|u\|^{p_2^-} - 2T \Big) \\ &\quad - \frac{\lambda a_1}{\alpha^-} \Big(TC_{\alpha^-}\|u\|^{\alpha^-} + TC_{\alpha^+}\|u\|^{\alpha^+} \Big) - a_2 \\ &\geq \frac{1}{p_{\max}^+} \Big(T^{\frac{2-p_{\max}^+}{2}} \|u\|^{p_{\min}^-} - 2T \Big) - a_3\|u\|^{\alpha^+} - a_2 \to +\infty, \end{aligned}$$

since $p_{\min}^- > \alpha^+$, the functional I_λ is coercive, also condition (ii) holds. So, for each $\lambda \in \Lambda_w$, the functional I_λ has at least three distinct critical points that are weak solutions of (1.1). $\square$

Example 3.4. For $T = 2$, consider the problem

$$\begin{gathered} -\Delta\Big(\Big(|\Delta u(0)|^{p_1(0)-2} + |\Delta u(0)|^{p_2(0)-2}\Big)\Delta u(0)\Big) = -2\lambda(u(1)-1) \\ -\Delta\Big(\Big(|\Delta u(1)|^{p_1(1)-2} + |\Delta u(1)|^{p_2(1)-2}\Big)\Delta u(1)\Big) = -2\lambda(u(2)-2) \\ u(0) = u(3) = 0, \end{gathered} \tag{3.14}$$

where $f(k,t) = -2(t-k)$ for $k = 1,2$ and for

$$p_1(k) = \frac{1}{2}k + 2, \quad p_2(k) = -\frac{1}{2}k + 4 \quad \text{for } k = 0,1,2.$$

Then one has

$$p_1^- = 2, \quad p_2^- = 3, \quad p_1^+ = 3, \quad p_2^+ = 4, \quad p_{\min}^- = 2, \quad p_{\max}^+ = 4.$$

In fact, if we choose, for example $\delta = 1$ and $\gamma = 6\sqrt{3}$ such that (3.1) is verified, we obtain $\sigma(\gamma) = 7/2$ and condition (3.2) holds. Moreover, one has

$$\frac{\sum_{k=1}^{2} \max_{|t|\le 6\sqrt{3}} F(k,t)}{7/2} = \frac{10}{7} < 2 = \frac{p_{\min}^- \sum_{k=1}^{2} F(k,1)}{4\delta^{p_{\max}^+}}.$$

Then, owing to Theorem 3.1, for each $\lambda \in \left]\frac{1}{2}, \frac{7}{10}\right[$, problem (3.14) admits at least one nontrivial solution $\overline{u}$, such that $|\overline{u}| < 6\sqrt{3}$.

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