Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 19, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BOUNDARY BEHAVIOR OF SOLUTIONS TO A SINGULAR DIRICHLET PROBLEM WITH A NONLINEAR CONVECTION

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Abstract. In this article we analyze the exact boundary behavior of solutions to the singular nonlinear Dirichlet problem

$$-\Delta u = b(x)g(u) + \lambda|\nabla u|^q + \sigma, \quad u > 0,\ x \in \Omega,$$
$$u\big|_{\partial\Omega} = 0,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $q \in (0,2]$, $\sigma > 0$, $\lambda > 0$, $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$, g is decreasing on $(0, s_0)$ for some $s_0 > 0$, $b \in C^\alpha_{\rm loc}(\Omega)$ for some $\alpha \in (0,1)$, is positive in Ω, but may be vanishing or singular on the boundary. We show that $\lambda|\nabla u|^q$ does not affect the first expansion of classical solutions near the boundary.

1. Introduction

In this article, we consider the boundary behavior of solutions to the singular nonlinear Dirichlet problem

$$-\Delta u = b(x)g(u) + \lambda|\nabla u|^q + \sigma, \quad u > 0,\ x \in \Omega, \quad u\big|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $q \in (0,2]$, $\lambda > 0$, $\sigma > 0$, b satisfies

(B1) $b \in C^\alpha_{\rm loc}(\Omega)$ for some $\alpha \in (0,1)$, is positive in Ω,

and g satisfies

(G1) $g \in C^1((0,\infty),(0,\infty))$ and $\lim_{s\to 0} g(s) = \infty$;
(G2) there exists $s_0 > 0$ such that $g'(s) < 0$, for all $s \in (0, s_0)$;
(G3) there exists $C_g \geq 0$ such that

$$\lim_{s\to 0^+} g'(s)\int_0^s \frac{d\tau}{g(\tau)} = -C_g.$$

A typical example of functions which satisfy (G1)-(G3) is

$$g(s) = s^{-\gamma} + \mu s^p,\ s > 0,$$

where $\gamma, p, \mu > 0$. In this case, $C_g = \gamma/(1+\gamma)$. A complete characterization of g in (G1)-(G3) is provided in Lemma 2.14.

2000 *Mathematics Subject Classification.* 35J65, 35B05, 35J25, 60J50.
Key words and phrases. Semilinear elliptic equation; singular Dirichlet problem; nonlinear convection term; classical solution; boundary behavior.

Submitted December 3, 2014. Published January 20, 2015.

For convenience, we denote by ψ the solution to the problem

$$\int_0^{\psi(t)} \frac{ds}{g(s)} = t, \quad \forall t > 0. \tag{1.2}$$

When $\lambda = 0$, (1.1) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials (see, for instance, [9, 17, 21, 36, 38, 41]) and has been discussed by many authors and in many contexts. With regard to the existence, nonexistence, uniqueness, multiplicity, regularity, local (near the boundary) and global estimates of (classical or weak) solutions, see, for instance, [1]-[4], [6, 8, 9, 12, 16, 17, 21], [23]-[27], [29]-[31], [35], [39]-[46], [51] and the references therein.

When $\lambda > 0$, $b \equiv 1$ in Ω and $g(u) = u^{-\gamma}$ with $\gamma > 0$, the authors [47] considered the existence and regularities of the unique solution to (1.1). Cui [11] established a sub-supersolution method to more general problem than (1.1).

When $\lambda = 1$, $\sigma = 0$, $0 < q < 2$, $b \equiv 1$ in Ω and the function $g : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [19] showed that if g satisfies the following conditions

- (G01) $\int_0^1 g(s)ds = \infty$, $\int_1^\infty g(s)ds < \infty$;
- (G02) there exist positive constants δ and M with $M > 1$ such that $G(s) < MG(2s)$, for all $s \in (0, \delta)$, $G(s) := \int_s^\infty g(\tau)d\tau$, $s > 0$,

then the unique solution u to (1.1) has the following properties:

- (I1) $|u(x) - \phi(d(x))| < c_0 d(x)$, for all $x \in \Omega$ for $0 < q \le 1$;
- (I2) $|u(x) - \phi(d(x))| < c_0 d(x)[G(\phi(d(x)))]^{(q-1)/2}$, for all $x \in \Omega$ for $1 < q < 2$;

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, c_0 is a suitable positive constant and $\phi \in C[0, \infty) \cap C^2(0, \infty)$ is the unique solution of the problem

$$\int_0^{\phi(t)} \frac{ds}{\sqrt{2G(s)}} = t, \quad t > 0. \tag{1.3}$$

For further works, see [10], [13]-[15], [18, 20, 28, 37], [48]-[50] and the references therein.

We introduce two types of functions. First, we denote by K the set of all Karamata functions $\hat{L}$ which are **normalized** slowly varying at zero (see, Bingham-Goldie-Teugels's book [5] and Maric's book [32]) defined on $(0, \eta]$ for some $\eta > 0$ by

$$\hat{L}(s) = c_0 \exp\Big(\int_s^\eta \frac{y(\tau)}{\tau} d\tau \Big), \quad s \in (0, \eta], \tag{1.4}$$

where $c_0 > 0$ and the function $y \in C([0, \eta])$ with $y(0) = 0$.

Next let Λ denote the set of all positive monotonic functions θ in $C^1(0, \delta_0) \cap L^1(0, \delta_0)$ ($\delta_0 > 0$) which satisfy

$$\lim_{t \to 0} \frac{d}{dt}\big(\frac{\Theta(t)}{\theta(t)}\big) := C_\theta \in [0, \infty), \quad \Theta(t) := \int_0^t \theta(s)ds. \tag{1.5}$$

The set Λ was first introduced by Cîrstea and Rădulescu [7] for non-decreasing functions and by Mohammed [34] for non-increasing functions to study the boundary behavior of solutions to boundary blow-up elliptic problems.

We assume that b satisfies

(B2) there exists $\theta \in \Lambda$ such that

$$0 < b_1 := \lim_{d(x)\to 0} \inf \frac{b(x)}{\theta^2(d(x))} \le b_2 := \lim_{d(x)\to 0} \sup \frac{b(x)}{\theta^2(d(x))} < \infty.$$

Recently, for g satisfying (G1) and decreasing on $(0,\infty)$, the authors [50] considered the two cases

(i) $q \in (0,2)$, $b \equiv 1$ in Ω, g satisfies (G3) with $C_g > 1/2$;
(ii) $q = 2$, b satisfies (B1) and (B2), g satisfies (G3) with

$$C_\theta + 2C_g > 2, \tag{1.6}$$

and one of the following two conditions holds
(S01) $C_g > 0$;
(S02) $C_g = 0$ and $\lambda \limsup_{s\to 0^+} \frac{g(s)}{|g'(s)|} < 1$

and obtained the boundary behavior of the unique solutions to (1.1).

In this article, we extend [50] for more general g and b. We first establish a local comparison principle for $q \in (0,1)$ under (G2). More precisely, we show the first exact asymptotic behaviour of any classical solution near the boundary to (1.1) and reveal that the nonlinear gradient term $\lambda|\nabla u|^q$ does not affect the behaviour. For $q \in [1,2]$, by using a nonlinear change, the local comparison principle and the results in [51] and [30], we show the same results as $q \in (0,1)$. Our main results are summarized as follows.

Theorem 1.1. *For fixed $\lambda > 0$, let g satisfy* (G1)–(G3)*, b satisfy* (B1)–(B2)*. If both* (1.6) *and one of the following conditions hold*

(S1) $q \in (0,1)$*;*
(S2) $q \in [1,2]$ *and* $C_g > 0$*;*
(S3) $q \in [1,2]$*,* $C_g = 0$ *and*

$$\lambda \limsup_{s\to 0^+} \frac{g(s)}{|g'(s)|} < 1,$$

then for any classical solution u_λ to (1.1)*, it holds*

$$\xi_1^{1-C_g} \le \lim_{d(x)\to 0} \inf \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \lim_{d(x)\to 0} \sup \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \xi_2^{1-C_g}, \tag{1.7}$$

where

$$\xi_1 = \frac{b_1}{2(C_\theta + 2C_g - 2)}, \quad \xi_2 = \frac{b_2}{2(C_\theta + 2C_g - 2)}. \tag{1.8}$$

In particular,

(i) *when* $C_g = 1$*,* u_λ *satisfies*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} = 1;$$

(ii) *when* $C_g < 1$ *and* $b_1 = b_2 = b_0$ *in* (B2)*,* u_λ *satisfies*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(d^2(x)\theta^2(d(x)))} = (\xi_{01} C_\theta^2)^{1-C_g},$$

where

$$\xi_{01} = \frac{b_0}{2(C_\theta + 2C_g - 2)}.$$

Theorem 1.2. *For fixed $\lambda > 0$, let $q \in (0, 2]$, g satisfy* (G1)–(G3), *and let b satisfy* (B1) *and*

(B3) *there exists $\hat{L} \in K$ with $\int_0^\eta \frac{\hat{L}(s)}{s} ds < \infty$ such that*
$$0 < b_1 := \lim_{d(x)\to 0} \inf \frac{b(x)}{a^2(d(x))} \le b_2 := \lim_{d(x)\to 0} \sup \frac{b(x)}{a^2(d(x))} < \infty,$$
where
$$a^2(t) = t^{-2}\hat{L}(t), \ t \in (0, \eta]. \tag{1.9}$$

If one of (S1), (S2), (S3) *holds, then for any classical solution u_λ to* (1.1), *it holds*
$$b_1^{1-C_g} \le \lim_{d(x)\to 0} \inf \frac{u_\lambda(x)}{\psi(h_1(d(x)))} \le \lim_{d(x)\to 0} \sup \frac{u_\lambda(x)}{\psi(h_1(d(x)))} \le b_2^{1-C_g}, \tag{1.10}$$
where
$$h_1(t) = \int_0^t \frac{\hat{L}(s)}{s} ds, \quad t \in (0, \eta). \tag{1.11}$$

In particular,

(i) *when $C_g = 1$, u_λ satisfies*
$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(h_1(d(x)))} = 1;$$

(ii) *when $C_g < 1$ and $b_1 = b_2 = b_0$ in* (B_3), *u_λ satisfies*
$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(h_1(d(x)))} = b_0^{1-C_g}.$$

Theorem 1.3. *For fixed $\lambda > 0$, let $q \in (0, 2]$, b satisfy* (B1), *g satisfy* (G1) *and $g(s) = s^{-\gamma} + \mu s^p$, $s \in (0, s_0)$, for some $s_0 > 0$, where γ, $p, \mu > 0$. If b satisfies*

(B4) *there exists $\hat{L} \in K$ with $\int_0^\eta \frac{\hat{L}(s)}{s} ds = \infty$ such that*
$$0 < b_1 := \lim_{d(x)\to 0} \inf \frac{b(x)}{(d(x))^{\gamma-1}\hat{L}(d(x))} \le b_2 := \lim_{d(x)\to 0} \sup \frac{b(x)}{(d(x))^{\gamma-1}\hat{L}(d(x))} < \infty,$$

then for any classical solution u_λ to (1.1), *it holds*
$$\begin{aligned} (b_1(1+\gamma))^{1/(1+\gamma)} &\le \lim_{d(x)\to 0} \inf \frac{u_\lambda(x)}{d(x)(h_2(d(x)))^{1/(1+\gamma)}} \\ &\le \lim_{d(x)\to 0} \sup \frac{u_\lambda(x)}{d(x)(h_2(d(x)))^{1/(1+\gamma)}} \\ &\le (b_2(1+\gamma))^{1/(1+\gamma)}, \end{aligned} \tag{1.12}$$
where
$$h_2(t) = \int_t^\eta \frac{L(\tau)}{\tau} d\tau, \quad t \in (0, \eta). \tag{1.13}$$

Remark 1.4. Some basic examples of functions which satisfy (G1)–(G3) with $C_g = 0$ and $\lim_{s\to 0^+} \frac{g(s)}{|g'(s)|} = 0$ are

(i) $g(s) = (-\ln s)^\gamma$, $\gamma > 0$, $s \in (0, s_0)$;
(ii) $g(s) = (\ln(-\ln s))^\gamma$, $\gamma > 0$, $s \in (0, s_0)$;
(iii) $g(s) = e^{(-\ln s)^\gamma}$, $0 < \gamma < 1$, $s \in (0, s_0)$, where $s_0 > 0$ sufficiently small.

Remark 1.5. When $\gamma > 0$, we note that $C_g = \frac{\gamma}{1+\gamma}$ and $C_\theta = \frac{2}{\gamma+1}$ in Theorem 1.3, i.e., $C_\theta + 2C_g = 2$.

The outline of this paper is as follows. In section 2, we present some basic facts from Karamata regular variation theory and some preliminaries. Some comparison principles are given in section 3. In section 4, we prove Theorems 1.1–1.3.

2. Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic processes (see Bingham, Goldie and Teugels' book [5], Maric's book [32] and the references therein). In this section, we present some basic facts from Karamata regular variation theory.

Definition 2.1. A positive continuous function g defined on $(0, \eta]$, for some $\eta > 0$, is called **regularly varying at zero** with index ρ, denoted by $g \in RVZ_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s \to 0^+} \frac{g(\xi s)}{g(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, g is called **slowly varying at zero**.

Clearly, if $g \in RVZ_\rho$, then $L(s) := g(s)/s^\rho$ is slowly varying at zero.

Definition 2.2. A positive continuous function g defined on $(0, \eta]$, for some $\eta > 0$, is called **rapidly varying to infinity at zero** if for each $\xi \in (0, 1)$

$$\lim_{s \to 0^+} \frac{g(\xi s)}{g(s)} = \infty. \tag{2.2}$$

Definition 2.3. A positive function $g \in C(0, \eta]$ with $\lim_{s\to 0^+} g(s) = 0$, for some $\eta > 0$, is called **rapidly varying to zero at zero** if for each $\xi \in (0, 1)$

$$\lim_{s \to 0^+} \frac{g(\xi s)}{g(s)} = 0. \tag{2.3}$$

Proposition 2.4 (Uniform convergence theorem)**.** *If $g \in RVZ_\rho$, then* (2.1) *holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2.5 (Representation theorem)**.** *A function L is slowly varying at zero if and only if it may be written in the form*

$$L(s) = l(s) \exp\Big(\int_s^\eta \frac{y(\tau)}{\tau} d\tau \Big), \quad s \in (0, \eta], \tag{2.4}$$

where the functions l and y are continuous and for $s \to 0^+$, $y(s) \to 0$ and $l(s) \to c_0$, with $c_0 > 0$.

Note that

$$\hat{L}(s) = c_0 \exp\Big(\int_s^\eta \frac{y(\tau)}{\tau} d\tau \Big), \quad s \in (0, \eta], \tag{2.5}$$

is **normalized** slowly varying at zero, and

$$g(s) = s^\rho \hat{L}(s), \quad s \in (0, \eta], \tag{2.6}$$

is **normalized** regularly varying at zero with index ρ (and denoted by $g \in NRVZ_\rho$).

A function $g \in NRVZ_\rho$ if and only if

$$g \in C^1(0, \eta], \text{ for some } \eta > 0 \text{ and } \lim_{s \to 0^+} \frac{s g'(s)}{g(s)} = \rho. \tag{2.7}$$

Proposition 2.6. *If functions L, L_1 are slowly varying at zero, then*

(i) L^ρ *for every* $\rho \in \mathbb{R}$, $c_1L + c_2L_1$ $(c_1 \ge 0,\ c_2 \ge 0$ with $c_1 + c_2 > 0)$, $L \cdot L_1$, $L \circ L_1$ (*if* $L_1(s) \to 0$ *as* $s \to 0^+$) , *are also slowly varying at zero;*
(ii) *For every* $\rho > 0$ *and* $s \to 0^+$, $s^\rho L(s) \to 0$, $s^{-\rho} L(s) \to \infty$*;*
(iii) *For* $\rho \in \mathbb{R}$ *and* $s \to 0^+$, $\ln(L(s))/\ln s \to 0$ *and* $\ln(s^\rho L(s))/\ln s \to \rho$.

Proposition 2.7. *If* $g_1 \in RVZ_{\rho_1}$, $g_2 \in RVZ_{\rho_2}$ *with* $\lim_{s\to 0} g_2(s) = 0$, *then* $g_1 \circ g_2 \in RVZ_{\rho_1\rho_2}$.

Proposition 2.8 (Asymptotic behavior)**.** *If a function* L *is slowly varying at zero, then for* $\eta > 0$ *and* $t \to 0^+$,
(i) $\int_0^t s^\rho L(s)ds \cong (1+\rho)^{-1} t^{1+\rho} L(t)$, *for* $\rho > -1$*;*
(ii) $\int_t^\eta s^\rho L(s)ds \cong (-\rho-1)^{-1} t^{1+\rho} L(t)$, *for* $\rho < -1$.

Proposition 2.9. *Let* $g \in C^1(0,\eta]$ *be positive and*
$$\lim_{s\to 0^+} \frac{sg'(s)}{g(s)} = +\infty.$$
Then g *is rapidly varying to zero at zero.*

Proposition 2.10. *Let* $g \in C^1(0,\eta)$ *be positive and*
$$\lim_{s\to 0^+} \frac{sg'(s)}{g(s)} = -\infty.$$
Then g *is rapidly varying to infinity at zero.*

Proposition 2.11 ([46, Lemma 2.3])**.** *Let* $\hat{L}$ *be defined on* $(0,\eta]$ *and be normalized slowly varying at zero. Then*
$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(\tau)}{\tau} d\tau} = 0.$$
If further $\int_0^\eta \frac{L(\tau)}{\tau} d\tau$ *converges, then*
$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(\tau)}{\tau} d\tau} = 0.$$

Our results in the section are summarized in the following lemmas.

Lemma 2.12. *Let* $\theta \in \Lambda$.
(i) $\lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} = 0$*;*
(ii) $\lim_{t\to 0^+} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{\Theta(t)}{\theta(t)}\Big) = 1 - C_\theta$, *and* $C_\theta \in [0,1]$ *when* θ *is non-decreasing,* $C_\theta \ge 1$ *provided* θ *is non-increasing;*
(iii) *when* $C_\theta > 0$, $\theta \in NRVZ_{(1-C_\theta)/C_\theta}$ *and* $\Theta \in NRVZ_{1/C_\theta}$.

Proof. For an arbitrary $\theta \in \Lambda$, we have:

(i) When θ is non-decreasing, we have that $0 < \Theta(t) \le t\theta(t)$, for all $t \in (0,\delta_0)$ and (i) holds; when θ is non-increasing, it follows by $\theta \in L^1(0,\delta_0)$ that
$$\lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} = \lim_{t\to 0^+} \frac{1}{\theta(t)} \lim_{t\to 0^+} \Theta(t) = 0.$$

(ii) Since
$$\lim_{t\to 0^+} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\big(\frac{\Theta(t)}{\theta(t)}\big) = 1 - C_\theta, \tag{2.8}$$

it follows that $C_\theta \in [0,1]$ when θ is non-decreasing, and $C_\theta \geq 1$ provided θ is non-increasing;

(iii) (1.5) and the l'Hospital's rule imply

$$\lim_{t\to 0^+} \frac{\Theta(t)}{t\theta(t)} = \lim_{t\to 0^+} \frac{\frac{\Theta(t)}{\theta(t)}}{t} = \lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{\Theta(t)}{\theta(t)}\Big) = C_\theta. \tag{2.9}$$

So, when $C_\theta > 0$, $\Theta \in NRVZ_{C_\theta^{-1}}$ and it follows by (2.8) and (2.9) that

$$\lim_{t\to 0} \frac{t\theta'(t)}{\theta(t)} = \lim_{t\to 0} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} \lim_{t\to 0} \frac{t\theta(t)}{\Theta(t)} = \frac{1 - C_\theta}{C_\theta}, \tag{2.10}$$

i.e., $\theta \in NRVZ_{(1-C_\theta)/C_\theta}$. □

Lemma 2.13 ([51, Lemma 2.2])**.** *Let g satisfy* (G1), (G2).

(i) *If g satisfies* (G3)*, then $C_g \leq 1$;*
(ii) (G3) *holds with $C_g \in (0,1)$ if and only if $g \in NRVZ_{-C_g/(1-C_g)}$;*
(iii) (G3) *holds with $C_g = 0$ if and only if g is normalized slowly varying at zero;*
(iv) *if* (G3) *holds with $C_g = 1$, then g is rapidly varying to infinity at zero.*

Lemma 2.14 ([51, Lemma 2.3])**.** *Let g satisfy* (G1), (G2) *and let ψ be uniquely determined by*

$$\int_0^{\psi(t)} \frac{d\tau}{g(\tau)} = t, \ t \in [0,\infty).$$

Then

(i) $\psi'(t) = g(\psi(t))$*,* $\psi(t) > 0$*,* $t > 0$*,* $\psi(0) = 0$ *and* $\psi''(t) = g(\psi(t))g'(\psi(t))$*,* $t > 0$*;*
(ii) $\lim_{t\to 0^+} tg(\psi(t)) = 0$ *and* $\lim_{t\to 0^+} tg'(\psi(t)) = -C_g$*;*
(iii) $\psi \in NRVZ_{1-C_g}$ *and* $\psi' \in NRVZ_{-C_g}$*;*
(iv) *when* $C_\theta + 2C_g > 2$ *and* $\theta \in \Lambda$*,* $\lim_{t\to 0^+} \frac{t}{\psi(\xi\Theta^2(t))} = 0$ *uniformly for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$*, where* Θ *is given as in* (1.5)*;*
(v) $\lim_{t\to 0^+} \frac{t}{\psi(\xi h_1(t))} = 0$ *uniformly for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$*, where* h_1 *is given in* (1.11)*.*

Lemma 2.15. *Let $q \in (0,1)$. If $C_\theta + 2C_g > 2$, then*

$$\lim_{s\to 0^+} (g(\psi(\Theta^2(t))))^{q-1} \frac{(\Theta(t))^q}{(\theta(t))^{2-q}} = 0, \quad \lim_{s\to 0^+} g(\psi(\Theta^2(t)))\theta^2(t) = \infty.$$

Proof. Using Proposition 2.7, Lemma 2.13 (iii) and Lemma 2.15 (iii), we see that $g(\psi(\Theta^2(t)))\theta^2(t)$ belongs to $NRVZ_{\rho_1}$ with

$$\rho_1 = \frac{-2C_g}{C_\theta} + \frac{2(1-C_\theta)}{C_\theta} = -\frac{C_\theta + 2C_g - 2 + C_\theta}{C_\theta} < 0,$$

and $(g(\psi(\Theta^2(t))))^{q-1} \frac{(\Theta(t))^q}{(\theta(t))^{2-q}}$ belongs to $NRVZ_{\rho_2}$ with

$$\begin{aligned}\rho_2 &= \frac{q}{C_\theta} - \frac{2C_g(q-1)}{C_\theta} - \frac{(2-q)(1-C_\theta)}{C_\theta} \\ &= \frac{C_\theta + 2C_g - 2 + C_\theta(1-q) + 2q(1-C_g)}{C_\theta} > 0.\end{aligned}$$

Thus the results follow by Proposition 2.6 (ii). □

3. Local comparison principles

In this section we give some comparison principles near the boundary. For any $\delta > 0$, we define

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\}, \quad \Gamma_\delta := \{x \in \Omega : d(x) = \delta\}.$$

Since $\partial\Omega \in C^2$, there exists a constant $\delta \in (0, \min\{s_0, \delta_0\})$ which only depends on Ω such that (see, [22, Lemmas 14.16 and 14.17])

$$d \in C^2(\Omega_\delta), \quad |\nabla d(x)| = 1, \quad \Delta d(x) = -(N-1)H(\bar{x}) + o(1), \quad \forall x \in \Omega_\delta, \tag{3.1}$$

where $\bar{x}$ is the nearest point to x on $\partial\Omega$, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at $\bar{x}$.

Next let $v_0 \in C^{2+\alpha}(\Omega) \cap C^1(\bar{\Omega})$ be the unique solution of the problem

$$-\Delta v = 1, \quad v > 0, \quad x \in \Omega, \quad v|_{\partial\Omega} = 0. \tag{3.2}$$

By the Höpf maximum principle in [22], we see that

$$\nabla v_0(x) \neq 0, \quad \forall x \in \partial\Omega \quad \text{and} \quad c_1 d(x) \leq v_0(x) \leq c_2 d(x), \quad \forall x \in \Omega, \tag{3.3}$$

where c_1, c_2 are positive constants. We have the lower bound estimations near the boundary of solutions to (1.1).

Lemma 3.1 (A local comparison principle)**.** *For fixed $\lambda > 0$, let $q \in (0, 2]$, g satisfy* (G1), (G2)*, b satisfy* (B1)*, and let $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ be an arbitrary solution to* (1.1) *and $u_0 \in C^2(\Omega) \cap C(\bar{\Omega})$ be an arbitrary solution to the problem*

$$-\Delta u = b(x)g(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.4}$$

Then there exists a positive constant M_0 such that

$$u_0(x) \leq u_\lambda(x) + M_0 v_0(x), \quad x \in \Omega_\delta, \tag{3.5}$$

where $\delta > 0$ sufficiently small such that

$$u_0(x), \quad u_\lambda(x) \in (0, s_0), \quad x \in \Omega_\delta,$$

where s_0 is given as in (G2).

Proof. First, by $u_\lambda(x) = v_0(x) = u_0(x) = 0$, for all $x \in \partial\Omega$, and

$$u_0, v_0, u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega}), \tag{3.6}$$

we can choose a large M_0 such that

$$u_0(x) \leq u_\lambda(x) + M_0 v_0(x), \quad x \in \Gamma_\delta. \tag{3.7}$$

Now we prove (3.5). Assume the contrary, there exists $x_0 \in \Omega_\delta$ such that

$$u_0(x_0) - (u_\lambda(x_0) + M_0 v_0(x_0)) > 0.$$

It follows that there exists $x_1 \in \Omega_\delta$ such that

$$0 < u_0(x_1) - (u_\lambda(x_1) + M_0 v_0(x_1)) = \max_{x \in \bar{\Omega}_\delta}(u_0(x) - (u_\lambda(x) + M_0 v_0(x))).$$

Then ([22, Theorem 2.2])

$$\Delta(u_0 - (u_\lambda + M_0 v_0))(x_1) \leq 0.$$

On the other hand, we see by (B1), (G1) and (G2) that

$$\begin{aligned}
&\Delta(u_0 - (u_\lambda + M_0 v_0))(x_1) \\
&= -\Delta u_\lambda(x_1) + M_0 + \Delta u_0(x_1)
\end{aligned}$$

$$= b(x_1)(g(u_\lambda(x_1)) - g(u_0(x_1))) + M_0 + \lambda|\nabla u_\lambda(x_1)|^q + \sigma > 0,$$

which is a contradiction. Hence (3.5) holds. □

Next we consider the upper bound estimations near the boundary to u_λ. For $q \in (0,1)$, we have the following lemma.

Lemma 3.2 (A local comparison principle). *For fixed $\lambda > 0$, let g satisfy* (G1), (G2), *b satisfy* (B1), *and let $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ be an arbitrary solution to* (1.1), *$\bar{u}_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta)$ satisfy*

$$-\Delta \bar{u}_\lambda \geq b(x)g(\bar{u}_\lambda) + \lambda|\nabla \bar{u}_\lambda|^q + \sigma, \quad \bar{u}_\lambda > 0, \quad x \in \Omega_\delta, \quad \bar{u}_\lambda|_{\partial\Omega} = 0, \tag{3.8}$$

where $\delta > 0$ sufficiently small such that

$$\bar{u}_\lambda(x), \quad u_\lambda(x) \in (0, s_0), \quad x \in \Omega_\delta,$$

where s_0 is given as in (G2). *Then there exists a positive constant M_0 such that*

$$u_\lambda(x) \leq \bar{u}_\lambda(x) + \lambda M_0 v_0(x), \quad x \in \Omega_\delta. \tag{3.9}$$

Proof. From $u_\lambda(x) = \bar{u}_\lambda(x) = v_0(x) = 0$, for all $x \in \partial\Omega$, and

$$u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega}), \quad v_0 \in C^2(\Omega) \cap C^1(\bar{\Omega}), \quad \bar{u}_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta), \tag{3.10}$$

we can choose a large M_0 such that

$$u_\lambda(x) \leq \bar{u}_\lambda(x) + \lambda M_0 v_0(x), \quad x \in \Gamma_\delta, \tag{3.11}$$

$$M_0^{1-q} \geq \lambda^q \max_{x\in\bar{\Omega}} |\nabla v_0(x)|^q. \tag{3.12}$$

Now we prove (3.9). Assume the contrary, there exists $x_0 \in \Omega_\delta$ such that

$$u_\lambda(x_0) - (\bar{u}_\lambda(x_0) + \lambda M_0 v_0(x_0)) > 0.$$

It follows that there exists $x_1 \in \Omega_\delta$ such that

$$0 < u_\lambda(x_1) - (\bar{u}_\lambda(x_1) + \lambda M_0 v_0(x_1)) = \max_{x\in\bar{\Omega}_\delta}(u_\lambda(x) - (\bar{u}_\lambda(x) + \lambda M_0 v_0(x))).$$

Then ([22, Theorem 2.2])

$$\nabla(u_\lambda - (\bar{u}_\lambda + \lambda M_0 v_0))(x_1) = 0 \quad \text{and} \quad \Delta(u_\lambda - (\bar{u}_\lambda + \lambda M_0 v_0))(x_1) \leq 0.$$

On the other hand, using the basic inequality for $q \in (0,1)$

$$|s_2^q - s_1^q| \leq |s_2 - s_1|^q, \quad \forall s_2, s_1 \geq 0,$$

it follows by (B1), (G1) and (G2) that

$$\begin{aligned}
&\Delta\big(u_\lambda - (\bar{u}_\lambda + \lambda M_0 v_0)\big)(x_1)\\
&= -\Delta \bar{u}_\lambda(x_1) + \lambda M_0 + \Delta u_\lambda(x_1)\\
&\geq b(x_1)(g(\bar{u}_\lambda(x_1)) - g(u_\lambda(x_1))) + \lambda(M_0 + |\nabla \bar{u}_\lambda(x_1)|^q - |\nabla u_\lambda(x_1)|^q)\\
&> \lambda(M_0 - \lambda^q M_0^q |\nabla v_0(x_1)|^q) > 0,
\end{aligned}$$

which is a contradiction. Hence (3.9) holds. □

For $q \in [1,2]$ and an arbitrary positive constant C, by using the following inequality [47, (3.10)]

$$s^q \leq \frac{s^2}{C^{1-q/2}} + C^{q/2}, \quad \forall s \geq 0,$$

we see that

$$-\Delta u_\lambda \leq b(x)g(u_\lambda) + \lambda C^{q/2-1}|\nabla u_\lambda|^2 + \lambda C^{q/2} + \sigma, \; u_\lambda > 0, \; x \in \Omega, \; u_\lambda|_{\partial\Omega} = 0. \tag{3.13}$$

We can choose C such that the problem

$$\begin{gathered}-\Delta\bar{u}_\lambda = b(x)g(\bar{u}_\lambda) + \lambda C^{q/2-1}|\nabla\bar{u}_\lambda|^2 + \lambda C^{q/2} + \sigma,\ \bar{u}_\lambda > 0,\ x \in \Omega,\\ \bar{u}_\lambda|_{\partial\Omega} = 0,\end{gathered} \tag{3.14}$$

has one classical solution $\bar{u}_\lambda$ ([48, Theorem 4.1]).

For a fixed λ, let u_λ and $\bar{u}_\lambda$ be arbitrary solutions to (3.13) and (3.14), we see that the nonlinear changes of variable

$$w_\lambda = \exp(\eta u_\lambda) - 1 \quad \text{and} \quad \bar{w}_\lambda = \exp(\eta\bar{u}_\lambda) - 1$$

transform problems (3.13) and (3.14) into the equivalent problems

$$-\Delta w_\lambda \le b(x)\tilde{g}(w_\lambda) + \eta f(w_\lambda),\ w_\lambda > 0, \quad x \in \Omega, \quad w_\lambda|_{\partial\Omega} = 0, \tag{3.15}$$

and

$$-\Delta\bar{w}_\lambda = b(x)\tilde{g}(\bar{w}_\lambda) + \eta f(\bar{w}_\lambda), \quad \bar{w}_\lambda > 0, \quad x \in \Omega, \quad \bar{w}_\lambda|_{\partial\Omega} = 0, \tag{3.16}$$

respectively. Where

$$\tilde{g}(s) = \eta(1+s)g(\eta^{-1}\ln(1+s)), \quad \eta = \lambda C^{q/2-1}, \tag{3.17}$$

$$f(s) = (\eta C + \sigma)(1+s). \tag{3.18}$$

Lemma 3.3. *For fixed $\lambda > 0$. Let g satisfy* (G1)–(G3). *Then*

(i) $\tilde{g} \in C^1((0,\infty),(0,\infty))$ *and* $\lim_{s\to 0}\tilde{g}(s) = \infty$;

(ii) *when one of the following conditions holds*

(S01) $C_g > 0$;

(S02) $C_g = 0$ *and* $\lambda\limsup_{s\to 0^+}\frac{g(s)}{|g'(s)|} < 1$,

there exists $s_1 > 0$ such that $\tilde{g}'(s) < 0$, $\forall s \in (0,s_1)$;

(iii)
$$\lim_{s\to 0^+}\tilde{g}'(s)\int_0^s \frac{d\tau}{\tilde{g}(\tau)} = -C_g.$$

Proof. By (G1), (i) is obvious. (ii) follows by [50, Lemma 3.1]. (iii) Since g satisfies (G1) and is decreasing on $(0,s_0)$, we see that

$$0 < \int_0^s \frac{d\tau}{g(\tau)} < \frac{s}{g(s)}, \quad \forall s \in (0,s_0),$$

i.e.,

$$0 < g(s)\int_0^s \frac{d\tau}{g(\tau)} < s, \quad \forall s \in (0,s_0), \tag{3.19}$$

$$\lim_{s\to 0^+} g(s)\int_0^s \frac{d\tau}{g(\tau)} = 0. \tag{3.20}$$

Let $\upsilon = \eta^{-1}\ln(1+\tau)$ and $\varsigma = \eta^{-1}\ln(1+s)$. It follows by (3.20) and (G3) that

$$\begin{aligned}&\lim_{s\to 0^+}\tilde{g}'(s)\int_0^s \frac{d\tau}{\tilde{g}(\tau)}\\ &= \lim_{s\to 0^+}\big(g'(\eta^{-1}\ln(1+s)) + \eta g(\eta^{-1}\ln(1+s))\big)\int_0^s \frac{d\tau}{\eta(1+\tau)g(\eta^{-1}\ln(1+\tau))}\\ &= \lim_{\varsigma\to 0^+}\big(g'(\varsigma)\int_0^\varsigma \frac{d\upsilon}{g(\upsilon)} + \eta g(\varsigma)\int_0^\varsigma \frac{d\upsilon}{g(\upsilon)}\big) = -C_g.\end{aligned}$$

□

Thus we have the following comparison principle.

Lemma 3.4 ([50, Lemma 3.1]). *For fixed $\lambda > 0$, let $f \in C([0,\infty),[0,\infty))$, g satisfy* (G1), (G2), *b satisfy* (B1). *Then there exists a positive constant M_0 such that*

$$w_\lambda(x) \le \bar{w}_\lambda(x) + M_0(\eta C + \sigma)v_0(x), \quad x \in \Omega_\delta, \tag{3.21}$$

where $\delta > 0$ sufficiently small such that

$$w_\lambda(x),\ \bar{w}_\lambda(x) \in (0, s_1), \quad x \in \Omega_\delta,$$

where s_1 is as in Lemma 3.3.

4. Boundary behavior

In this section we prove Theorems 1.1–1.3. First we have the statement in [30, Theorem 1.1] with $a \equiv 1$ in Ω.

Lemma 4.1. *For a fixed $\lambda > 0$, let $f \in C([0,\infty),[0,\infty))$, g satisfy* (G1)–(G3), *and let b satisfy* (B1), (B2). *If*

$$C_\theta + 2C_g > 2, \tag{4.1}$$

then for any classical solution V_λ to the problem

$$-\Delta V = b(x)g(V) + \lambda a(x)f(V), \quad V > 0, \quad x \in \Omega, \quad V|_{\partial\Omega} = 0, \tag{4.2}$$

it holds that

$$\xi_1^{1-C_g} \le \liminf_{d(x)\to 0} \frac{V_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \limsup_{d(x)\to 0} \frac{V_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \xi_2^{1-C_g}, \tag{4.3}$$

where ψ is the solution to (1.2), *ξ_1 and ξ_2 are given as in* (1.8). *In particular,* (i) *and* (ii) *in Theorem 1.1 hold.*

Next we have the statement in [51, Theorems 1.2] with $a \equiv 1$ in Ω.

Lemma 4.2. *For a fixed $\lambda > 0$, let $f \in C([0,\infty),[0,\infty))$, g satisfy* (G1)–(G3), *and let b satisfy* (B1). *If b satisfies* (B3), *then any classical solution V_λ to* (4.2) *satisfies* (1.10).

Next we have the statement in [51, Theorems 1.3] with $a \equiv b$ in Ω.

Lemma 4.3. *For a fixed $\lambda > 0$, let $f \in C([0,\infty),[0,\infty))$, g satisfy* (G1) *and $g(s) = s^{-\gamma} + \mu s^p$, $s \in (0, s_0)$ for some $s_0 > 0$ and $\gamma, p, \mu > 0$, and let b satisfy* (B1). *If b satisfies* (B4), *then any classical solution V_λ to* (4.2) *satisfies* (1.12).

Remark 4.4. Obviously, when $f \equiv 0$ on $[0,\infty)$, a solution V_λ to (4.2) is a solution to (3.4).

Proof of Theorem 1.1. Let $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ be an arbitrary solution to (1.1). Using (3.3), Lemmas 3.1, 3.3, 3.4, 4.1 and 2.14 (iv), we obtain that for $q \in (0,2]$,

$$\xi_1^{1-C_g} \le \liminf_{d(x)\to 0} \frac{u_0(x)}{\psi(\Theta^2(d(x)))} \le \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))}, \tag{4.4}$$

and for $q \in [1,2]$,

$$\limsup_{d(x)\to 0} \frac{w_\lambda(x)}{\psi_1(\Theta^2(d(x)))} \le \limsup_{d(x)\to 0} \frac{\bar{w}_\lambda(x)}{\psi_1(\Theta^2(d(x)))} \le \xi_2^{1-C_g}. \tag{4.5}$$

where $w_\lambda(x)=\exp(\eta u_\lambda(x))-1$, $\bar w_\lambda(x)=\exp(\eta\bar u_\lambda(x))-1$, $\eta=\lambda C^{q/2-1}$, ψ_1 is the solution to the problem

$$\int_0^{\psi_1(t)}\frac{ds}{\tilde g(s)}=t,\quad \forall t>0, \tag{4.6}$$

and $\tilde g$ is given in (3.17).

From

$$\psi(t)=\eta^{-1}\ln(1+\psi_1(t)),\quad \forall t>0,$$
$$\exp(\eta s)-1\cong\eta s\quad \text{as } s\to 0,$$

it follows that $q\in[1,2]$,

$$\lim_{d(x)\to 0}\sup\frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))}\le\xi_2^{1-C_g}. \tag{4.7}$$

Thus (1.7) holds for $q\in[1,2]$.

Next we structure an appropriate supersolution near the boundary to (1.1) in the case $q\in(0,1)$. Let $\varepsilon\in(0,b_1/4)$ and let

$$\tau_1=\xi_2+2\varepsilon\xi_2/b_2,$$

where ξ_2 is given in (1.8). It follows that $\xi_2<\tau_1<2\xi_2$, $\lim_{\varepsilon\to0}\tau_1=\xi_2$, and

$$-4\tau_1C_g+2\tau_1(2-C_\theta)+b_2=-2\varepsilon. \tag{4.8}$$

By (B2), (3.1), Lemmas 2.12, 2.14 and 2.15, we see that

$$\lim_{d(x)\to 0}\tau_1\Theta^2(d(x))g'(\psi(\tau_1\Theta^2(d(x))))=-C_g,$$
$$\lim_{d(x)\to0}\Big(\frac{\theta'(d(x))\Theta(d(x))}{\theta^2(d(x))}+1+\frac{\Theta(d(x))}{\theta(d(x))}\Delta d(x)\Big)=2-C_\theta,$$
$$\lim_{d(x)\to0}\Big(\lambda\tau_1^q2^q\frac{(\Theta(d(x)))^q}{(\theta(d(x)))^{2-q}(g(\psi(\tau_1\Theta^2(d(x)))))^{1-q}}+\frac{\sigma}{\theta^2(d(x))g(\psi(\tau_1\Theta^2(d(x))))}\Big)=0,$$
$$\limsup_{d(x)\to0}\frac{b(x)}{\theta^2(d(x))}\le b_2.$$

Thus, corresponding to ε, s_0 and δ, where s_0 is given in (G2) and δ in Lemma 3.1, respectively, there is $\delta_\varepsilon\in(0,\delta)$ sufficiently small such that for $x\in\Omega_{\delta_\varepsilon}$,

$$\bar u_\varepsilon=\psi(\tau_1\Theta^2(d(x)))$$

satisfies

$$\bar u_\varepsilon(x)\in(0,s_0),\quad x\in\Omega_{\delta_\varepsilon}, \tag{4.9}$$

and

$$\begin{aligned}
&\Delta\bar u_\varepsilon(x)+b(x)g(\bar u_\varepsilon(x))+\lambda|\bar u_\varepsilon(x)|^q+\sigma\\
&=\psi''(\tau_1\Theta^2(d(x)))(2\tau_1\Theta(d(x))\theta(d(x)))^2+2\tau_1\psi'(\tau_1\Theta^2(d(x)))\\
&\quad\times\big(\theta^2(d(x))+\Theta(d(x))\theta'(d(x))+\Theta(d(x))\theta(d(x))\Delta d(x)\big)\\
&\quad+b(x)g(\psi(\tau_1\Theta^2(d(x))))+\lambda(2\tau_1)^q(\theta(d(x))\Theta(d(x)))^q(g(\psi(\tau_1\Theta^2(d(x)))))^q+\sigma\\
&=g(\psi(\tau_1\Theta^2(d(x))))\theta^2(d(x))\Big(4\tau_1\tau_1\Theta^2(d(x))g'(\psi(\tau_1\Theta^2(d(x))))
\end{aligned}$$

$$
\begin{aligned}
&+ 2\tau_1\big(\frac{\theta'(d(x))\Theta(d(x))}{\theta^2(d(x))} + 1 + \frac{\Theta(d(x))}{\theta(d(x))}\Delta d(x)\big) + \frac{b(x)}{\theta^2(d(x))} \\
&+ \lambda\tau_1^q 2^q \frac{(\Theta(d(x)))^q}{(\theta(d(x)))^{2-q}(g(\psi(\tau_1\Theta^2(d(x)))))^{1-q}} + \frac{\sigma}{\theta^2(d(x))g(\psi(\tau_1\Theta^2(d(x))))}\Big) \\
&\le 0;
\end{aligned}
$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of equation (1.1) in $\Omega_{\delta_\varepsilon}$.

Let $u_\lambda \in C(\bar{\Omega}) \cap C^{2+\alpha}(\Omega)$ be an arbitrary classical solution to (1.1). By Lemma 3.2, we see that there exists $M_0 > 0$ such that for $x \in \Omega_{\delta_\varepsilon}$,

$$u_\lambda(x) \le \bar{u}_\varepsilon(x) + \lambda M_0 v_0(x),$$

i.e.,

$$\frac{u_\lambda(x)}{\psi(\tau_1\Theta^2(d(x)))} \le 1 + \lambda M_0 \frac{v_0(x)}{\psi(\tau_1\Theta^2(d(x)))}, \quad x \in \Omega_{\delta_\varepsilon}.$$

It follows by (3.3) and Lemma 2.14 (iv) that

$$\lim_{d(x)\to 0}\sup \frac{u_\lambda(x)}{\psi(\tau_2\Theta^2(d(x)))} \le 1.$$

Using Lemma 2.14 again, we have

$$\lim_{d(x)\to 0} \frac{\psi(\tau_1\Theta^2(d(x)))}{\psi(\Theta^2(d(x)))} = {\tau_1}^{1-C_g}.$$

Moreover, since $C_\theta > 0$, by (2.9) and Lemma 2.14, we obtain that

$$\lim_{d(x)\to 0} \frac{\psi(\Theta^2(d(x)))}{\psi(d^2(x)\theta^2(d(x)))} = C_\theta^{2(1-C_g)}.$$

Thus letting $\varepsilon \to 0$, we have

$$\lim_{d(x)\to 0}\sup \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} \le \xi_2^{1-C_g}. \tag{4.10}$$

Combining (4.10) with (4.4), we obtain (1.7). In particular, when $C_g = 1$, u_λ satisfies

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\Theta^2(d(x)))} = 1;$$

and, when $C_g < 1$ and $b_1 = b_2 = b_0$ in (b1), u_λ satisfies

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(d^2(x)\theta^2(d(x)))} = (\xi_{01}C_\theta^2)^{1-C_g}.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ be an arbitrary solution to (1.1). For $q \in [1, 2]$, in a similar way as that of Theorem 1.1, by using (3.1), (3.3), Lemmas 3.1, 3.3, 3.4, 4.2 and 2.14 (v), we can show that Theorem 1.2 holds.

Next we construct an appropriate supersolution near the boundary to (1.1) in the case of $q \in (0, 1)$. Let $\varepsilon \in (0, b_1/4)$ and let $\tau_2 = b_2 + 2\varepsilon$. It follows that

$$b_1/2 < \tau_2 < 2b_2.$$

By (B3), (G1), (3.1), (3.3), Propositions 2.6 (iii) and 2.11, and Lemma 2.14, we derive that

$$\sigma \lim_{d(x)\to 0} \frac{d^2(x)}{\hat{L}(d(x))g(\psi(\tau_2 h_1(d(x))))} = 0,$$

$$\begin{gathered}
\lambda\tau_2^q \lim_{d(x)\to 0}\big((d(x))^{2-q}(\hat{L}(d(x)))^{q-1}(g(\psi(\tau_2 h_1(d(x)))))^{1-q}\big)=0,\\
\lim_{d(x)\to 0}\tau_2 h_1(d(x))g'(\psi(\tau_2 h_1(d(x))))=-C_g,\\
\lim_{d(x)\to 0}\frac{\hat{L}(d(x))}{h_1(d(x))}=0,\quad \limsup_{d(x)\to 0}\frac{b(x)}{d^{-2}(x)\hat{L}(d(x))}\le b_2,\\
\lim_{d(x)\to 0}\tau_2\big(1-\frac{d(x)\hat{L}'(d(x))}{\hat{L}(d(x))}\big)=\tau_2,\quad \lim_{d(x)\to 0}\tau_2 d(x)\Delta d(x)=0.
\end{gathered}$$

Thus, corresponding to ε, s_0 and δ, where s_0 is given as in (G2) and δ in Lemma 3.1, respectively, there is $\delta_\varepsilon\in(0,\delta)$ sufficiently small such that for $x\in\Omega_{\delta_\varepsilon}$

$$\bar{u}_\varepsilon=\psi(\tau_2 h_1(d(x)))$$

satisfies (4.9) and

$$\begin{aligned}
&\Delta\bar{u}_\varepsilon(x)+b(x)g(\bar{u}_\varepsilon(x))+\lambda|\bar{u}_\varepsilon(x)|^q+\sigma\\
&=\psi''(\tau_2 h_1(d(x)))\tau_2^2 h_1^2(d(x))+\psi'(\tau_2 h_1(d(x)))\big(\tau_2 h_1''(d(x))+\tau_2^2 h_1'(d(x))\Delta d(x)\big)\\
&\quad+b(x)g(\psi(\tau_2 h(d(x))))+\lambda\tau^q\big(\frac{\hat{L}(d(x))}{d(x)}\big)^q(g(\psi(\tau_2 h_1(d(x)))))^q+\sigma\\
&=(d(x))^{-2}\hat{L}(d(x))g(\psi(\tau_2 h_1(d(x))))\\
&\quad\times\Big(\tau_2\big(\tau_2 h_1(d(x))g'(\psi(\tau_2 h_1(d(x))))\big)\frac{\hat{L}(d(x))}{h_1(d(x))}-\tau_2\Big(1-\frac{d(x)\hat{L}'(d(x))}{\hat{L}(d(x))}\Big)\\
&\quad+\tau_2 d(x)\Delta d(x)+\frac{b(x)}{d^{-2}(x)\hat{L}(d(x))}+\sigma\frac{d^2(x)}{\hat{L}(d(x))}\frac{1}{g(\psi(\tau_2 h_1(d(x))))}\\
&\quad+\lambda\tau_2^q(d(x))^{2-q}(\hat{L}(d(x)))^{q-1}\frac{1}{(g(\psi(\tau_2 h_1(d(x)))))^{1-q}}\Big)\\
&\le 0,
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution to equation (1.1) in $\Omega_{\delta_\varepsilon}$.

The rest of the proof is the same as that Theorem 1.1 and is omitted. □

Proof of Theorem 1.3. Let $u_\lambda\in C^2(\Omega)\cap C(\bar{\Omega})$ be an arbitrary solution to (1.1).

For $q\in[1,2]$, in a similar way as that of Theorem 1.1, by using (3.1), (3.3), Lemmas 3.1, 3.3, 3.4 and 4.3, we can show that Theorem 1.3 holds.

Next we construct an appropriate supersolution near the boundary to (1.1) in the case of $q\in(0,1)$. Let $\varepsilon\in(0,1)$. Let τ_3 be the unique positive solution to the problem

$$b_2 t^{-\gamma}-\frac{t}{1+\gamma}=-2\varepsilon,$$

it follows by the properties of the function $b_i t^{-\gamma}-\frac{t}{1+\gamma}$ $(i=1,2)$ that

$$(b_1(1+\gamma))^{1/(1+\gamma)}<\tau_3<\zeta_0,\quad \lim_{\varepsilon\to 0}\tau_3=(b_2(1+\gamma))^{1/(1+\gamma)},$$

where ζ_0 is the unique positive solution to the problem

$$b_2 t^{-\gamma}-\frac{t}{1+\gamma}=-2.$$

Since $\hat{L}$ and $\int_t^\eta \frac{\hat{L}(\tau)}{\tau}d\tau$ are slowly varying at zero, we see that by (3.3), (B4), Propositions 2.6 and 2.11 that

$$\lim_{d(x)\to 0}\frac{d(x)}{\hat{L}(d(x))}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau-\frac{1}{1+\gamma}\hat{L}(d(x))\Big)\Delta d(x)=0,$$

$$\lim_{d(x)\to 0}\frac{\gamma}{(1+\gamma)^2}\frac{\hat{L}(d(x))}{\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau}=0,\quad \lim_{d(x)\to 0}\frac{1}{1+\gamma}\frac{d(x)\hat{L}'(d(x))}{\hat{L}(d(x))}=0,$$

$$\lim_{d(x)\to 0}\sup\frac{b(x)}{(d(x))^{\gamma-1}\hat{L}(d(x))}\le b_2,$$

and, using (G1) and (B4), there holds

$$\begin{aligned}
&\sigma\tau_3^{-1}\lim_{d(x)\to 0}\frac{d(x)}{\hat{L}(d(x))}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{1/(1+\gamma)}=0;\\
&\mu\tau_3^{p-1}\lim_{d(x)\to 0}(d(x))^{\gamma+p}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{(\gamma+p)/(1+\gamma)}=0;\\
&\lambda\tau_3^{q-1}\lim_{d(x)\to 0}\Big(d(x)\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{(q+\gamma)/(1+\gamma)}\\
&\times\Big|1-\frac{1}{1+\gamma}\hat{L}(d(x))\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{-1}\Big|^q\Big)=0.
\end{aligned}$$

Thus, corresponding to ε, s_0 and δ, where s_0 is given as in (G2) and δ in Lemma 3.1, respectively, there is $\delta_\varepsilon\in(0,\delta)$ sufficiently small such that for $x\in\Omega_{\delta_\varepsilon}$

$$\bar{u}_\varepsilon=\tau_3 d(x)\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{1/(1+\gamma)}$$

satisfies (4.2) and

$$\begin{aligned}
&\Delta\bar{u}_\varepsilon(x)+b(x)((\bar{u}_\varepsilon(x))^{-\gamma}+\mu(\bar{u}_\varepsilon(x))^p)+\lambda|\bar{u}_\varepsilon(x)|^q+\sigma\\
&=\tau_3\frac{\hat{L}(d(x))}{d(x)}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{-\gamma/(1+\gamma)}\Big(-\frac{1}{1+\gamma}-\frac{\gamma}{(1+\gamma)^2}\frac{\hat{L}(d(x))}{\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau}\\
&\quad-\frac{1}{1+\gamma}\frac{d(x)\hat{L}'(d(x))}{\hat{L}(d(x))}+\frac{d(x)}{\hat{L}(d(x))}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau-\frac{1}{1+\gamma}\hat{L}(d(x))\Big)\Delta d(x)\\
&\quad+\frac{b(x)}{(d(x))^{\gamma-1}\hat{L}(d(x))}\Big(\tau_3^{-\gamma-1}+\mu\tau_3^{p-1}(d(x))^{\gamma+p}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{(\gamma+p)/(1+\gamma)}\Big)\\
&\quad+\sigma\tau_3^{-1}\frac{d(x)}{\hat{L}(d(x))}\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{\gamma/(1+\gamma)}\\
&\quad+\lambda\tau_3^{q-1}d(x)\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{(q+\gamma)/(1+\gamma)}\Big|1-\frac{1}{1+\gamma}\hat{L}(d(x))\Big(\int_{d(x)}^{\eta}\frac{\hat{L}(\tau)}{\tau}d\tau\Big)^{-1}\Big|^q\Big)\\
&\le 0,
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of (1.1) in $\Omega_{\delta_\varepsilon}$. The conclusion follows as in the proof of Theorem 1.1. □

4.1. **Acknowledgments.** This work is supported in part by NNSF of China under grant 11301301.

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