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# SINGULAR LIMIT SOLUTIONS FOR 4-DIMENSIONAL STATIONARY KURAMOTO-SIVASHINSKY EQUATIONS WITH EXPONENTIAL NONLINEARITY

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Abstract. Let $\Omega$ be a bounded domain in $\mathbb{R}^4$ with smooth boundary, and let $x_1, x_2, \dots, x_m$ be points in $\Omega$. We are concerned with the singular stationary non-homogenous Kuramoto-Sivashinsky equation

$$\Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 = \rho^4 f(u),$$

where $f$ is a function that depends only the spatial variable. We use a nonlinear domain decomposition method to give sufficient conditions for the existence of a positive weak solution satisfying the Dirichlet-like boundary conditions $u = \Delta u = 0$, and being singular at each $x_i$ as the parameters $\lambda, \gamma$ and $\rho$ tend to 0. An analogous problem in two-dimensions was considered in [2] under condition (A1) below. However we do not assume this condition.

## 1. Introduction and statement of results

First, we introduce a model arising in the growth of amorphous surfaces which is a partial differential equation, called the non-homogenous Kuramoto-Sivashinsky (KS) equation,

$$\partial_t u + \Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 = f(u).$$

on $\mathbb{R}^d$ with $d \geq 1$, where $\lambda$ and $\gamma$ are real parameters and $f(u)$ is a nonlinear function. The Kuramoto-Sivashinsky equation was independently created by Kuramoto and Tsuzuki [14], and by Sivashinsky [25] in the study of a reaction-diffusion system and flame front propagation, respectively. This equation is also found in the study of $2D$ Kolmogorov fluid flows [26]. This form of the Kuramoto-Sivashinsky equation is sometimes called the integrated version of the Kuramoto-Sivashinsky equations (KSE), which arises in several models for surface growth. Most mathematical results concern the case $n \leq 3$ and essentially for $n = 1$. Subject to appropriate initial and boundary conditions has been introduced in [13] and some reference therein in studying phase turbulence and the flame front propagation in combustion theory. This type of version equation is suggested in [21, 22] (and some reference therein) as a phenomenological model for the growth of an amorphous

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surface ($Z_{r65}Al_{7,5}, Cu_{27,5}$). Winkler and Stein [28] used Rothe's method to verify the existence of a global weak solution, this result has been recently extended by Winkler [30] to the two-dimensional case of (KS) equation, using energy type estimates for $\int e^u dx$.

The non homogeneous Kuramoto-Sivashinsky equation with exponential nonlinearity is a generalization of the fourth-order one-dimensional semilinear parabolic equation arises also in several other models for surface growth see for example [9] for the equation

$$u_t + u_{xxxx} - \beta[(u_x)^3]_x = e^u$$

with a parameter $\beta \geq 0$, which is a model equation from explosion-convection theory of which the fourth-order extension of the Frank-Kamenetskii equation,

$$u_t + u_{xxxx} = e^u$$

(a solid fuel model) is a limiting case.

Recently Chen and McKenna [6] suggested to investigate the equation

$$u_{xxxx} + cu_{xx} = e^u, \tag{1.1}$$

where they give some existence and nonexistence results. In a note on an exponential semilinear equation of the fourth order, Mugnai [19] considered the related problem to (1.1). More precisely he considered, without non linear gradient term, the problem

$$\begin{gathered} \Delta^2 u + c\Delta u = b(e^u - 1) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

where $\Omega$ is a bounded and smooth domain of $\mathbb{R}^n$, $c \in \mathbb{R}$ and $b \in \mathbb{R}$. The author prove some existence and nonexistence results for (1.2) via variational techniques. Such equations may occur while studying traveling waves in suspension bridges. For more general problem see [24], for the Navier boundary-value problem

$$\begin{gathered} \Delta^2 u + c\Delta u = f(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

in $\mathbb{R}^n$, $n \geq 4$ and $f$ is non linear growth function. In conformal dimensional i.e $n = 4$ and $f$ has the subcritical (exponential) growth on $\Omega$, i.e.,

$$\lim_{t \to +\infty} \frac{|f(x,t)|}{\exp(\alpha t)} = 0$$

uniformly on $x \in \Omega$ for all $\alpha > 0$ and in some cases and hypothesis and using Adams inequality, (see [15]), for the fourth-order derivative, namely,

$$\sup_{\{u \in H^2(\Omega) \cap H_0^1(\Omega), \|u\| \leq 1\}} \int_\Omega e^{32\pi^2 u^2} dx \leq C|\Omega|,$$

the authors show that the problem (1.3) has at least two nontrivial solutions (for more details see Theorem 1.3 in [15]) or infinitely many nontrivial solutions (for more details see [15, Theorem 1.4]).

A fundamental goal in the study of non-linear initial boundary value problems involving partial differential equations is to determine whether solutions to a given equation develop a singularity. Resolving the issue of blow-up is important, in part because it can have bearing on the physical relevance and validity of the underlying model. However, determining the answer to this question is notoriously difficult for a wide range of equations such fourth order equation like stationary non homogenous Kuramoto-Sivashinsky equation with strong nonlinearity like exponential $e^u$. One route is to try to simplify or modify the boundary conditions in an attempt to gain evidence for or against the occurrence of blow-up. A second route is to modify the equations in some way, and to study the modified equations with the hope of gaining insight into the blow-up of solutions to the original equations: see problems (1.4)-(1.5) bellow and the effect of the presence of the second-order backward diffusion term $-\gamma\Delta u$ and the nonlinear term $-\lambda|\nabla u|^2$ in (1.4). The occurrence and type of blow-up depends on the parameters $\lambda, \gamma$ and the domain. Studding this type of equations, we will answer for different basic questions. We concentrate next on the analysis of the main questions raised in the study of blow-up for such equations. This list can be suitably adapted to other singularity formation problems. We will examine several case studies related to such approaches where basic list includes the questions of, where and how. We propose here an expanded list of three items: (i) Does blow-up occur? (ii) Where? (iii) How? For the first question, the blow-up problem is properly formulated only when a suitable class of solutions is chosen for all solutions in the given class or only for some solutions (which should be identified) or other kinds of generalized solutions can be more natural to a given problem and which equations and problems do exhibit blow-up. The second question, is concerned with where finite number of points, or regional blow-up, are localized: The set of blow-up is defined by

$$S := \{x \in \Omega : \exists x_n \to x \text{ such that } u_n(x_n) \to +\infty\}.$$

For the third question, we are concerned just by calculate the rate at which solution diverges as $x$ approaches to the set $S$ of blow-up point and to calculate the blow-up profiles as limits of solution at the non-blowing points. A major aim of the present work is to provide examples which demonstrate that one must be extremely cautious in generalizing claims about the blow-up of problems studied in idealized settings to claims about the blow-up of the original problem and to the nonlinearity of a problem which can cause the formation of a singularity, where no such singularity is present in the unaltered equation. However, many such studies have tried to search for singularities of the solutions of the equations in the setting of different types of boundary conditions like periodic boundary conditions related to the solution of Kuramoto-Sivashinsky equation. The question of blow up of of solutions of stationary Kuramoto-Sivashinsky equation is still an open question in dimensions fourth and in higher cases.

For the stationary Kuramoto-Sivashinsky equation, the reader is referred to [7] and some refrence therein, where the author give some explicit estimates for the $L^\infty$-norm of the periodic solutions of the time-independent non homogeneous Kuramoto-Sivashinsky equation

$$\Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 = f(u)$$

in $\mathbb{R}^n$ and its dependence on $f(u)$. In particular, they give an estimate of the Michelson's upper bound of all periodic solutions on space $x$ in $\mathbb{R}$ of the time-independent homogeneous Kuramoto-Sivashinsky equation which is the case with non linearity exponential i.e: a solutions of such equation under steady with $f(u) = e^u$ is invariant under the group of translations $a \to u(\cdot + a)$.

One of the purposes of this article is to present a rather efficient method to solve such singularly perturbed problems of the time-independent Kuramoto-Sivashinsky equation called also the integrated version of the homogeneous steady state KSE. This method has already been used successfully in geometric context (constant mean curvature surfaces, constant scalar curvature metrics, extremal Kähler metrics, manifolds with special holonomy, ...) The techniques developed and used here are inspired by the work of [1]. Motivated by the above discussion, we felt that, given the interest in singular perturbation problems, it was worth illustrating this on the non Homogenous stationary Kuramoto-Sivashinsky equation: $\Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 = \rho^2 f(u)$ in $\Omega \subset \mathbb{R}^4$ under the physical Dirichlet-like boundary conditions $u = \Delta u = 0$ on $\partial\Omega$, given by the following problem.

Let $\Omega \subset \mathbb{R}^4$ be a regular bounded open domain in $\mathbb{R}^4$. We are interested in the positive solution of

$$\begin{gathered} \Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 = \rho^4 e^u \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

which is singular at each point $x_i$ as the parameters $\lambda, \gamma$ and $\rho$ tend to 0. This problem in some way a generalization of a fourth order Liouville problem

$$\begin{gathered} \Delta^2 u = \rho^4 e^u \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.5}$$

in the case $(\gamma, \lambda) = (0, 0)$, when the parameters $\rho$ tends to 0. (See for example [1]). Also problem (1.4) can be considered as a higher order counterpart of the problem

$$\begin{gathered} -\Delta u - \lambda|\nabla u|^2 = \rho^2 e^u \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.6}$$

when the parameter $\rho$ tends to 0 ($\rho \sim \epsilon$ as $\epsilon$ tends to 0). This is a particular case of non homogenous viscous Hamilton-Jacobian equation [27],

$$\begin{gathered} \partial_t u - \Delta u - \lambda|\nabla u|^p = f(u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^p$, $p \geq 1$.

Problem (1.6) was studied by Baraket et al. in [2] for the existence of $v_{\varepsilon,\lambda}$ a sequence of solutions which converges to some singular function as the parameters $\varepsilon$ and $\lambda$ tend to 0, under the assumption

(A1) If $0 < \varepsilon < \lambda$, then $\lambda^{1+\delta/2}\varepsilon^{-\delta} \to 0$ as $\lambda \to 0$, for any $\delta \in (0, 1)$.

In particular, if we take $\lambda = \mathcal{O}(\varepsilon^{2/3})$, then condition (A1) is satisfied. With assumption (A1), problem (1.6), can be treated as a perturbation of the Liouville equation

$$-\Delta u = \rho^2 e^u \quad \text{in } \Omega \subset \mathbb{R}^2.$$

This last equation was studied by Baraket and frank in [3] as $\rho$ tends to 0. As observed by Ren and Wei [23], problem (1.6), can be reduced to a problem without gradient term. Indeed, if $u$ is a solution of (1.6), then the function

$$w = (\lambda\rho^2 e^u)^\lambda,$$

satisfies

$$\begin{gathered} -\Delta w = w^{\frac{\lambda+1}{\lambda}} \quad \text{in } \Omega \\ w = (\lambda\rho^2)^\lambda \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

since the exponent $p = (\lambda + 1)/\lambda$ tends to infinity as $\lambda$ tends to 0, see also [8].

Note that Ghergu and Radulescu [10] studied a more general problem on a domain $\Sigma \subset \mathbb{R}^n$, $n \geq 2$:

$$\begin{gathered} -\Delta u - \lambda |\nabla u|^a = g(u) + \mu f(x,u) \quad \text{in } \Sigma \\ u = 0 \quad \text{on } \partial\Sigma, \end{gathered} \tag{1.8}$$

with $0 < a \leq 2$, $\lambda, \mu > 0$ and some assumptions on $f$ and $g$. Problems of the type (1.8) arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically conducting materials. See also [11]. It includes also some simple prototype models from boundary-layer theory of viscous fluids [31].

The question we would like to study is concerned with the existence of other branches of solutions of (1.4) as $\rho, \lambda$ and $\gamma$ tend to 0. To describe our result, let us denote by $G(x,\cdot)$ the solution of

$$\begin{gathered} \Delta^2 G(x,\cdot) = 64\pi^2 \delta_x \quad \text{in } \Omega \\ G(x,\cdot) = \Delta G(x,\cdot) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.9}$$

It is easy to check that the function

$$R(x,y) := G(x,y) + 8\log|x-y| \tag{1.10}$$

is a smooth function.

We define

$$W(x^1,\dots,x^m) := \sum_{j=1}^m R(x^j,x^j) + \sum_{j\neq\ell} G(x^j,x^\ell). \tag{1.11}$$

In dimension 4, Wei [29], studied the behavior of solutions to the nonlinear eigenvalue problem for the biharmonic operator $\Delta^2$ in $\mathbb{R}^4$,

$$\begin{gathered} \Delta^2 u = \lambda\, f(u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.12}$$

and $u^*$ the solution of

$$\begin{gathered} \Delta^2 u^* = 64\pi^2 \sum_{i=1}^m \delta_{x^i} \quad \text{in } \Omega \\ u^* = \Delta u^* = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.13}$$

The author proved the following result.

**Theorem 1.1** ([29]). *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^4$ and $f$ a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \quad \text{and} \quad e^{-u}\int_0^u f(s)ds \text{ tend to } 1, \text{ as } u \to +\infty. \tag{1.14}$$

*For $u_\lambda$ solution of (1.12), denote by $\Sigma_\lambda = \lambda\int_\Omega f(u_\lambda)dx$. Then many cases occur:*

- (i) $\Sigma_\lambda \to 0$ *therefore,* $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ *as* $\lambda \to 0$.
- (ii) $\Sigma_\lambda \to +\infty$ *then* $u_\lambda \to +\infty$ *as* $\lambda \to 0$.
- (iii) $\Sigma_\lambda \to 64\pi^2 m$, *for some positive integer $m$. Then the limiting function $u^* = \lim_{\lambda\to 0} u_\lambda$ has $m$ blow-up points, $\{x^1,\dots,x^m\}$, where $u_\lambda(x^i) \to +\infty$ as $\lambda \to 0$. Moreover, $(x^1,\dots,x^m)$ is a critical point of $W$.*

Our main result reads as follows.

**Theorem 1.2.** *Let $\alpha \in (0,1)$ and $\Omega$ be an open smooth bounded domain of $\mathbb{R}^4$. Assume that $(x^1, \dots, x^m) \in \Omega^m$ is a* nondegenerate *critical point of $W$, then there exist $\rho_0 > 0$, $\lambda_0 > 0$, $\gamma_0 > 0$ and $\{u_{\rho,\lambda,\gamma}\}$ with $0 < \rho < \rho_0$, $0 < \lambda < \lambda_0$, $0 < \gamma < \gamma_0$, a one parameter family of solutions of* (1.4)*, such that*

$$\lim_{\rho\to 0,\, \lambda\to 0,\, \gamma\to 0} u_{\rho,\lambda,\gamma} = \sum_{j=1}^{m} G(x_j, \cdot)$$

*in* $\mathcal{C}^{4,\alpha}_{\rm loc}(\Omega - \{x^1, \dots, x^m\})$.

Our result reduces the study of nontrivial branches of solutions of (1.4) to the search for critical points of the function $W$ defined in (1.11). Observe that the assumption on the nondegeneracy of the critical point is a rather mild assumption since it is certainly fulfilled for generic choice of the open domain $\Omega$.

Semilinear equations involving fourth-order elliptic operator and exponential nonlinearity appear naturally in conformal geometry and in particular in the prescription of the so called $Q$-curvature on 4-dimensional Riemannian manifolds [4], [5]

$$Q_g = \frac{1}{12}\big( - \Delta_g S_g + S_g^2 - 3|\operatorname{Ric}_g|^2\big)$$

where $\operatorname{Ric}_g$ denotes the Ricci tensor and $S_g$ is the scalar curvature of the metric $g$. Recall that the $Q$-curvature changes under a conformal change of metric

$$g_w = e^{2w} g,$$

according to

$$P_g w + 2Q_g = 2\tilde{Q}_{g_w} e^{4w} \tag{1.15}$$

where

$$P_g := \Delta_g^2 + \delta\Big(\frac{2}{3} S_g I - 2\operatorname{Ric}_g\Big) d \tag{1.16}$$

is the Panietz operator, which is an elliptic 4-th order partial differential operator [5] and which transforms according to

$$e^{4w} P_{e^{2w} g} = P_g, \tag{1.17}$$

under a conformal change of metric $g_w := e^{2w} g$. In the special case where the manifold is the Euclidean space, the Panietz operator is simply given by

$$P_{g_{eucl}} = \Delta^2$$

in which case (1.15) reduces to

$$\Delta^2 w = \tilde{Q} e^{4w}$$

the solutions of which give rise to conformal metric $g_w = e^{2w} g_{eucl}$ whose $Q$-curvature is given by $\tilde{Q}$. There is by now an extensive literature about this problem and we refer to [5] and [16] for references and recent developments.

We briefly describe the plan of the paper : In Section 2 we discuss rotationally symmetric solutions of (1.4). In Section 3 we study the linearized operator about the radially symmetric solution defined in the previous section. In Section 4, we recall some Known results about the analysis of the bi-Laplace operator in weighted spaces. Both section strongly use the b-operator which has been developed by Melrose [18] in the context of weighted Sobolev spaces and by Mazzeo [17] in the context of weighted Hölder spaces (see also [20]).

A first nonlinear problem is studied in Section 5 where the existence of an infinite dimensional family of solutions of (1.4) which are defined on a large ball and which are close to the rotationally symmetric solution is proven. In Section 6, we prove the existence of an infinite dimensional family of solutions of (1.4) which are defined on $\Omega$ with small ball removed. Finally, in Section 7, we show how elements of these infinite dimensional families can be connected to produce solutions of (1.4) described in Theorem 1.2. In Section 7, we patch these pieces, in the two last sections, together via a nonlinear version of the Cauchy data matching. Throughout the paper, the symbol $c_\kappa > 0$ (which can depend only on $\kappa$) denotes always a positive constant independent of $\varepsilon, \lambda$ and $\gamma$ which might change from one line to another.

## 2. Rotationally symmetric solutions

We first describe the rotationally symmetric approximate solutions of

$$\Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 = \rho^4 e^u \tag{2.1}$$

in $\mathbb{R}^4$ which will play a central role in our analysis. For this raison given $\varepsilon > 0$, we define

$$u_\varepsilon(x) := 4\log(1+\varepsilon^2) - 4\log(\varepsilon^2 + |x|^2).$$

which is clearly a solution of

$$\Delta^2 u - \rho^4 e^u = 0, \tag{2.2}$$

when

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}. \tag{2.3}$$

Let us notice that (2.2) is invariant under some dilation in the following sense: If $u$ is a solution of (2.2) and $\tau > 0$, then $u(\tau\cdot) + 4\log\tau$ is also a solution of (2.2). With this observation in mind, we define, for all $\tau > 0$

$$u_{\varepsilon,\tau}(x) := 4\log(1+\varepsilon^2) + 4\log\tau - 4\log(\varepsilon^2 + \tau^2|x|^2). \tag{2.4}$$

## 3. A linear fourth-order elliptic operator on $\mathbb{R}^4$

We define the linear fourth-order elliptic operator

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+|x|^2)^4} \tag{3.1}$$

which corresponds to the linearization of (2.2) about the solution $u_1 (= u_{\varepsilon=1})$ which has been defined in the previous section.

We are interested in the classification of bounded solutions of $\mathbb{L}w = 0$ in $\mathbb{R}^4$. Some solutions are easy to find. For example, we can define

$$\phi_0(x) := r\partial_r u_1(x) + 4 = 4\frac{1-r^2}{1+r^2},$$

where $r = |x|$. Clearly $\mathbb{L}\phi_0 = 0$ and this reflects the fact that (2.2) is invariant under the group of dilations $\tau \to u(\tau\cdot) + 4\log\tau$. We also define, for $i = 1, \dots, 4$

$$\phi_i(x) := -\partial_{x_i} u_1(x) = \frac{8x_i}{1+|x|^2},$$

which are also solutions of $\mathbb{L}\phi_j = 0$ since these solutions correspond to the invariance of the equation under the group of translations $a \to u(\cdot + a)$.

The following result classifies all bounded solutions of $\mathbb{L}w = 0$ which are defined in $\mathbb{R}^4$.

**Lemma 3.1** ([1])**.** *Any bounded solution of $\mathbb{L}w = 0$ defined in $\mathbb{R}^4$ is a linear combination of $\phi_i$ for $i = 0, 1, \dots, 4$.*

Let $B_r$ denote the ball of radius $r$ centered at the origin in $\mathbb{R}^4$.

**Definition 3.2.** Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathbb{C}^{k,\alpha}_{\mu}(\mathbb{R}^4)$ as the space of functions $w \in \mathbb{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$ for which the norm

$$\|w\|_{\mathbb{C}^{k,\alpha}_{\mu}(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1)} + \sup_{r \geq 1}\Big((1+r^2)^{-\mu/2}\|w(r\cdot)\|_{\mathbb{C}^{k,\alpha}_{\mu}(\bar{B}_1 - B_{1/2})}\Big),$$

is finite.

We also define

$$\mathcal{C}^{k,\alpha}_{\mathrm{rad},\mu}(\mathbb{R}^4) = \{f \in \mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^4);\ f(x) = f(|x|), \forall x \in \mathbb{R}^4\}.$$

As a consequence of the result in Lemma 3.1, we have:

**Proposition 3.3** ([1])**.** *(i) Assume that $\mu > 1$ and $\mu \notin \mathbb{N}$, then*

$$\begin{gathered} L_\mu : \mathbb{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \to \mathbb{C}^{0,\alpha}_{\mu-4}(\mathbb{R}^4) \\ w \mapsto \mathbb{L}w \end{gathered}$$

*is surjective.*

*(ii) Assume that $\delta > 0$ and $\delta \notin \mathbb{N}$ then*

$$\begin{gathered} L_\delta : \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4) \to \mathcal{C}^{0,\alpha}_{\mathrm{rad},\delta-4}(\mathbb{R}^4) \\ w \mapsto \mathbb{L}w \end{gathered}$$

*is surjective.*

We set $\bar{B}_1^* = \bar{B}_1 - \{0\}$.

**Definition 3.4.** Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathbb{C}^{k,\alpha}_{\mu}(\bar{B}_1^*)$ as the space of functions in $\mathbb{C}^{k,\alpha}_{\mathrm{loc}}(\bar{B}_1^*)$ for which the norm

$$\|u\|_{\mathbb{C}^{k,\alpha}_{\mu}(\bar{B}_1^*)} = \sup_{r \leq 1/2}\big(r^{-\mu}\|u(r\cdot)\|_{\mathbb{C}^{k,\alpha}(\bar{B}_2 - B_1)}\big)$$

is finite.

Then we define the subspace of radial functions in $\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\bar{B}_1^*)$ by

$$\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\bar{B}_1^*) = \{f \in \mathcal{C}^{k,\alpha}_{\delta}(\mathbb{R}^4); f(x) = f(|x|), \forall x \in \bar{B}_1^*\}.$$

For $\varepsilon, \tau, \lambda > 0$, we define

$$R_{\varepsilon,\lambda,\gamma} := \tau r_{\varepsilon,\lambda,\gamma}/\varepsilon$$

where

$$r_{\varepsilon,\lambda,\gamma} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}, \sqrt{\gamma}) \tag{3.2}$$

We would like to find a solution $u$ of

$$\Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^2 - \rho^4 e^u = 0 \tag{3.3}$$

in $\bar{B}_{r_{\varepsilon,\lambda,\gamma}}$. Recall that in the polar coordinates if we assume that $\varphi$ is a radially symmetric function, we get the usual formulas $|\nabla\varphi| = (\nabla\varphi,\nabla\varphi)^{1/2}$ where $(\cdot,\cdot)$ is the usual Euclidian dot product in $\mathbb{R}^n$. Then

$$|\nabla\varphi|^2 = (\frac{\partial\varphi}{\partial r})^2,$$
$$\Delta\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{n-1}{r}\frac{\partial\varphi}{\partial r},$$
$$\Delta^2\varphi = \frac{\partial^4\varphi}{\partial r^4} + \frac{2(n-1)}{r}\frac{\partial^3\varphi}{\partial r^3} + \frac{(n-1)(n-3)}{r^2}\frac{\partial^2\varphi}{\partial r^2} - \frac{(n-1)(n-3)}{r^3}\frac{\partial\varphi}{\partial r}.$$

Using the transformation

$$v(x) = u\big(\frac{\varepsilon}{\tau}x\big) + 8\log\varepsilon - 4\log\big(\tau(1+\varepsilon^2)/2\big),$$

then (3.3) is equivalent to

$$\Delta^2 v - \Big(\frac{\varepsilon}{\tau}\Big)^2\big(\gamma\Delta v + \lambda|\nabla v|^2\big) - 24e^v = 0 \tag{3.4}$$

in $\bar{B}_{R_{\varepsilon,\lambda,\gamma}}$. Now we look for a solution of (3.4) of the form

$$v(x) = u_1(x) + h(x),$$

this amounts to solving

$$\mathbb{L}h = \frac{384}{(1+|x|^2)^4}(e^h - h - 1) + \Big(\frac{\varepsilon}{\tau}\Big)^2\big(\gamma\Delta(u_1+h) + \lambda|\nabla(u_1+h)|^2\big) \tag{3.5}$$

in $\bar{B}_{R_{\varepsilon,\lambda,\gamma}}$.

**Definition 3.5.** Given $\bar{r} \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_\mu(B_{\bar{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\bar{r}})$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\bar{B}_{\bar{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1\leq r\leq\bar{r}}\Big(r^{-\mu}\|w(r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})}\Big).$$

For $\sigma \geq 1$, we denote $\mathcal{E}_\sigma : \mathcal{C}^{0,\alpha}_\mu(\bar{B}_\sigma) \to \mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)$ the extension operator defined by

$$\mathcal{E}_\sigma(f)(x) = \begin{cases} f(x) & \text{for } |x| \leq \sigma \\ \chi\Big(\frac{|x|}{\sigma}\Big)\, f\Big(\sigma\frac{x}{|x|}\Big) & \text{for } |x| \geq \sigma, \end{cases} \tag{3.6}$$

where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \leq 1$ and identically equal to 0 for $t \geq 2$. It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\mathcal{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)} \leq c\|w\|_{\mathcal{C}^{0,\alpha}_\mu(\bar{B}_\sigma)}. \tag{3.7}$$

We fix $\delta \in (0,1)$ and denote by $\mathcal{G}_\delta$ to be a right inverse of $\mathbb{L}_\delta$ provided by Proposition 3.3. To find a solution of (3.5) it is enough to find a fixed point $h$, in a small ball of $\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$, solution of

$$h = \aleph(h) \tag{3.8}$$

where

$$\aleph(h) := \mathcal{G}_\delta \circ \mathcal{E}_\delta \circ \mathfrak{R}(h),$$
$$\mathfrak{R}(h) = \frac{384}{(1+|x|^2)^4}(e^h - h - 1) + \Big(\frac{\varepsilon}{\tau}\Big)^2\Big(\gamma\Delta(u_1+h) + \lambda|\nabla(u_1+h)|^2\Big).$$

We have

$$|\mathfrak{R}(0)| = \Big(\frac{\varepsilon}{\tau}\Big)^2 \Big(\gamma\Delta u_1 + \lambda|\nabla u_1|^2\Big).$$

For $|x| = r$, we have

$$\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\ |\mathfrak{R}(0)| \le \Big(\frac{\varepsilon}{\tau}\Big)^2 \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\Big(\gamma\Delta u_1 + \lambda|\nabla u_1|^2\Big).$$

Using

$$\begin{aligned}\gamma\Delta u_1 + \lambda|\nabla u_1|^2 &= -16\gamma\frac{2+r^2}{(1+r^2)^2} + 64\lambda\frac{r^2}{(1+r^2)^2}\\ &= \frac{-32\gamma}{(1+r^2)^2} + 16\,(4\lambda-\gamma)\,\frac{r^2}{(1+r^2)^2}\,,\end{aligned}$$

this implies that for each $\kappa > 0$, there exist $c_\kappa > 0$ (which can depend only on $\kappa$), such that for $\delta \in (0,1)$, we have

$$\begin{aligned}\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\ |\mathfrak{R}(0)| &\le c_\kappa\gamma\varepsilon^2 + c_\kappa(4\lambda+\gamma)\varepsilon^2 R^{2-\delta}_{\varepsilon,\lambda,\gamma}\\ &\le c_\kappa\gamma\varepsilon^2 + c_\kappa(4\lambda+\gamma)\varepsilon^\delta r^{2-\delta}_{\varepsilon,\lambda,\gamma}\\ &\le c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}.\end{aligned}$$

Then there exist $\bar{c}_\kappa > 0$ (which can depend only on $\kappa$), such that

$$\|\aleph(0)\|_{C^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le \bar{c}_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$$

Using Proposition 3.3 and (3.7), we conclude that

$$\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le 2\bar{c}_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}. \tag{3.9}$$

Now, let $h_1, h_2$ be in $B(0, 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma})$ of $\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$. Then for each $\kappa > 0$, there exist $c_\kappa > 0$ (which can depend only on $\kappa$), such that for $\delta \in (0,1)$, we have

$$\begin{aligned}&\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\ |\mathfrak{R}(h_2) - \mathfrak{R}(h_1)|\\ &\le c_\kappa \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}(1+|x|^2)^{-4}\left|e^{h_2} - e^{h_1} + h_1 - h_2\right|\\ &\quad + c_\kappa\lambda\varepsilon^2 \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}(|\nabla(u_1+h_2)|^2 - |\nabla(u_1+h_1)|^2)\\ &\quad + c_\kappa\gamma\varepsilon^2 \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\Big|\Delta(u_1+h_2) - \Delta(u_1+h_1)\Big|\\ &\le c_\kappa \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{-4-\delta}|h_2-h_1||h_2+h_1|\\ &\quad + c_\kappa\lambda\varepsilon^2 \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\Big(|\nabla(h_2-h_1)|\,(|\nabla(h_2+h_1)| + 2|\nabla u_1|)\Big)\\ &\quad + c_\kappa\gamma\varepsilon^2 \sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\Big|\Delta(h_2-h_1)\Big|\\ &\le c_\kappa \sum_{i=1}^2 \|h_i\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + c_\kappa\gamma\varepsilon^2 R^4_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\\ &\quad + c_\kappa\lambda\varepsilon^2 R^4_{\varepsilon,\lambda,\gamma}\Big(R^\delta_{\varepsilon,\lambda,\gamma}\sum_{i=1}^2\|h_i\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + 1\Big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

Provided $h_i \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ satisfies $\|h_i\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le 2c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$, then the last estimate, is given by

$$\begin{aligned}
&\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(h_2)-\mathfrak{R}(h_1)| \\
&\le c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + c_\kappa r^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \\
&\quad + c_\kappa r^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.
\end{aligned}$$

Similarly, using Proposition 3.3 and (5.4), we conclude that for each $\kappa > 0$, there exist $\varepsilon_\kappa, \lambda_\kappa$, $\gamma_\kappa$ and $\bar{c}_\kappa > 0$ (only depend on $\kappa$) such that

$$\|\aleph(h_2)-\aleph(h_1)\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le \bar{c}_\kappa r^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}. \tag{3.10}$$

Reducing $\varepsilon_\kappa, \lambda_\kappa$ and $\gamma_\kappa$ if necessary, we can assume that

$$\bar{c}_\kappa r^2_{\varepsilon,\lambda,\gamma} \le \frac{1}{2}$$

for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\gamma \in (0,\gamma_\kappa)$. Then, (3.9) and (3.10) are enough to show that $h \mapsto \aleph(h)$ is a contraction from the ball

$$\{h \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4) : \|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}\}$$

into itself and hence has a unique fixed point $h$ in this set. This fixed point is a solution of (3.8) in $\bar{B}_{R_{\varepsilon,\lambda,\gamma}}$. We summarize this in the following proposition.

**Proposition 3.6.** *For each $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$ and $c_\kappa > 0$ (which can depend only on $\kappa$) such that for all for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\gamma \in (0,\gamma_\kappa)$ and for $\delta \in (0,1)$, there exists a unique solution $h \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ of (3.8) such that*

$$v(x) = u_1(x) + h(x)$$

*solves (3.4) in $\bar{B}_{R_{\varepsilon,\lambda,\gamma}}$. In addition*

$$\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}.$$

## 4. Known results [1]

### 4.1. **Analysis of the bi-Laplace operator in weighted spaces.**

Given $x^1,\dots,x^m \in \Omega$ we define $X := (x^1,\dots,x^m)$ and

$$\bar{\Omega}^*(X) := \bar{\Omega} - \{x^1,\dots,x^m\},$$

and we choose $r_0 > 0$ so that the balls $B_{r_0}(x^i)$ of center $x^i$ and radius $r_0$ are mutually disjoint and included in $\Omega$. For all $r \in (0,r_0)$ we define

$$\bar{\Omega}_r(X) := \bar{\Omega} - \cup_{j=1}^m B_r(x^j)$$

With these notation, we have the following definition.

**Definition 4.1.** Given $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\bar{\Omega}^*(X))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\bar{\Omega}^*(X))$ which is endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\bar{\Omega}^*(X))} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{\Omega}_{r_0/2}(X))} + \sum_{j=1}^m \sup_{r\in(0,r_0/2)} \left(r^{-\nu}\|w(x^j+r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2-B_1)}\right),$$

is finite.

When $k \geq 2$, we denote by $[\mathbb{C}^{k,\alpha}_\nu(\bar\Omega^*(X))]_0$ be the subspace of functions $w \in \mathbb{C}^{k,\alpha}_\nu(\bar\Omega^*(X))$ satisfying $w = \Delta w = 0$.

**Proposition 4.2** ([1]). *Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\begin{aligned} \mathcal{L}_\nu : [\mathbb{C}^{4,\alpha}_\nu(\bar\Omega^*(X))]_0 &\to \mathbb{C}^{0,\alpha}_{\nu-4}(\bar\Omega^*(X)) \\ w &\mapsto \Delta^2 w \end{aligned}$$

*is surjective.*

4.2. **Bi-harmonic extensions.** Given $\varphi \in \mathbb{C}^{4,\alpha}(S^3)$ and $\psi \in \mathbb{C}^{2,\alpha}(S^3)$ we define $H^i(= H^i(\varphi,\psi;\cdot))$ as the solution of

$$\begin{gathered} \Delta^2 H^i = 0 \quad \text{in } B_1 \\ H^i = \varphi \quad \text{on } \partial B_1 \\ \Delta H^i = \psi \quad \text{on } \partial B_1, \end{gathered} \tag{4.1}$$

where, as already mentioned, $B_1$ denotes the unit ball in $\mathbb{R}^4$.

We set $B_1^* = B_1 - \{0\}$. As in the previous section, we have a definition.

**Definition 4.3.** Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathbb{C}^{k,\alpha}_\mu(\bar B_1^*)$ as the space of function in $\mathbb{C}^{k,\alpha}_{\rm loc}(\bar B_1^*)$ for which the norm

$$\|u\|_{\mathbb{C}^{k,\alpha}_\mu(\bar B_1^*)} = \sup_{r \leq 1/2} \left( r^{-\mu} \|u(r\cdot)\|_{\mathbb{C}^{k,\alpha}(\bar B_2 - B_1)} \right),$$

is finite.

This corresponds to the space and norm already defined in the previous section when $\Omega = B_1$, $m = 1$ and $x^1 = 0$.

Let $e_1, \dots, e_4$ be the coordinate functions on $S^3$.

**Lemma 4.4.** [1] *Assume that*

$$\int_{S^3} (8\varphi - \psi) dv_{S^3} = 0 \quad \text{and} \quad \int_{S^3} (12\varphi - \psi) e_\ell dv_{S^3} = 0 \tag{4.2}$$

*for $\ell = 1, \dots, 4$. Then there exists $c > 0$ such that*

$$\|H^i(\varphi,\psi;\cdot)\|_{\mathcal{C}^{4,\alpha}_2(\bar B_1^*)} \leq c(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}).$$

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$, we define (when it exsits) $H^e$ $(= H^e(\varphi,\psi;\cdot))$ to be the solution of

$$\begin{gathered} \Delta^2 H^e = 0 \quad \text{in } \mathbb{R}^4 - B_1 \\ H^e = \varphi \quad \text{on } \partial B_1 \\ \Delta H^e = \psi \quad \text{on } \partial B_1 \end{gathered}$$

which decays at infinity.

**Definition 4.5.** Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathbb{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1)$ as the space of functions $w \in \mathbb{C}^{k,\alpha}_{\rm loc}(\mathbb{R}^4 - B_1)$ for which the norm

$$\|w\|_{\mathbb{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1)} = \sup_{r \geq 1} \left( r^{-\nu} \|w(r\cdot)\|_{\mathbb{C}^{k,\alpha}_\nu(\bar B_2 - B_1)} \right),$$

is finite.

**Lemma 4.6** ([1]). *Assume that*

$$\int_{S^3} \psi dv_{S^3} = 0. \tag{4.3}$$

*Then there exists $c > 0$ such that*

$$\|H^e(\varphi, \psi; \cdot)\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1)} \leq c(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}).$$

**Lemma 4.7** ([1]). *The mapping*

$$\begin{aligned} \mathcal{P} : \mathcal{C}^{4,\alpha}(S^3)^\perp \times \mathcal{C}^{2,\alpha}(S^3)^\perp &\to \mathcal{C}^{3,\alpha}(S^3)^\perp \times \mathcal{C}^{1,\alpha}(S^3)^\perp \\ (\varphi, \psi) &\mapsto (\partial_r H^i - \partial_r H^e, \partial_r \Delta H^i - \partial_r \Delta H^e) \end{aligned}$$

*where $H^i = H^i(\varphi, \psi; \cdot)$ and $H^e = H^e(\varphi, \psi; \cdot)$, is an isomorphism.*

## 5. First nonlinear Dirichlet problem

Recall for $\varepsilon, \tau, \lambda, \gamma > 0$, we define $R_{\varepsilon,\lambda,\gamma} := \tau r_{\varepsilon,\lambda,\gamma}/\varepsilon$, where

$$r_{\varepsilon,\lambda,\gamma} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}, \sqrt{\gamma}). \tag{5.1}$$

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfying (4.2), we define

$$\mathbf{u} := u_1 + h + H^i(\varphi, \psi; (\cdot/R_{\varepsilon,\lambda,\gamma})).$$

We would like to find a solution $u$ of

$$\Delta^2 u - \gamma\big(\frac{\varepsilon}{\tau}\big)^2 \Delta u - \lambda\big(\frac{\varepsilon}{\tau}\big)^2 |\nabla u|^2 - 24e^u = 0 \tag{5.2}$$

which is defined in $B_{R_{\varepsilon,\lambda,\gamma}}$ and which is a perturbation of $\mathbf{u}$. Writing $u = \mathbf{u} + v$, this amounts to solve the equation

$$\begin{aligned} \mathbb{L}v &= \frac{384}{(1+r^2)^4} e^h (e^{H^i(\varphi,\psi;(\cdot/R_{\varepsilon,\lambda,\gamma}))+v} - 1 - v) + \frac{384}{(1+r^2)^4}(e^h - 1)v \\ &\quad + \gamma\big(\frac{\varepsilon}{\tau}\big)^2 \Delta\Big(u_1 + h + H^i(\varphi, \psi; (\cdot/R_{\varepsilon,\lambda,\gamma})) + v\Big) - \gamma\big(\frac{\varepsilon}{\tau}\big)^2 \Delta(u_1 + h) \\ &\quad + \lambda\big(\frac{\varepsilon}{\tau}\big)^2 \Big|\nabla\Big(u_1 + h + H^i(\varphi, \psi; (\cdot/R_{\varepsilon,\lambda,\gamma})) + v\Big)\Big|^2 - \lambda\big(\frac{\varepsilon}{\tau}\big)^2 |\nabla(u_1 + h)|^2, \end{aligned} \tag{5.3}$$

since $H^i$ is bi-harmonic. In the following, we will denote by $\mathcal{K}(v)$ the right hand side of (5.3).

**Definition 5.1.** Given $\bar{r} \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_\mu(B_{\bar{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\bar{r}})$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\bar{B}_{\bar{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1 \leq r \leq \bar{r}} \Big(r^{-\mu} \|w(r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})}\Big).$$

For $\sigma \geq 1$, we denote by $\mathcal{E}_\sigma : \mathcal{C}^{0,\alpha}_\mu(\bar{B}_\sigma) \to \mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)$ the extension operator defined by

$$\mathcal{E}_\sigma(f)(x) = \chi\Big(\frac{|x|}{\sigma}\Big) f\Big(\sigma \frac{x}{|x|}\Big),$$

where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \geq 2$ and identically equal to 0 for $t \leq 1$. It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\mathcal{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)} \leq c\|w\|_{\mathcal{C}^{0,\alpha}_\mu(\bar{B}_\sigma)}. \tag{5.4}$$

We fix $\mu \in (1,2)$ and denote by $\mathcal{G}_\mu$ a right inverse provided by Proposition 3.3. To find $\underline{a}$ solution of (5.3), it is enough to find $v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solution of

$$v = N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v) \tag{5.5}$$

where we have defined

$$N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v) := \mathcal{G}_\mu \circ \mathcal{E}_{R_{\varepsilon,\lambda,\gamma}}\left(\mathcal{K}(v)\right)$$

Given $\kappa > 1$ (whose value will be fixed later on), we now further assume that the functions $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ and the constant $\tau > 0$ satisfy

$$\begin{gathered} \frac{1}{\log 1/r^2_{\varepsilon,\lambda,\gamma}}|\log(\tau/\tau_*)| \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\varphi\|_{\mathbb{C}^{4,\alpha}(S^3)} \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \\ \|\psi\|_{\mathbb{C}^{2,\alpha}(S^3)} \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \end{gathered} \tag{5.6}$$

where $\tau_* > 0$ is fixed later.

**Lemma 5.2.** *For each $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in (0,1)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\gamma \in (0,\gamma_\kappa)$,*

$$\|N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}. \tag{5.7}$$

*Moreover,*

$$\|N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v_2) - N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v_1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \bar{c}_\kappa r^2_{\varepsilon,\lambda,\gamma}\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \tag{5.8}$$

*provided $\tilde{v} = v_1, v_2 \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$, $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfy*

$$\begin{gathered} \|\tilde{v}\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}, \quad \|\varphi\|_{\mathbb{C}^{4,\alpha}(S^3)} \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \\ \|\psi\|_{\mathbb{C}^{2,\alpha}(S^3)} \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad |\log(\tau/\tau_*)| \le \kappa r^2_{\varepsilon,\lambda,\gamma}\log 1/r^2_{\varepsilon,\lambda,\gamma}. \end{gathered}$$

*Proof.* The estimates follow from Lemma 4.4 together with the assumption on the norms of $\varphi$ and $\psi$. Let $c^{(i)}_\kappa$ denote constants which only depend on $\kappa$ (provided $\varepsilon$, $\lambda$ and $\gamma$ are chosen small enough).

It follows from Lemma 4.4 and the estimates given by (5.6) that

$$\begin{aligned} &\|H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\ &\le c_\kappa R^{-2}_{\varepsilon,\lambda,\gamma}(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}) \\ &\le c_\kappa \varepsilon^2 \end{aligned} \tag{5.9}$$

Therefore, using the fact that for each $x \in \bar{B}_{R_{\varepsilon,\lambda,\gamma}}$, we have $|h(x)| \le c_\kappa\ r^{2+\delta}_{\varepsilon,\lambda,\gamma}$, which tends to 0 as $\varepsilon,\lambda$ and $\gamma$ tend to 0, we obtain

$$\|(1+|\cdot|^2)^{-4}e^h\left(e^{H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})} - 1\right)\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \le c_\kappa \varepsilon^2,$$

for $\mu \in (1,2)$, we have

$$\begin{aligned} &\left\|\varepsilon^2\left[|\nabla\left(u_1 + h + H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\right)|^2 - |\nabla(u_1+h)|^2\right]\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\ &\le \left\|\varepsilon^2|\nabla\left(H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\right)|\left[2|\nabla(u_1+h)|\right.\right. \\ &\quad \left.\left.+ |\nabla\left(H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\right)|\right]\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\ &\le c_\kappa \varepsilon^\mu r^{4-\mu}_{\varepsilon,\lambda,\gamma}\left[1 + r^\delta_{\varepsilon,\lambda,\gamma}\varepsilon^{-\delta}\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + r^2_{\varepsilon,\lambda,\gamma}\varepsilon^{-2}\|H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})}\right] \end{aligned}$$

Provided $h \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ satisfy $\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le 2c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$, and from the asymptotic behavior of $H^i$ given by the estimate (5.9) and $\mu \in (1,2)$, we deduce that

$$\Big\|\varepsilon^2\Big[|\nabla\left(u_1 + h + H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\right)|^2 - |\nabla(u_1+h)|^2\Big]\Big\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \le c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}$$

and

$$\left\|\varepsilon^2\Delta\left(H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \le c_\kappa \varepsilon^\mu r^{4-\mu}_{\varepsilon,\lambda,\gamma} \le c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}.$$

Using Proposition 3.3 and (5.4), we conclude that

$$\|N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}.$$

To derive the second estimate, we use the fact that for each $x \in \bar{B}_{R_{\varepsilon,\lambda,\gamma}}$, we have $|h(x)| \le c_\kappa r^{2+\delta}_{\varepsilon,\lambda,\gamma}$, which tends to 0 as $\varepsilon, \lambda$ and $\gamma$ tend to 0. For each $\kappa > 0$, there exists $c_\kappa > 0$ such that

$$\begin{aligned}
&\left\|(1+|\cdot|^2)^{-4} e^{H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})+h}\left(e^{v_2} - e^{v_1} - v_2 + v_1\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\
&\le c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|(1+|\cdot|^2)^{-4} e^{h}\left(e^{H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})} - 1\right)(v_2 - v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\
&\le c_\kappa \varepsilon^2 \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},
\end{aligned}$$

and for $\mu \in (1,2)$, we obtain

$$\|\varepsilon^2\Delta(v_2 - v_1)\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \le c_\kappa r^2_{\varepsilon,\lambda,\gamma}\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.$$

Provided $h \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ satisfy $\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \le 2c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$, we deduce that

$$\begin{aligned}
\left\|(1+|\cdot|^2)^{-4}(e^h - 1)(v_2 - v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} &\le c_\kappa \|h\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},
\end{aligned}$$

and

$$\begin{aligned}
&\Big\||\varepsilon^2\Big(\Big|\nabla(u_1 + h + H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma}) + v_2)\Big|^2 \\
&- \Big|\nabla(u_1 + h + H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma}) + v_1)\Big|^2\Big)\Big\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\
&\le \|\varepsilon^2|\nabla(v_2 - v_1)|\Big(|\nabla(v_2 + v_1)| + 2|\nabla(u_1 + h + H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma}))|\Big)\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})} \\
&\le c_\kappa \varepsilon^2 R^2_{\varepsilon,\lambda,\gamma}\Big(R^\mu_{\varepsilon,\lambda,\gamma}\sum_{i=1}^2 \|v_i\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + 1 \\
&\quad + R^2_{\varepsilon,\lambda,\gamma}\|H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})}\Big)\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le c_\kappa r^2_{\varepsilon,\lambda,\gamma}\Big(r^\mu_{\varepsilon,\lambda,\gamma}\varepsilon^{-\mu}\sum_{i=1}^2 \|v_i\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + 1 \\
&\quad + r^2_{\varepsilon,\lambda,\gamma}\varepsilon^{-2}\|H^i(\varphi,\psi;\cdot/R_{\varepsilon,\lambda,\gamma})\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda,\gamma}})}\Big)\|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}$$

Provided $v_1, v_2 \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfy $\|v_i\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}$ and from the asymptotic behavior of $H^i$ given by the estimate (5.9), we derive the desired estimate, using Proposition 3.3 and (5.4). $\square$

Reducing $\varepsilon_\kappa, \lambda_\kappa$ and $\gamma_\kappa$ if necessary, we can assume that

$$\bar{c}_\kappa r^2_{\varepsilon,\lambda,\gamma} \leq \frac{1}{2}. \tag{5.10}$$

Then there exist $\varepsilon_\kappa > 0, \lambda_\kappa > 0$, $\gamma_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$. Then, (5.7) and (5.8) in Lemma 5.2 are sufficient to show that

$$v \mapsto N(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; v)$$

is a contraction from $\{v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) : \|v\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}\}$ into itself and hence has a unique fixed point $v(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; \cdot)$ in this set. This fixed point is a solution of (5.5) in $B_{R_{\varepsilon,\lambda,\gamma}}$. We summarize this as follows.

**Proposition 5.3.** *For each $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$ and $c_\kappa > 0$ (only depending on $\kappa$) such that given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfying* (4.2) *and $\tau > 0$ satisfying*

$$|\log(\tau/\tau_*)| \leq \kappa r^2_{\varepsilon,\lambda,\gamma} \log 1/r^2_{\varepsilon,\lambda,\gamma}, \quad \|\varphi\|_{\mathbb{C}^{4,\alpha}(S^3)} \leq \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\psi\|_{\mathbb{C}^{2,\alpha}(S^3)} \leq \kappa r^2_{\varepsilon,\lambda,\gamma},$$

*the function*

$$u(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; \cdot) := u_1 + h + H^i(\varphi, \psi; \cdot/R_{\varepsilon,\lambda,\gamma}) + v(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; \cdot),$$

*solves* (5.2) *in $B_{R_{\varepsilon,\lambda,\gamma}}$. In addition*

$$\|v(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; \cdot)\|_{\mathbb{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}. \tag{5.11}$$

Observe that the function $v(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; \cdot)$ being obtained as a fixed point for contraction mapping, it depends continuously on the parameter $\tau$.

## 6. Second nonlinear Dirichlet problem

For $(\varepsilon, \lambda, \gamma) \in (0, r_0^2)^3$, we recall that

$$r_{\varepsilon,\lambda,\gamma} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}, \sqrt{\gamma}).$$

Recall that $G(x, \cdot)$ denotes the unique solution of

$$\Delta^2 G(x, \cdot) = 64\pi^2 \delta_x$$

in $\Omega$, with $G(x, \cdot) = \Delta G(x, \cdot) = 0$ on $\partial\Omega$. In addition, the following decomposition holds

$$G(x, y) = -8 \log |x - y| + R(x, y)$$

where $y \mapsto R(x, y)$ is a smooth function.

We recall in this section a result which concerns the properties of the Greens function in the following lemma.

**Lemma 6.1.** *There exists $C > 0$ such that for all $x, y \in \Omega, x \neq y$, we have*

$$|\nabla^i G(x, y)| \leq C|x - y|^{-i}, i \geq 1.$$

The estimate in the above lemma is originally due to Krasovskiĭ [12].

Given $x^1, \dots, x^m \in \Omega$. We need the following data:

(i) Points $Y := (y^1, \dots, y^m) \in \Omega^m$ close enough to $X := (x^1, \dots, x^m)$.
(ii) Parameters $\tilde{\eta} := (\tilde{\eta}^1, \dots, \tilde{\eta}^m) \in \mathbb{R}^m$ close to 0.

(iii) Boundary data $\Phi := (\varphi^1, \dots, \varphi^m) \in (\mathcal{C}^{4,\alpha}(S^3))^m$ and $\Psi := (\psi^1, \dots, \psi^m) \in (\mathcal{C}^{2,\alpha}(S^3))^m$ each of which satisfies (4.3).

With all these data, we define

$$\tilde{\mathbf{u}} := \sum_{j=1}^m (1+\tilde{\eta}^j) G(y^j, \cdot) + \sum_{j=1}^m \chi_{r_0}(\cdot - y^j) H^e(\varphi^j, \psi^j; (\cdot - y^j)/r_{\varepsilon,\lambda,\gamma}) \tag{6.1}$$

where $\chi_{r_0}$ is a cutoff function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside $B_{r_0}$.

We define $\rho > 0$ by

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

We would like to find a solution of the equation

$$\Delta^2 u - \gamma \Delta u - \lambda |\nabla u|^2 - \rho^4 e^u = 0, \tag{6.2}$$

which is defined in $\bar{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)$ and which is a perturbation of $\tilde{\mathbf{u}}$. Writing $u = \tilde{\mathbf{u}} + \tilde{v}$, this amounts to solve

$$\Delta^2 \tilde{v} = \rho^4 e^{\tilde{\mathbf{u}}+\tilde{v}} - \Delta^2 \tilde{\mathbf{u}} + \gamma \Delta(\tilde{\mathbf{u}} + \tilde{v}) + \lambda |\nabla(\tilde{\mathbf{u}} + \tilde{v})|^2. \tag{6.3}$$

We need to define an auxiliary weighed space.

**Definition 6.2.** Given $\bar{r} \in (0, r_0/2)$, $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\bar{\Omega}_{\bar{r}}(X))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}(\bar{\Omega}_{\bar{r}}(X))$ which is endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\bar{\Omega}_{\bar{r}}(X))} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{\Omega}_{r_0/2}(X))} + \sum_{j=1}^m \sup_{r \in [\bar{r}, r_0/2)} \left( r^{-\nu} \|w(x^j + r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2 - B_1)} \right).$$

For all $\sigma \in (0, r_0/2)$ and all $Y \in \Omega^m$ such that $\|X - Y\| \le r_0/2$, we denote by

$$\tilde{\mathcal{E}}_{\sigma,Y} : \mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}_\sigma(Y)) \to \mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}^*(Y)),$$

the extension operator defined by $\tilde{\mathcal{E}}_{\sigma,Y}(f) = f$ in $\bar{\Omega}_\sigma(Y)$

$$\tilde{\mathcal{E}}_{\sigma,Y}(f)(y^i + x) = \tilde{\chi}\left(\frac{|x|}{\sigma}\right) f\left(y^i + \sigma \frac{x}{|x|}\right)$$

for each $j = 1, \dots, m$ and $\tilde{\mathcal{E}}_{\sigma,Y}(f) = 0$ in each $B_{\sigma/2}(y^j)$, where $t \mapsto \tilde{\chi}(t)$ is a cutoff function identically equal to 1 for $t \ge 1$ and identically equal to 0 for $t \le 1/2$. It is easy to check that there exists a constant $c = c(\nu) > 0$ only depending on $\nu$ such that

$$\|\tilde{\mathcal{E}}_{\sigma,Y}(w)\|_{\mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}^*(X))} \le c \|w\|_{\mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}_\sigma(X))}. \tag{6.4}$$

We fix $\nu \in (-1, 0)$, and denote by $\tilde{\mathcal{G}}_{\nu,Y}$ the right inverse provided by Proposition 4.2. Clearly, it is enough to find $\tilde{v} \in \mathcal{C}^{4,\alpha}_\nu(\Omega^*(Y))$ solution of

$$\tilde{v} = \tilde{N}(\varepsilon, \lambda, \gamma, \tilde{\eta}, Y, \Phi, \Psi; \tilde{v}) \tag{6.5}$$

where we have defined

$$\begin{aligned}
\tilde{N}(\tilde{v}) &:= \tilde{N}(\varepsilon, \lambda, \gamma, \tilde{\eta}, Y, \Phi, \Psi; \tilde{v}) \\
&:= \tilde{\mathcal{G}} \circ \tilde{\mathcal{E}}_{r_{\varepsilon,\lambda,\gamma},Y} \left( \rho^4 e^{\tilde{\mathbf{u}}+\tilde{v}} - \Delta^2 \tilde{\mathbf{u}} + \gamma \Delta(\tilde{\mathbf{u}} + \tilde{v}) + \lambda |\nabla(\tilde{\mathbf{u}} + \tilde{v})|^2 \right) \\
&:= \tilde{\mathcal{G}}_{\nu,Y} \circ \tilde{\mathcal{E}}_{r_{\varepsilon,\lambda,\gamma},Y} \big( \tilde{S}(v) \big)
\end{aligned}$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that $\Phi$ and $\Psi$ satisfy

$$\|\Phi\|_{(\mathbb{C}^{4,\alpha}(S^3))^m} \leq \kappa r_{\varepsilon,\lambda,\gamma}^2, \quad \text{and} \quad \|\Psi\|_{(\mathbb{C}^{2,\alpha}(S^3))^m} \leq \kappa r_{\varepsilon,\lambda,\gamma}^2. \tag{6.6}$$

Moreover, we assume that the parameters $\tilde{\eta}$ and the points $Y$ are chosen to satisfy

$$|\tilde{\eta}| \leq \kappa r_{\varepsilon,\lambda,\gamma}^2, \quad \text{and} \quad \|Y - X\| \leq \kappa r_{\varepsilon,\lambda,\gamma}. \tag{6.7}$$

Then, the following result holds.

**Lemma 6.3.** *For each $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa), \lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$, we have*

$$\|\tilde{N}(\varepsilon, \lambda, \gamma, \tilde{\eta}, Y, \Phi, \Psi; 0)\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(Y))} \leq c_\kappa r_{\varepsilon,\lambda,\gamma}^2. \tag{6.8}$$

*Moreover,*

$$\begin{aligned} &\|\tilde{N}(\varepsilon, \lambda, \gamma, \tilde{\eta}, Y, \Phi, \Psi; \tilde{v}_2) - \tilde{N}(\varepsilon, \lambda, \gamma, \tilde{\eta}, Y, \Phi, \Psi; \tilde{v}_1)\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(Y))} \\ &\leq \bar{c}_\kappa r_{\varepsilon,\lambda,\gamma}^2 \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(Y))} \end{aligned} \tag{6.9}$$

*provided $\tilde{v} = v_1, v_2 \in \mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(Y))$, $\tilde{\Phi} = \Phi_1, \Phi_2 \in (\mathcal{C}^{4,\alpha}(S^3))^m$, $\tilde{\Psi} = \Psi_1, \Psi_2 \in (\mathcal{C}^{2,\alpha}(S^3))^m$ satisfy*

$$\begin{gathered} \|\tilde{v}\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(Y))} \leq 2c_\kappa r_{\varepsilon,\lambda,\gamma}^2, \quad \|\tilde{\Phi}\|_{(\mathbb{C}^{4,\alpha}(S^3))^m} \leq \kappa r_{\varepsilon,\lambda,\gamma}^2, \\ \|\tilde{\Psi}\|_{(\mathbb{C}^{2,\alpha}(S^3))^m} \leq \kappa r_{\varepsilon,\lambda,\gamma}^2, \quad |\tilde{\eta}| \leq \kappa r_{\varepsilon,\lambda,\gamma}^2, \quad \|Y - X\| \leq \kappa r_{\varepsilon,\lambda,\gamma}. \end{gathered}$$

*Proof.* The first estimate follows from the asymptotic behavior of $H^e$ together with the assumption on the norm of boundary data $\tilde{\varphi}^i$ given by (6.6). Indeed, let $c_\kappa$ be a constant depending only on $\kappa$ (provided $\varepsilon$, $\lambda$ and $\gamma$ are chosen small enough) it follows from the estimate of $H^e$, given by lemma 4.6, that

$$|H^e_{\tilde{\varphi}^j, \tilde{\psi}^j}((x - y^j)/r_{\varepsilon,\lambda,\gamma})| \leq c_\kappa r_{\varepsilon,\lambda,\gamma}^3 r^{-1}. \tag{6.10}$$

Recall that $\tilde{N}(\tilde{v}) = \tilde{\mathcal{G}}_\nu \circ \tilde{\xi}_{r_{\varepsilon,\lambda,\gamma}} \circ \tilde{S}(\tilde{v})$, we will estimate $\tilde{N}(0)$ in different subregions of $\bar{\Omega}^*$.

- In $B_{r_0/2}(y^j)$ for $1 \leq j \leq m$, we have $\chi_{r_0}(x - y^j) = 1$ and $\Delta^2 \tilde{\mathbf{u}} = 0$ so that

$$\begin{aligned} |\tilde{S}(0)| &\leq c_\kappa \varepsilon^4 \prod_{j=1}^m \Big[ e^{(1+\tilde{\eta}^j) G_{y^j}(x) + H^e_{\tilde{\varphi}^j, \tilde{\psi}^j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})} + \gamma |\Delta \tilde{\mathbf{u}}| + \lambda |\nabla \tilde{\mathbf{u}}|^2 \\ &\leq c_\kappa \varepsilon^4 \prod_{j=1}^m |x - y^j|^{-8(1+\tilde{\eta}^j)} + \gamma |\Delta(\tilde{\mathbf{u}})| + \lambda |\nabla(\tilde{\mathbf{u}})|^2 \\ &\leq c_\kappa \varepsilon^4 |x - y^j|^{-8(1+\tilde{\eta}^j)} \prod_{\ell=1, \ell \neq j}^m |x - y^\ell|^{-8(1+\tilde{\eta}^j)} \\ &\quad + c_\kappa \gamma (1 + \tilde{\eta}^j) \sum_{j=1}^m \Delta R(x, y^j) + c_\kappa \gamma (1 + \tilde{\eta}^j) |x - y^j|^{-2} \sum_{\ell=1, \ell \neq j}^m |x - y^\ell|^{-2} \\ &\quad + c_\kappa \gamma r_{\varepsilon,\lambda,\gamma}^3 |x - y^j|^{-3} \sum_{\ell=1, \ell \neq j}^m |x - y^\ell|^{-3} \\ &\quad + c_\kappa \lambda (1 + \tilde{\eta}^j) \sum_{j=1}^m |\nabla R(x, y^j)|^2 + c_\kappa \lambda (1 + \tilde{\eta}^j) |x - y^j|^{-2} \sum_{\ell=1, \ell \neq j}^m |x - y^\ell|^{-2} \end{aligned}$$

$$+ c_\kappa \lambda r^3_{\varepsilon,\lambda,\gamma} |x-y^j|^{-4} \sum_{\ell=1,\ell\neq j}^m |x-y^\ell|^{-4}.$$

Hence, for $\nu \in (-1,0)$ and $\tilde\eta^j$ small enough, we get

$$\begin{aligned}\|\tilde N(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\bigcup_{j=1}^m B(y^j,r_0/2))} &\le \sup_{r_{\varepsilon,\lambda,\gamma}\le r\le r_0/2} r^{4-\nu}|\tilde N(0)| \\ &\le c_\kappa \varepsilon^4 r^{-4}_{\varepsilon,\lambda,\gamma} + 2c_\kappa\gamma + c_\kappa\gamma r^3_{\varepsilon,\lambda,\gamma} + c_\kappa\lambda + c_\kappa\gamma r^3_{\varepsilon,\lambda,\gamma}.\end{aligned}$$

• In $\Omega_{r_0,\tilde x}$ (recall that $\Omega_{r_0,\tilde x} = \Omega \setminus \cup_j B_{r_0}(\tilde x^j)$), we have $\chi_{r_0}(x-y^j) = 0$ and $\Delta^2 \tilde{\mathbf{u}} = 0$, then

$$\begin{aligned}|\tilde S(0)| \le{}& c_\kappa \varepsilon^4 |x-y^j|^{-8(1+\tilde\eta^j)} \prod_{\ell=1,\ell\neq j}^m |x-y^\ell|^{-8(1+\tilde\eta^j)} \\ &+ c_\kappa\gamma(1+\tilde\eta^j)\sum_{j=1}^m \Delta R(x,y^j) + c_\kappa\gamma(1+\tilde\eta^j)|x-y^j|^{-2}\sum_{\ell=1,\ell\neq j}^m |x-y^\ell|^{-2} \\ &+ c_\kappa\lambda(1+\tilde\eta^j)\sum_{j=1}^m |\nabla R(x,y^j)|^2 + c_\kappa\lambda(1+\tilde\eta^j)|x-y^j|^{-2}\sum_{\ell=1,\ell\neq j}^m |x-y^\ell|^{-2}.\end{aligned}$$

Thus

$$\|\tilde N(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega_{r_0,\tilde x})} \le c_\kappa \sup_{r\ge r_0} r^{4-\nu}|\tilde S(0)| \le c_\kappa\varepsilon^4 + c_\kappa\gamma + c_\kappa\lambda.$$

• In $B_{r_0}(y^j) - B_{r_0/2}(y^j)$, for $j = 1,\dots,m$, we have

$$\begin{aligned}|\tilde S(0)| \le{}& c_\kappa\varepsilon^4 \Big| \prod_{j=1}^m e^{(1+\tilde\eta^j)G_{y^j}} e^{\chi_{r_0}(x-y^j)H^e_{\tilde\varphi^j,\tilde\psi^j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})} + \Delta^2\tilde{\mathbf{u}} + \Delta\tilde{\mathbf{u}} + |\nabla\tilde{\mathbf{u}}|^2\Big| \\ \le{}& c_\kappa\varepsilon^4|x-y^j|^{-8(1+\tilde\eta^j)} \prod_{\ell=1,\ell\neq 1}^m |x-y^\ell|^{-8(1+\tilde\eta^\ell)} \\ &+ c_\kappa\varepsilon^4 \sum_{j=1}^m |[\Delta^2,\chi_{r_0}(x-y^j)]||H^{\rm ext}_{\tilde\varphi_j,\tilde\psi_j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})| \\ &+ c_\kappa\gamma\sum_{j=1}^m |[\Delta,\chi_{r_0}(x-y^j)]||H^{\rm ext}_{\tilde\varphi_j,\tilde\psi_j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})| \\ &+ c_\kappa\gamma(1+\tilde\eta^j)\sum_{j=1}^m |\Delta R(x,y^j)| \\ &+ c_\kappa\lambda\sum_{j=1}^m \Big|[\nabla,\chi_{r_0}(x-y^j)]\Big|^2 |H^{\rm ext}_{\tilde\varphi_j,\tilde\psi_j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})| \\ &+ c_\kappa\lambda(1+\tilde\eta^j)\sum_{j=1}^m |\nabla R(x,y^j)|^2 \\ &+ c_\kappa\gamma(1+\tilde\eta^j)|x-y^\ell|^{-2}\sum_{\ell=1,\ell\neq j}^m |x-y^j|^{-2} \\ &+ c_\kappa\lambda(1+\tilde\eta^j)|x-y^\ell|^{-2}\sum_{\ell=1,\ell\neq j}^m |x-y^j|^{-2}.\end{aligned}$$

Here

$$\begin{aligned}&[\nabla, \chi_{r_0}]w\\&= \nabla\chi_{r_0}\cdot w + \chi_{r_0}\cdot\nabla w[\Delta^2, \chi_{r_0}]w\\&= 2\Delta\chi_{r_0}\Delta w + w\Delta^2\chi_{r_0} + 4\nabla\chi_{r_0}\cdot\nabla(\Delta w) + 4\nabla w\cdot\nabla(\Delta\chi_{r_0}) + 4\nabla^2\chi_{r_0}\cdot\nabla^2 w.\end{aligned}$$

So,

$$\|\tilde N(0)\|_{\mathcal{C}^{4,\alpha}_\nu(B(y^j,r_0)-B(y^j,r_0/2))} \le c_\kappa \sup_{r_0/2\le r\le r_0} r^{4-\nu}|\tilde N(0)| \le c_\kappa r^2_{\varepsilon,\lambda,\gamma}.$$

Finally

$$\|\tilde N(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega-\bigcup_{j=1}^m B(y^j,r_{\varepsilon,\lambda,\gamma}))} \le c_\kappa(\varepsilon^4 + r^3_{\varepsilon,\lambda,\gamma} + \gamma + \lambda).$$

To derive the second estimate, we use the fact that for $\tilde v_1$ and $\tilde v_2 \in \tilde B_{r^2_{\varepsilon,\lambda,\gamma}}$ of $C^{4,\alpha}_\nu(\bar\Omega^*)$, we obtain

$$\|\tilde N(\tilde v_1) - \tilde N(\tilde v_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega_{r_{\varepsilon,\lambda,\gamma},y^j})} \le \Big\|\tilde{\mathcal{G}}_{\nu,Y}\circ\tilde\xi_{r_{\varepsilon,\lambda,\gamma}}\Big(\tilde S(\tilde v_1) - \tilde S(\tilde v_2)\Big)\Big\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega_{r_{\varepsilon,\lambda,\gamma},y^j})}.$$

Using(6.4) and Proposition 4.2, we conclude that

$$\|\tilde N(\tilde v_1) - \tilde N(\tilde v_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega_{r_{\varepsilon,\lambda,\gamma},y^j})} \le c_\kappa r^2_{\varepsilon,\lambda,\gamma}\|\tilde v_1 - \tilde v_2\|_{\mathcal{C}^{4,\alpha}_\nu(\bar\Omega^*)}.$$

□

Reducing $\varepsilon_\kappa, \lambda_\kappa$ and $\gamma_\kappa$ if necessary, we can assume that

$$\bar c_\kappa r^2_{\varepsilon,\lambda,\gamma} \le \frac{1}{2}$$

for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\gamma \in (0,\gamma_\kappa)$. Then, (6.8) and (6.9) are sufficient to show that $\tilde v \mapsto \tilde N(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\Phi,\Psi\tilde v)$ is a contraction from

$$\{\tilde v \in \mathcal{C}^{4,\alpha}_\nu(\bar\Omega^*(Y)) : \|\tilde v\|_{\mathcal{C}^{4,\alpha}_\nu(\bar\Omega^*(Y))} \le 2c_\kappa r^2_{\varepsilon,\lambda,\gamma}\}$$

into itself and hence has a unique fixed point $\tilde v(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\Phi,\Psi;\cdot)$ in this set. This fixed point is a solution of (6.3).

We summarize these resutls as follows.

**Proposition 6.4.** *For each $\kappa > 0$, there exists $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$, and $c_\kappa > 0$ (only depending on $\kappa$) such that for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$, $\gamma \in (0,\gamma_\kappa)$ and for all set of parameters $\tilde\eta$, points $Y$ satisfying*

$$|\tilde\eta| \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \text{and} \quad \|Y - X\| \le \kappa r_{\varepsilon,\lambda,\gamma}$$

*and boundary functions $\Phi$ and $\Psi$ satisfying (4.3) and*

$$\|\Phi\|_{(\mathbb{C}^{4,\alpha}(S^3))^m} \le \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\Psi\|_{(\mathbb{C}^{2,\alpha}(S^3))^m} \le \kappa r^2_{\varepsilon,\lambda,\gamma}.$$

*The function*

$$\begin{aligned}\tilde u(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\Phi,\Psi;\cdot) := \sum_{j=1}^m (1+\tilde\eta^j)G_{y^j} + \sum_{j=1}^m \chi_{r_0}(\cdot - y^j)H^e(\varphi^j,\psi^j;(\cdot - y^j)/r_{\varepsilon,\lambda,\gamma})\\ + \tilde v(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\Phi,\Psi;\cdot),\end{aligned}$$

*is a solution to (6.2) in $\bar\Omega_{r_{\varepsilon,\lambda,\gamma}}(Y)$. In addition*

$$\|\tilde v(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\Phi,\Psi;\cdot)\|_{\mathbb{C}^{4,\alpha}_\nu(\bar\Omega^*)} \le 2c_\kappa r^2_{\varepsilon,\lambda,\gamma}. \tag{6.11}$$

Observe that the function $\tilde v(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\Phi,\Psi)$ being obtained as a fixed point for contraction mapping, it depends continuously on the parameters $\tilde\eta$ and the points $Y$.

## 7. Nonlinear Cauchy-data matching

Keeping the notations of the previous sections, we gather the results of the Proposition 5.3 and Proposition 6.4. From now let $\kappa>1$ is fixed large enough (we will shortly see how) and assume that $\varepsilon\in(0,\varepsilon_\kappa)$, $\lambda\in(0,\lambda_\kappa)$ and $\gamma\in(0,\gamma_\kappa)$.

Assume that $X=(x^1,\dots,x^m)\in\Omega^m$ is a nondegenerate critical point of the function $W$ defined in the introduction. For all $j=1,\dots,m$, we define $\tau_*^j>0$ by

$$-4\log\tau_*^j=R(x^j,x^j)+\sum_{\ell\neq j}G(x^\ell,x^j). \tag{7.1}$$

We assume that we are given:

(i) points $Y:=(y^1,\dots,y^m)\in\Omega^m$ close to $X:=(x^1,\dots,x^m)$ satisfying (6.7).
(ii) parameters $\tilde\eta:=(\tilde\eta^1,\dots,\tilde\eta^m)\in\mathbb{R}^m$ satisfying (6.7).
(iii) parameters $T:=(\tau^1,\dots,\tau^m)\in(0,\infty)^m$ satisfying (5.6) (where, for each $j=1,\dots,m$, $\tau_*$ is replaced by $\tau_*^j$).

We set

$$R^j_{\varepsilon,\lambda,\gamma}:=\tau^j/r_{\varepsilon,\lambda,\gamma}$$

First, we consider the boundary data

$$\Phi:=(\varphi^1,\dots,\varphi^m)\in(\mathbb{C}^{4,\alpha}(S^3))^m\quad\text{and}\quad\Psi:=(\psi^1,\dots,\psi^m)\in(\mathbb{C}^{2,\alpha}(S^3))^m$$

satisfying (4.2) and (5.6).

Thanks to the result in Proposition 5.3, we can find $u_{\rm int}$ a solution of

$$\Delta^2u-\lambda\Delta u-\lambda|\nabla u|^2-\rho^4e^u=0$$

in each $B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$, which can be decomposed as

$$\begin{aligned}
&u_{\rm int}(\varepsilon,\lambda,\gamma,T,Y,\Phi,\Psi;x)\\
&:=u_{\varepsilon,\tau^j}(x-y^j)+h(R^j_{\varepsilon,\lambda,\gamma}(x-y^j)/r_{\varepsilon,\lambda,\gamma})+H^i(\varphi^j,\psi^j;(x-y^j)/r_{\varepsilon,\lambda,\gamma})\\
&\quad+v(\varepsilon,\lambda,\gamma,\tau^j,\varphi^j,\psi^j;R^j_{\varepsilon,\lambda,\gamma}(x-y^j)/r_{\varepsilon,\lambda,\gamma})
\end{aligned}$$

in $B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$. Similarly, given the boundary data

$$\tilde\Phi:=(\tilde\varphi^1,\dots,\tilde\varphi^m)\in(\mathbb{C}^{4,\alpha}(S^3))^m\quad\text{and}\quad\tilde\Psi:=(\tilde\psi^1,\dots,\tilde\psi^m)\in(\mathbb{C}^{2,\alpha}(S^3))^m$$

satisfying (4.3) and (6.6), we use the result of Proposition 6.4, to find $u_{\rm ext}$ a solution of

$$\Delta^2-\lambda\Delta u-\lambda|\nabla u|^2-\rho^4e^u=0$$

in $\bar\Omega_{r_{\varepsilon,\lambda,\gamma}}(Y)$, which can be decomposed as

$$\begin{aligned}
&u_{\rm ext}(\varepsilon,\lambda,\gamma,\tilde\eta,\tilde\Phi,\tilde\Psi;x)\\
&=\sum_{j=1}^m(1+\tilde\eta^j)G(y^j,x)+\sum_{j=1}^m\chi_{r_0}(x-y^j)H^e(\tilde\varphi^j,\tilde\psi^j;(x-y^j)/r_{\varepsilon,\lambda,\gamma})\\
&\quad+\tilde v(\varepsilon,\lambda,\gamma,\tilde\eta,Y,\tilde\Phi,\tilde\Psi;x).
\end{aligned}$$

It remains to determine the parameters and the boundary functions in such a way that the function which is equal to $u_{int}$ in $\cup_jB_{r_{\varepsilon,\lambda,\gamma}}(y^j)$ and which is equal to $u_{\rm ext}$

in $\bar{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1, \dots, m$

$$u_{\rm int} = u_{\rm ext}, \quad \partial_r u_{\rm int} = \partial_r u_{\rm ext}, \quad \Delta u_{\rm int} = \Delta u_{\rm ext}, \quad \partial_r \Delta u_{\rm int} = \partial_r \Delta u_{\rm ext}, \tag{7.2}$$

on $\partial B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$. Assuming we have already done so, this provides for each $\varepsilon, \lambda$ and $\gamma$ are small enough a function $w_{\varepsilon,\lambda,\gamma} \in \mathcal{C}^{4,\alpha}(\bar{\Omega})$ (which is obtained by patching together the function $u_{\rm int}$ and the function $u_{\rm ext}$) solution of $\Delta^2 u - \lambda\Delta u - \lambda|\nabla u|^2 - \rho^4 e^u = 0$ and elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result since, as $\varepsilon, \lambda$ and $\gamma$ tend to 0, the sequence of solutions we have obtained satisfies the required properties, namely, away from the points $x^j$ the sequence $w_{\varepsilon,\lambda,\gamma}$ converges to $\sum_j G(x^j, \cdot)$.

Before, we proceed, some remarks are due. First it will be convenient to observe that the functions $u_{\varepsilon,\tau^j}$ can be expanded as

$$u_{\varepsilon,\tau^j}(x) = -8\log|x| - 4\log\tau^j + \mathcal{O}(r^2_{\varepsilon,\lambda,\gamma}) \tag{7.3}$$

near $\partial B_{r_{\varepsilon,\lambda,\gamma}}$. Also, the function

$$\sum_{j=1}^{m}(1+\tilde{\eta}^j)G(y^j, x)$$

which appears in the expression of $u_{\rm ext}$ can be expanded as

$$\sum_{\ell=1}^{m}(1+\tilde{\eta}^\ell)G(y^\ell, y^j+x) = -8(1+\tilde{\eta}^j)\log|x| + E_j(Y;y^j) + \nabla E_j(Y;y^j)\cdot x + \mathcal{O}(r^2_{\varepsilon,\lambda,\gamma}) \tag{7.4}$$

near $\partial B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$. Here, we have defined

$$E_j(Y;\cdot) := R(y^j, \cdot) + \sum_{\ell\neq j} G(y^\ell, \cdot).$$

In (7.2), all functions are defined on $\partial B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$, nevertheless, it will be convenient to solve, instead of (7.2) the following set of equations

$$\begin{gathered}
(u_{\rm int} - u_{\rm ext})(y^j + r_{\varepsilon,\lambda,\gamma}\cdot) = 0, \quad (\partial_r u_{int} - \partial_r u_{\rm ext})(y^j + r_{\varepsilon,\lambda,\gamma}\cdot) = 0, \\
(\Delta u_{int} - \Delta u_{\rm ext})(y^j + r_{\varepsilon,\lambda,\gamma}\cdot) = 0, \quad (\partial_r \Delta u_{int} - \partial_r \Delta u_{\rm ext})(y^j + r_{\varepsilon,\lambda,\gamma}\cdot) = 0,
\end{gathered} \tag{7.5}$$

on $S^3$. Here all functions are considered as functions of $z \in S^3$ and we have simply used the change of variables $x = y^j + r_{\varepsilon,\lambda,\gamma} z$ to parameterize $\partial B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$.

Since the boundary data satisfy (4.2) and (4.3), we decompose

$$\begin{gathered}
\Phi = \Phi_0 + \Phi_1 + \Phi^\perp, \quad \Psi = 8\Phi_0 + 12\Phi_1 + \Psi^\perp, \\
\tilde{\Phi} = \tilde{\Phi}_0 + \tilde{\Phi}_1 + \tilde{\Phi}^\perp, \quad \tilde{\Psi} = \tilde{\Psi}_1 + \tilde{\Psi}^\perp
\end{gathered}$$

where the components of $\Phi_0, \tilde{\Phi}_0$ are constant functions on $S^3$, the components of $\Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1$ belong to $\ker(\Delta_{S^3} + 3) = \operatorname{span}\{e_1, \dots, e_4\}$ and where the components of $\Phi^\perp, \Psi^\perp, \tilde{\Phi}^\perp, \tilde{\Psi}^\perp$ are $L^2(S^3)$ orthogonal to the constant function and the functions $e_1, \dots, e_4$. Observe that the components of $\Psi$ over the constant functions or functions in $\ker(\Delta_{S^3} + 3)$ are determined by the corresponding components of $\Phi$. Moreover, $\tilde{\Psi}$ has no component over constant functions.

We first consider the $L^2(S^3)$-orthogonal projection of (7.5) onto the space of functions which are orthogonal to the constant function and the functions $e_1, \dots, e_4$. This yields the system

$$\begin{aligned} \varphi^{j,\perp} - \tilde{\varphi}^{j,\perp} &= M_0^{(j)}(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}) \\ \partial_r H^i(\varphi^{j,\perp}, \psi^{j,\perp}; \cdot) - \partial_r H^e(\tilde{\varphi}^{j,\perp}, \tilde{\psi}^{j,\perp}; \cdot) &= M_1^{(j)}(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}) \\ \psi^{j,\perp} - \tilde{\psi}^{j,\perp} &= M_2^{(j)}(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}) \\ &\partial_r \Delta H^i(\varphi^{j,\perp}, \psi^{j,\perp}; \cdot) - \partial_r \Delta H^e(\tilde{\varphi}^{j,\perp}, \tilde{\psi}^{j,\perp}; \cdot)) \\ &= M_3^{(j)}(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}) \end{aligned} \tag{7.6}$$

where the functions $M_k^{(j)}$ are nonlinear functions of the parameters $\varepsilon$, $\tilde{\eta}$, $Y$, $T$ and the boundary data $\Phi$, $\tilde{\Phi}$, $\Psi$ and $\tilde{\Psi}$. Moreover, using (7.3) and (7.4) and also (5.11) (keeping in mind that $\mu \in (1,2)$) and (6.11) (keeping in mind that $\nu \in (-1,0)$), we conclude that, for each $j = 1, \dots, m$ and $k = 0, 1, 2, 3$

$$\|M_k^{(j)}\|_{\mathcal{C}^{4-k,\alpha}(S^3)} \le c r_{\varepsilon,\lambda,\gamma}^2 \tag{7.7}$$

for some constant $c > 0$ independent of $\kappa$ (provided $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$).

Thanks to the result of Lemma 4.7 and (7.7), this last system can be re-written as

$$(\Phi^\perp, \tilde{\Phi}^\perp, \Psi^\perp, \tilde{\Psi}^\perp) = M(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi, \tilde{\Phi}, \Psi, \tilde{\Psi})$$

where

$$\|M\|_{(\mathcal{C}^{4,\alpha}(S^3))^{2m} \times (\mathcal{C}^{2,\alpha}(S^3))^{2m}} \le c r_{\varepsilon,\lambda,\gamma}^2$$

for some constant $c > 0$ independent of $\kappa$ (provided $(\varepsilon_\kappa, \lambda_\kappa, \gamma_\kappa) \in (0, \varepsilon_\kappa)^3$). Moreover, (5.8) and (6.9) imply (reducing $\varepsilon_\kappa$, $\lambda_\kappa$ and $\gamma_\kappa$ if necessary) that, the mapping $M$ is a contraction from the ball of radius $\kappa r_{\varepsilon,\lambda,\gamma}^2$ in $(\mathcal{C}^{4,\alpha}(S^3))^{2m} \times (\mathcal{C}^{2,\alpha}(S^3))^{2m}$ into itself and as such has a unique fixed point in this set. Observe that this fixed point depends continuously on $\varepsilon, \lambda, \gamma$, $\tilde{\eta}$, $T$, $Y$ and also on $\Phi_0$, $\tilde{\Phi}_0$, $\Phi_1$, $\tilde{\Phi}_1$ and $\tilde{\Psi}_1$.

We insert this fixed point in (7.5) and now project the corresponding system over the set of functions spanned by $e_1, \dots, e_4$ and finally over the set of constant functions.

The first projection yields the system of equations

$$\begin{aligned} \Phi_1 &= \bar{M}_1(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1) \\ \tilde{\Phi}_1 &= \bar{M}_2(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1) \\ \Psi_1 &= \bar{M}_3(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1) \\ r_{\varepsilon,\lambda,\gamma} \nabla E_j(Y; y^j) &= \bar{M}_4^{(j)}(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1) \end{aligned} \tag{7.8}$$

where the functions $\bar{M}_k$ (and also $\bar{M}_4^{(j)}$) are nonlinear functions depending continuously on the parameters $\varepsilon, \lambda, \gamma$, $\tilde{\eta}$, $T$, $Y$ and the components of the boundary data $\Phi_0$, $\tilde{\Phi}_0$, $\Phi_1$, $\tilde{\Phi}_1$ and $\tilde{\Psi}_1$. Moreover,

$$|\bar{M}_k| \le c r_{\varepsilon,\lambda,\gamma}^2$$

for some constant $c > 0$ independent of $\kappa$ (provided provided $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$).

Let us comment briefly on how these equations are obtained. These equations simply come from (7.2) when expansions (7.3) and (7.4) are taken into account, together with the expression of $H^i(\varphi^j, \psi^j; \cdot)$ and $H^e(\tilde{\varphi}^j, \tilde{\psi}^j; \cdot)$ given in Lemma 4.4 and Lemma 4.6, and also the estimates (5.11) and (6.11). Observe that the projection of the term $x \to \nabla E_j(Y; y^j) \cdot x$ which arises in (7.4), as well as the projection of its partial derivative with respect to $r$, over the set of constant function is equal to 0. Moreover, this term projects identically over the set of functions spanned by $e_1, \dots, e_4$ as well as its derivative with respect to $r$. Finally, its Laplacian vanishes identically.

Recall that we have define in the introduction the function

$$W(Y) := \sum_{j=1}^{m} R(y^j, y^j) + \sum_{j_1 \neq j_2} G(y^{j_1}, y^{j_2})$$

Using the symmetries of the functions $G$ and $R$, namely the fact that

$$G(x, y) = G(y, x) \quad \text{and} \quad R(x, y) = R(y, x)$$

we obtain

$$\nabla W|_Y = 2(\nabla E_1(Y; y^1), \dots, \nabla E_m(Y, y^m)).$$

Now, we have assumed that the point $X = (x^1, \dots, x^m)$ is a nondegenerate critical point of the functional $W$ and hence $\nabla W|_X = 0$, and

$$(\mathbb{R}^4)^m \ni Z \mapsto D(\nabla W)|_X(Z) \in (\mathbb{R}^4)^m$$

is invertible. Therefore, the last equation can be rewritten as

$$r_{\varepsilon,\lambda,\gamma}(Y - X) = \bar{M}_5(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1)$$

The projection of (7.5) over the constant function, leads to the system

$$\begin{aligned}
(\log 1/r^2_{\varepsilon,\lambda,\gamma})^{-1} \log(\tau^j/\tau^j_*) &= \bar{M}_6(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \Psi_1, \tilde{\Psi}_1) \\
\tilde{\Phi}_0 &= \bar{M}_7(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \Psi_1, \tilde{\Psi}_1) \\
\Phi_0 &= \bar{M}_8(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \Psi_1, \tilde{\Psi}_1) \\
\tilde{\eta} &= \bar{M}_9(\varepsilon, \lambda, \gamma, \tilde{\eta}, T, Y, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \Psi_1, \tilde{\Psi}_1)
\end{aligned} \tag{7.9}$$

where the function $\bar{M}_k$ satisfy the usual properties. We are now in a position to define $\tau_-$ and $\tau^+$ since, according to the above, as $\varepsilon$, $\lambda$ and $\gamma$ tend to 0 we expect that $y_i$ will converge to $x_i$ and that $\tau_i$ will converge to $\tau_i^*$ satisfying (7.1) and hence it is enough to choose $\tau_-$ and $\tau^+$ in such a way that

$$4\log(\tau_-) < -\sup_i E_i(Y, x_i) \le -\inf_i E_i(Y, x_i) < 4\log(\tau^+).$$

So, if we define the parameters $U := (u^1, \dots, u^m)$ where

$$u^j = \frac{1}{\log 1/r^2_{\varepsilon,\lambda,\gamma}} \log(\tau^j/\tau^j_*), \quad Z = r_{\varepsilon,\lambda,\gamma}(Y - X)$$

so that the system we have to solve reads

$$(\varepsilon, \lambda, \gamma, U, \tilde{\eta}, Z, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1) = \bar{M}(\varepsilon, \lambda, \gamma, U, \tilde{\eta}, Z, \Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1). \tag{7.10}$$

where as usual, the nonlinear function $\bar{M}$ depends continuously on the parameters $T, \tilde{\eta}, Z$ and the functions $\Phi_0, \tilde{\Phi}_0, \Phi_1, \tilde{\Psi}_1$ and is bounded (in the appropriate norm) by a constant (independent of $\varepsilon, \lambda, \gamma$ and $\kappa$) time $r^2_{\varepsilon,\lambda,\gamma}$, provided $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$. Observe that

$$U, \tilde{\eta} \in \mathbb{R}^m, \quad Z \in (\mathbb{R}^4)^m, \quad \Phi_0, \tilde{\Phi}_0 \in \mathbb{R}^m$$

$$\Phi_1, \tilde{\Phi}_1, \tilde{\Psi}_1 \in (\ker(\Delta_{S^3} + 3))^m.$$

In addition, reducing $\varepsilon_\kappa, \lambda_\kappa$ and $\gamma_\kappa$ if necessary, this nonlinear mapping sends the ball of radius $\kappa r^2_{\varepsilon,\lambda,\gamma}$ (for the natural product norm) into itself, provided $\kappa$ is fixed large enough, $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$. Applying Schauder's fixed point Theorem in the ball of radius $\kappa r^2_{\varepsilon,\lambda,\gamma}$ in the product space where the entries live yields the existence of a solution of (7.10) and this completes the proof of Theorem 1.2.

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