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PIECEWISE WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS OF IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS VIA FRACTIONAL OPERATORS

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Abstract. In this article, we show sufficient conditions for the existence, uniqueness and attractivity of piecewise weighted pseudo almost periodic classical solution of nonlinear impulsive integro-differential equations. The working tools are based on the fixed point theorem and fractional powers of operators. An application to impulsive integro-differential equations is presented.

1. Introduction

It is well known that impulsive integro-differential equations or impulsive differential equations create an important subject of numerous mathematical investigations and constitute a significant branch of differential equations. It has great applications in actual modeling such as population dynamics, epidemic, engineering, optimal control, neural networks, economics, etc. With the help of several tools of functional analysis, topology and fixed point theorems, many authors have made important contributions to this theory [3, 5, 11, 15, 16, 20].

The concept of pseudo almost periodic function, which was introduced by Zhang [23, 24], is a natural and good generalization of the classical almost periodic functions in the sense of Bohr. Recently, weighted pseudo almost periodic function is investigated in [6] by the weighted function, which was more tricky and changeable than those of the classical functions. Many authors have made important contributions to this function. For more details on weighted pseudo almost periodic function and related topics, one can see [1, 4, 7, 8, 13] and the references therein.

For the integro-differential equations, the asymptotic properties of mild solutions have been studied from differential points, such as almost periodicity, almost automorphy, asymptotic stability, oscillation and so on. However, for the weighted pseudo almost periodicity of classical solutions, it is rarely investigated, particularly for the integro-differential equations with impulsive effects. The existence, uniqueness and attractivity of piecewise weighted pseudo almost periodic classical solutions for impulsive integro-differential equations is an untreated topic in the literature and this fact is the motivation of the present work.

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This article is organized as follows. In Section 2, we recall some fundamental results about the notion of piecewise almost periodic function. In Section 3, we introduce the concept of piecewise weighted pseudo almost periodic function and explore its properties. Sections 4 is devoted to the existence, uniqueness and attractivity of classical solution of (4.1) by fixed point theorem and fractional powers of operators. In Section 5, an application to impulsive integro-differential equations is presented to illustrate the main findings.

2. Preliminaries

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces, Ω be a subset of X and $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ stand for the set of natural numbers, integers and real numbers, respectively. For A being a linear operator on X, $\mathcal{D}(A)$ stands for the domain of A. Let T be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\kappa = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to-\infty} t_i = -\infty$.

To facilitate the discussion below, we further introduce the following notation:

- $C(\mathbb{R}, X)$: the set of continuous functions from $\mathbb{R}$ to X.
- $PC(\mathbb{R}, X)$: the space formed by all piecewise continuous functions $f : \mathbb{R} \to X$ such that $f(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$, $f(t_i^+)$, $f(t_i^-)$ exist, and $f(t_i^-) = f(t_i)$ for all $i \in \mathbb{Z}$.
- $PC(\mathbb{R} \times \Omega, X)$: the space formed by all piecewise continuous functions $f : \mathbb{R} \times \Omega \to X$ such that for any $x \in \Omega$, $f(\cdot, x) \in PC(\mathbb{R}, X)$ and for any $t \in \mathbb{R}$, $f(t, \cdot)$ is continuous at $x \in \Omega$.

It is possible to define fractional powers of A if $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ in a Banach space and $0 \in \rho(A)$. For $\alpha > 0$, define the fractional power $A^{-\alpha}$ of A by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)dt.$$

Operator $A^{-\alpha}$ is bounded, bijective and $A^\alpha = (A^{-\alpha})^{-1}$ is a closed linear operator, $\mathcal{D}(A^\alpha) = \mathcal{R}(A^{-\alpha})$, A^0 is the identity operator in X. For $0 \le \alpha \le 1$, $X_\alpha = \mathcal{D}(A^\alpha)$ with norm $\|x\|_\alpha = \|A^\alpha x\|$ is a Banach space.

Lemma 2.1 ([17]). *Let $-A$ be an infinitesimal operator of an analytic semigroup $T(t)$, then*

(i) $T(t) : X \to \mathcal{D}(A^\alpha)$ *for every $t > 0$ and $\alpha \ge 0$.*
(ii) *For every $x \in \mathcal{D}(A^\alpha)$, it follows that $T(t)A^\alpha x = A^\alpha T(t)x$.*
(iii) *For every $t > 0$, the operator $A^\alpha T(t)$ is bounded and*
$$\|A^\alpha T(t)\| \le M_\alpha t^{-\alpha} e^{-\lambda t}, \quad M_\alpha > 0,\ \lambda > 0. \tag{2.1}$$
(iv) *For $0 < \alpha \le 1$ and $x \in \mathcal{D}(A^\alpha)$, we have*
$$\|T(t)x - x\| \le C_\alpha t^\alpha \|A^\alpha x\|, \quad C_\alpha > 0.$$

Now, we recall the concepts of discrete almost periodic function, discrete weighted pseudo almost periodic function, piecewise almost periodic function.

Definition 2.2 ([10]). A function $f \in C(\mathbb{R}, X)$ is said to be almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t + \tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$. Denote by $AP(\mathbb{R}, X)$ the set of such functions.

Definition 2.3 ([19])**.** A sequence $\{x_n\}$ is called almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e. there exists a natural number $l = l(\varepsilon)$, such that for $k \in \mathbb{Z}$, there is at least one number p in $[k, k + l]$, for which inequality $\|x_{n+p} - x_n\| < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, X)$ the set of such sequences.

Let U_d denote the collection of functions (weights) $\rho : \mathbb{Z} \to (0, +\infty)$. For $\rho \in U_d$ and $m \in \mathbb{Z}^+ = \{n \in \mathbb{Z}, n \geq 0\}$, set $\mu(m, \rho) := \sum_{k=-m}^{m} \rho_k$. Denote $U_{d,\infty} := \{\rho \in U_d : \lim_{m\to\infty} \mu(m, \rho) = \infty\}$.

For $\rho \in U_{d,\infty}$, define

$$AAP_0(\mathbb{Z}, X) = \{x_n \in l^\infty(\mathbb{Z}, X) : \lim_{n\to\infty} \|x_n\| = 0\}.$$

$$WPAP_0(\mathbb{Z}, X, \rho) := \{x_n \in l^\infty(\mathbb{Z}, X),\ \lim_{m\to\infty} \frac{1}{\mu(m, \rho)} \sum_{k=-m}^{m} \|x_k\|\rho_k = 0\}.$$

Definition 2.4 ([18])**.** A sequence $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$ is called discrete asymptotically almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, X)$, $x_n^2 \in AAP_0(\mathbb{Z}, X)$. Denote by $AAP(\mathbb{Z}, X)$ the set of such sequences.

Definition 2.5 ([9])**.** Let $\rho \in U_{d,\infty}$. A sequence $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$ is called discrete weighted pseudo almost periodic if it can be expressed as $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, X)$ and $x_n^2 \in WPAP_0(\mathbb{Z}, X, \rho)$. The set of such functions denoted by $WPAP(\mathbb{Z}, X, \rho)$.

For $\{t_i\}_{i\in\mathbb{Z}} \in T$, $\{t_i^j\}$ defined by $\{t_i^j = t_{i+j} - t_i\}, i \in \mathbb{Z}, j \in \mathbb{Z}$.

Definition 2.6 ([19])**.** A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

(1) $\{t_i^j = t_{i+j} - t_i\}$, $i, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods common for all of the sequences $\{t_i^j\}$.
(2) For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $\|f(t') - f(t'')\| < \varepsilon$.
(3) For any $\varepsilon > 0$, there exists a relatively dense set Ω_ε in $\mathbb{R}$ such that if $\tau \in \Omega_\varepsilon$, then
$$\|f(t + \tau) - f(t)\| < \varepsilon$$
for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_i| > \varepsilon, i \in \mathbb{Z}$.

We denote by $AP_p(\mathbb{R}, X)$ the space of all piecewise almost periodic functions. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $\mathcal{U}PC(\mathbb{R}, X)$ be the space of all functions $f \in PC(\mathbb{R}, X)$ such that f satisfies the condition (2) in Definition 2.6.

Definition 2.7. $f \in PC(\mathbb{R} \times \Omega, X)$ is said to be piecewise almost periodic in t uniformly in $x \in \Omega$ if for each compact set $K \subseteq \Omega$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that $\|f(t + \tau, x) - f(t, x)\| \leq \varepsilon$ for all $x \in K, \tau \in \Omega_\varepsilon$ and $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$. Denote by $AP_p(\mathbb{R} \times \Omega, X)$ the set of all such functions.

Lemma 2.8 ([19]). *If the sequences $\{t_i^j\}$ are equipotentially almost periodic, then for each $j > 0$, there exists a positive integer N such that on each interval of length j, there are no more than N elements of the sequence $\{t_i\}$, i.e.,*

$$i(s,t) \leq N(t-s) + N,$$

where $i(s,t)$ is the number of the points $\{t_i\}$ in the interval $[s,t]$.

Lemma 2.9 ([19]). *Assume that $f \in AP_p(\mathbb{R}, X)$, $\{x_i\}_{i\in\mathbb{Z}} \in AP(\mathbb{Z}, X)$, and $\{t_i^j\}$, $j \in \mathbb{Z}$ are equipotentially almost periodic. Then for each $\varepsilon > 0$, there exist relatively dense sets Ω_ε of $\mathbb{R}$ and Q_ε of $\mathbb{Z}$ such that*

(i) *$\|f(t+\tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$ and $i \in \mathbb{Z}$.*
(ii) *$\|x_{i+q} - x_i\| < \varepsilon$ for all $q \in Q_\varepsilon$ and $i \in \mathbb{Z}$.*
(iii) *$|t_i^q - \tau| < \varepsilon$ for all $q \in Q_\varepsilon$, $\tau \in \Omega_\varepsilon$ and $i \in \mathbb{Z}$.*

Now, we give the generalized Gronwall-Bellman inequality which will be used later, one can see [12, Theorem 2.1] for more details.

Lemma 2.10 (generalized Gronwall-Bellman inequality). *Let a nonnegative function $u(t) \in PC(\mathbb{R}, X)$ satisfy for $t \geq t_0$*

$$u(t) \leq n(t) + \int_{t_0}^t v(\tau)u(\tau)d\tau + \sum_{t_0<t_i<t} \beta_i u(\tau_i),$$

with $n(t)$ a positive nondecreasing function for $t \geq t_0$, $\beta_i \geq 0$, $v(\tau) \geq 0$ and τ_i are the first kind discontinuity points of the functions $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq n(t) \prod_{t_0<t_i<t} (1+\beta_i) e^{\int_{t_0}^t v(\tau)d\tau}.$$

3. Piecewise weighted pseudo almost periodicity

In this section, we introduce the concept of piecewise weighted pseudo almost periodic function, explore its properties and establish the composition theorem.

Let U be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $r > 0$ and each $\rho \in U$, set

$$\mu(r, \rho) := \int_{-r}^r \rho(t)dt.$$

Define

$$U_\infty := \{\rho \in U : \lim_{r\to\infty} \mu(r,\rho) = \infty\},$$
$$U_B := \{\rho \in U_\infty : \rho \text{ is bounded and } \inf_{x\in\mathbb{R}} \rho(x) > 0\}.$$

It is clear that $U_B \subset U_\infty \subset U$.

Definition 3.1. Let $\rho_1, \rho_2 \in U_\infty$, ρ_1 is said to be equivalent to ρ_2 (i.e. $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$

It is trivial to show that "$\sim$" is a binary equivalence relation on U_∞. The equivalence class of a given weight $\rho \in U_\infty$ is denoted by $cl(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho\in U_\infty} cl(\rho)$.

For $\rho \in U_\infty$, define

$$PC_p^0(\mathbb{R}, X) = \{f \in PC(\mathbb{R}, X) : \lim_{t\to\infty} \|f(t)\| = 0\},$$

$$WPAP_p^0(\mathbb{R}, X, \rho) := \{f \in PC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t)\|f(t)\|dt = 0\},$$

$$\begin{aligned} &WPAP_p^0(\mathbb{R}\times\Omega, X, \rho) \\ &= \Big\{f \in PC(\mathbb{R}\times\Omega, X) : \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t)\|f(t,x)\|dt = 0 \text{ uniformly with} \\ &\quad \text{respect to } x \in K, \text{ where } K \text{ is an arbitrary compact subset of } \Omega\Big\}. \end{aligned}$$

Definition 3.2. A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise asymptotically almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_p(\mathbb{R}, X)$ and $\varphi \in PC_p^0(\mathbb{R}, X)$. Denote by $AAP_p(\mathbb{R}, X)$ the set of all such functions.

Definition 3.3. A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise weighted pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_p(\mathbb{R}, X)$ and $\varphi \in WPAP_p^0(\mathbb{R}, X, \rho)$. Denote by $WPAP_p(\mathbb{R}, X, \rho)$ the set of all such functions.

Definition 3.4. Let $WPAP_p(\mathbb{R}\times\Omega, X, \rho)$ consist of all functions $f \in PC(\mathbb{R}\times\Omega, X)$ such that $f = g + \varphi$, where $g \in AP_p(\mathbb{R}\times\Omega, X)$ and $\varphi \in WPAP_p^0(\mathbb{R}\times\Omega, X, \rho)$.

Let $\rho \in U_\infty, \tau \in \mathbb{R}$, and defined ρ^τ by $\rho^\tau(t) = \rho(t+\tau)$ for $t \in \mathbb{R}$. Define [25]

$$U_T = \{\rho \in U_\infty : \rho \sim \rho^\tau \text{ for each } \tau \in \mathbb{R}\}.$$

It is easy to see that U_T contains many of weights, such as $1, (1+t^2)/(2+t^2), e^t$, and $1+|t|^n$ with $n \in \mathbb{N}$ et al.

It is obvious that $(WPAP_p(\mathbb{R}, X, \rho), \|\cdot\|_\infty)$ (resp. $(WPAP_p(\mathbb{R}\times Y, X, \rho), \|\cdot\|_\infty)$), $\rho \in U_T$ is a Banach space when endowed with the sup norm.

Remark 3.5. (i) For $\rho \in U_T$, $WPAP_p^0(\mathbb{R}, X, \rho)$ is a translation invariant set of $PC(\mathbb{R}, X)$.
(ii) $PC_p^0(\mathbb{R}, X) \subset WPAP_p^0(\mathbb{R}, X, \rho)$ and $AAP_p(\mathbb{R}, X) \subset WPAP_p(\mathbb{R}, X, \rho)$

Similarly as the proof of [7, Lemma 2.5], one has the following lemma.

Lemma 3.6. *Let $\{f_n\}_{n\in\mathbb{N}} \subset WPAP_p^0(\mathbb{R}, X, \rho)$ be a sequence of functions. If f_n converges uniformly to f, then $f \in WPAP_p^0(\mathbb{R}, X, \rho)$.*

Similarly as the proof of [14], the following results and the composition theorems are hold for piecewise weighted pseudo almost periodic function.

Theorem 3.7. *Suppose the sequence of vector-valued functions $\{I_i\}_{i\in\mathbb{Z}}$ is weighted pseudo almost periodic, i.e, for any $x \in \Omega$, $\{I_i(x), i \in \mathbb{Z}\}$ is a weighted pseudo almost periodic sequence. Assume that the following conditions hold:*
(i) *$\{I_i(x), i \in \mathbb{Z}, x \in K\}$ is bounded for every bounded subset $K \subset \Omega$.*
(ii) *$I_i(x)$ is uniformly continuous in $x \in \Omega$ uniformly in $i \in \mathbb{Z}$.*

If $\phi \in WPAP_p(\mathbb{R}, X, \rho) \cap UPC(\mathbb{R}, X)$ such that $\mathcal{R}(\phi) \subset \Omega$, then $I_i(\phi(t_i))$ is weighted pseudo almost periodic.

Corollary 3.8. *Assume that the sequence of vector-valued functions $\{I_i\}_{i\in\mathbb{Z}}$ is weighted pseudo almost periodic, and there exists a constant $L_1 > 0$ such that*

$$\|I_i(u) - I_i(v)\| \le L_1\|u - v\|, \quad \textit{for all } u, v \in \Omega,\ i \in \mathbb{Z}.$$

if $\phi \in WPAP_p(\mathbb{R}, X, \rho) \cap UPC(\mathbb{R}, X)$ such that $\mathcal{R}(\phi) \subset \Omega$, then $I_i(\phi(t_i))$ is weighted pseudo almost periodic.

Theorem 3.9. *Suppose $f \in WPAP_p(\mathbb{R} \times \Omega, X, \rho)$. Assume that the following conditions hold:*

- (i) *$\{f(t,u) : t \in \mathbb{R}, u \in K\}$ is bounded for every bounded subset $K \subseteq \Omega$.*
- (ii) *$f(t,\cdot)$ is uniformly continuous in each bounded subset of Ω uniformly in $t \in \mathbb{R}$.*

If $\varphi \in WPAP_p(\mathbb{R}, X, \rho)$ such that $\mathcal{R}(\varphi) \subset \Omega$, then $f(\cdot, \varphi(\cdot)) \in WPAP_p(\mathbb{R}, X, \rho)$.

Corollary 3.10. *Let $f \in WPAP_p(\mathbb{R}\times\Omega, X, \rho)$, $\varphi \in WPAP_p(\mathbb{R}, X, \rho)$ and $\mathcal{R}(\varphi) \subset \Omega$. Assume that there exists a constant $L_f > 0$ such that*

$$\|f(t,u) - f(t,v)\| \le L_f \|u - v\|, \quad t \in \mathbb{R},\ u, v \in \Omega,$$

then $f(\cdot, \varphi) \in WPAP_p(\mathbb{R}, X, \rho)$.

4. Impulsive integro-differential equations

In this section, we investigate the existence, uniqueness and attractivity of piecewise weighted pseudo almost periodic classical solution of nonlinear impulsive integro-differential equations:

$$\begin{gathered}
\frac{du(t)}{dt} + Au(t) = f(t, u(t), (Ku)(t)), \quad t \in \mathbb{R},\ t \ne t_i,\ i \in \mathbb{Z}, \\
(Ku)(t) = \int_{-\infty}^{t} k(t-s) g(s, u(s)) ds, \\
\Delta u(t_i) := u(t_i^+) - u(t_i^-) = I_i(u(t_i)),
\end{gathered} \tag{4.1}$$

First, we make the following assumptions:

- (H1) $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ such that $\|T(t)\| \le Me^{-\omega t}$ for $t \ge 0$ and $0 \in \rho(A)$.
- (H2) $k \in C(\mathbb{R}^+, \mathbb{R})$ and $|k(t)| \le C_k e^{-\eta t}$ for some positive constants C_k, η.
- (H3) $g \in WPAP_p(\mathbb{R}\times X_\alpha, X, \rho)$, $\rho \in U_T$ and there exists a constant $L_g > 0$ such that
$$\|g(t,u) - g(t,v)\| \le L_g \|u - v\|_\alpha, \quad t \in \mathbb{R},\ u, v \in X_\alpha.$$
- (H4) $f \in WPAP_p(\mathbb{R}\times X_\alpha \times X_\alpha, X, \rho)$, $\rho \in U_T$ and there exists constants $L_f > 0$, $0 < \theta < 1$ such that
$$\|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)\| \le L_f(|t_1 - t_2|^\theta + \|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_\alpha),$$
for each $(t_i, u_i, v_i) \in \mathbb{R} \times X_\alpha \times X_\alpha,\ i = 1, 2.$
- (H5) $I_i \in WPAP(\mathbb{Z}, X, \rho)$ and there exists a constant $L_1 > 0$ such that
$$\|I_i(u) - I_i(v)\| \le L_1 \|u - v\|_\alpha, \quad t \in \mathbb{R},\ u, v \in X_\alpha,\ i \in \mathbb{Z}.$$

Before starting our main results, we recall the definition of the mild solution of (4.1).

Definition 4.1 ([19])**.** A function $u : \mathbb{R} \to X$ is called a mild solution of (4.1) if for any $t \in \mathbb{R}$, $t > \sigma$, $\sigma \ne t_i$, $i \in \mathbb{Z}$,

$$u(t) = T(t-\sigma)u(\sigma) + \int_\sigma^t T(t-s) f(s, u(s), (Ku)(s)) ds + \sum_{\sigma < t_i < t} T(t - t_i) I_i(u(t_i)), \tag{4.2}$$

Note that if (H1) holds, then (4.2) can be rewritten as

$$u(t)=\int_{-\infty}^{t}T(t-s)f(s,u(s),(Ku)(s))ds+\sum_{t_i<t}T(t-t_i)I_i(u(t_i)).$$

Lemma 4.2. *Assume that* (H1)–(H3) *hold, if* $u\in WPAP_p(\mathbb{R},X_\alpha,\rho)$, *then*

$$K(A^{-\alpha}u)(t):=\int_{-\infty}^{t}k(t-s)g(s,A^{-\alpha}u(s))ds\in WPAP_p(\mathbb{R},X,\rho).$$

Proof. Since $A^{-\alpha}$ is bounded, $\phi(\cdot)=g(s,A^{-\alpha}u(s))\in WPAP_p(\mathbb{R},X,\rho)$ by Corollary 3.10. Let $\phi=\phi_1+\phi_2$, where $\phi_1\in AP_p(\mathbb{R},X)$, $\phi_2\in WPAP_p^0(\mathbb{R},X,\rho)$, then

$$\begin{aligned}\int_{-\infty}^{t}k(t-s)g(s,A^{-\alpha}u(s))ds&=\int_{-\infty}^{t}k(t-s)\phi_1(s)ds+\int_{-\infty}^{t}k(t-s)\phi_2(s)ds\\&:=\Psi_1(t)+\Psi_2(t),\end{aligned}$$

where

$$\Psi_1(t)=\int_{-\infty}^{t}k(t-s)\phi_1(s)ds,\quad \Psi_2(t)=\int_{-\infty}^{t}k(t-s)\phi_2(s)ds.$$

(i) $\Psi_1\in AP_p(\mathbb{R},X)$. It is not difficult to see that $\Psi_1\in\mathcal{U}PC(\mathbb{R},X)$. Since $\phi_1\in AP_p(\mathbb{R},X)$, for $\varepsilon>0$, let Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε-periods of ϕ_1. If $\tau\in\Omega_\varepsilon$, $t\in\mathbb{R}$, $|t-t_i|>\varepsilon$, $i\in\mathbb{Z}$, then

$$\|\phi_1(t+\tau)-\phi_1(t)\|<\varepsilon.$$

Hence, by (H2), for $t\in\mathbb{R}$, $|t-t_i|>\varepsilon$, $i\in\mathbb{Z}$, one has

$$\begin{aligned}\|\Psi_1(t+\tau)-\Psi_1(t)\|&=\|\int_{-\infty}^{t+\tau}k(t+\tau-s)\phi_1(s)ds-\int_{-\infty}^{t}k(t-s)\phi_1(s)ds\|\\&=\|\int_{-\infty}^{t}k(t-s)(\phi_1(s+\tau)-\phi_1(s))ds\|\\&\le\int_{-\infty}^{t}C_ke^{-\eta(t-s)}\|\phi_1(s+\tau)-\phi_1(s)\|ds\\&<\frac{C_k}{\eta}\varepsilon,\end{aligned}$$

which implies that $\Psi_1\in AP_p(\mathbb{R},X)$.

(ii) $\Psi_2\in WPAP_p^0(\mathbb{R},X,\rho)$. In fact, for $r>0$, one has

$$\begin{aligned}\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\rho(t)\|\Psi_2(t)\|dt&=\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\rho(t)\|\int_{-\infty}^{t}k(t-s)\phi_2(s)ds\|dt\\&=\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\rho(t)\|\int_{0}^{\infty}k(s)\phi_2(t-s)ds\|dt\\&\le\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_{0}^{\infty}C_ke^{-\eta s}\rho(t)\|\phi_2(t-s)\|dsdt\\&\le\int_{0}^{\infty}C_ke^{-\eta s}\Phi_r(s)ds,\end{aligned}$$

where

$$\Phi_r(s)=\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\rho(t)\|\phi_2(t-s)\|dt.$$

Since $\phi_2 \in WPAP_p^0(\mathbb{R}, X, \rho)$, $\rho \in U_T$ it follows that $\phi_2(\cdot - s) \in WPAP_p^0(\mathbb{R}, X, \rho)$ for each $s \in \mathbb{R}$ by Remark 3.5, hence $\lim_{r\to\infty} \Phi_r(s) = 0$ for all $s \in \mathbb{R}$. By using the Lebesgue dominated convergence theorem, we have $\Psi_2 \in WPAP_p^0(\mathbb{R}, X, \rho)$. This completes the proof. $\square$

Lemma 4.3. *Assume that* (H1)–(H4) *hold, if* $u \in WPAP_p(\mathbb{R}, X, \rho)$, *then*

$$(\Lambda u)(t) := \int_{-\infty}^{t} A^\alpha T(t-s) f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s)))ds \in WPAA_p(\mathbb{R}, X, \rho).$$

Proof. We first show that Λu is well defined. In fact, if $u \in WPAA_p(\mathbb{R}, X, \rho)$, one has $K(A^{-\alpha}u) \in WPAA_p(\mathbb{R}, X, \rho)$ by Lemma 4.2, and $f(\cdot, A^{-\alpha}u(\cdot), K(A^{-\alpha}u(\cdot))) \in WPAA_p(\mathbb{R}, X, \rho)$ by Corollary 3.10. Hence $h(\cdot) = f(\cdot, A^{-\alpha}u(\cdot), K(A^{-\alpha}u(\cdot))) \in WPAA_p(\mathbb{R}, X, \rho)$, then $\|h\| := \sup_{t\in\mathbb{R}} \|h(t)\| < \infty$. By Lemma 2.1, one has

$$\begin{aligned}\|A^\alpha T(t-s) f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s)))\| &\le M_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} \|h(s)\| \\ &\le M_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} \|h\|,\end{aligned}$$

since

$$\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\lambda(t-s)} ds = \lambda^{\alpha-1}\Gamma(1-\alpha),$$

where Γ is the classical Gamma function. Hence

$$A^\alpha T(t-s) f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s)))$$

is integrable over $(-\infty, t)$ for $t \in \mathbb{R}$.

Now, let $h = h_1 + h_2$, where $h_1 \in AP_p(\mathbb{R}, X)$, $h_2 \in WPAP_p^0(\mathbb{R}, X, \rho)$, then

$$\begin{aligned}(\Lambda u)(t) &= \int_{-\infty}^{t} A^\alpha T(t-s) h_1(s) ds + \int_{-\infty}^{t} A^\alpha T(t-s) h_2(s) ds \\ &:= \Lambda_1(t) + \Lambda_2(t),\end{aligned}$$

where

$$\Lambda_1(t) = \int_{-\infty}^{t} A^\alpha T(t-s) h_1(s) ds, \quad \Lambda_2(t) = \int_{-\infty}^{t} A^\alpha T(t-s) h_2(s) ds.$$

(i) $\Lambda_1 \in \mathcal{U}PC(\mathbb{R}, X)$. Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$, then

$$\begin{aligned}&\Lambda_1(t') - \Lambda_1(t'') \\ &= \int_{-\infty}^{t'} A^\alpha T(t'-s) h_1(s) ds - \int_{-\infty}^{t''} A^\alpha T(t''-s) h_1(s) ds \\ &= \int_{-\infty}^{t''} A^\alpha (T(t'-s) - T(t''-s)) h_1(s) ds + \int_{t''}^{t'} A^\alpha T(t'-s) h_1(s) ds \\ &= \int_{-\infty}^{t''} [T(t'-t'') - I] A^\alpha T(t''-s) h_1(s) ds + \int_{t''}^{t'} A^\alpha T(t'-s) h_1(s) ds,\end{aligned}$$

It is easy to see that for any $\varepsilon > 0$, there exists

$$0 < \delta < \Big(\frac{(1-\alpha)\varepsilon}{2M_\alpha \|h_1\|}\Big)^{1/(1-\alpha)}$$

such that if t', t'' belongs to a same continuity and $0 < t' - t'' < \delta$, then

$$\|T(t'-t'') - I\| \le \frac{\varepsilon}{2M_\alpha \|h_1\| \lambda^{\alpha-1}\Gamma(1-\alpha)}.$$

So

$$
\begin{aligned}
&\|\Lambda_1(t') - \Lambda_1(t'')\| \\
&\leq \int_{-\infty}^{t''} \|[T(t'-t'') - I]A^\alpha T(t''-s)h_1(s)\|ds + \int_{t''}^{t'} \|A^\alpha T(t'-s)h_1(s)\|ds \\
&\leq \int_{-\infty}^{t''} \frac{\varepsilon}{2M_\alpha \lambda^{\alpha-1}\Gamma(1-\alpha)\|h_1\|} M_\alpha (t''-s)^{-\alpha} e^{-\lambda(t''-s)}\|h_1\|ds \\
&\quad + \int_{t''}^{t'} M_\alpha (t'-s)^{-\alpha} e^{-\lambda(t'-s)}\|h_1\|ds \\
&\leq \frac{\varepsilon}{2} + \frac{M_\alpha \|h_1\|\delta^{1-\alpha}}{1-\alpha} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

which implies that $\Lambda_1 \in \mathcal{U}PC(\mathbb{R}, X)$.

(ii) $\Lambda_1 \in AP_p(\mathbb{R}, X)$. Since $h_1 \in AP_p(\mathbb{R}, X)$, for $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that for $\tau \in \Omega_\varepsilon$, $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$,

$$
\|h_1(t+\tau) - h_1(t)\| < \varepsilon.
$$

Hence, by Lemma 2.1, for $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$, one has

$$
\begin{aligned}
&\|\Lambda_1(t+\tau) - \Lambda_1(t)\| \\
&= \| \int_{-\infty}^{t+\tau} A^\alpha T(t+\tau-s)h_1(s)ds - \int_{-\infty}^{t} A^\alpha T(t-s)h_1(s)ds\| \\
&= \| \int_{-\infty}^{t} A^\alpha T(t-s)(h_1(s+\tau) - h_1(s))ds\| \\
&\leq \int_{-\infty}^{t} \|A^\alpha T(t-s)(h_1(s+\tau) - h_1(s))\|ds \\
&\leq \int_{-\infty}^{t} M_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)}\|h_1(s+\tau) - h_1(s)\|ds \\
&< M_\alpha \lambda^{\alpha-1}\Gamma(1-\alpha)\varepsilon,
\end{aligned}
$$

that is $\Lambda_1 \in AP_p(\mathbb{R}, X)$.

(iii) $\Lambda_2 \in WPAP_p^0(\mathbb{R}, X)$. In fact, for $r > 0$, one has

$$
\begin{aligned}
\frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t)\|\Lambda_2(t)\|dt &= \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t)\| \int_{-\infty}^{t} A^\alpha T(t-s)h_2(s)ds\|dt \\
&= \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t)\| \int_{0}^{\infty} A^\alpha T(s)h_2(t-s)ds\|dt \\
&\leq \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \int_{0}^{\infty} \rho(t)\|A^\alpha T(s)h_2(t-s)\|dsdt \\
&\leq \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \int_{0}^{\infty} M_\alpha s^{-\alpha} e^{-\lambda s}\rho(t)\|h_2(t-s)\|dsdt \\
&\leq M_\alpha \int_{0}^{\infty} s^{-\alpha} e^{-\lambda s} H_r(s)ds,
\end{aligned}
$$

where

$$H_r(s) = \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \|h_2(t-s)\| dt.$$

Since $h_2 \in WPAP_p^0(\mathbb{R}, X, \rho)$, $\rho \in U_T$, it follows that $h_2(\cdot - s) \in WPAP_p^0(\mathbb{R}, X, \rho)$ for each $s \in \mathbb{R}$ by Remark 3.5, hence $\lim_{r\to\infty} H_r(s) = 0$ for all $s \in \mathbb{R}$. By using the Lebesgue dominated convergence theorem, we have $\Lambda_2 \in WPAP_p^0(\mathbb{R}, X, \rho)$. □

Theorem 4.4. *Assume that* (H1)–(H5) *hold and if* $\Theta < 1$, *where*

$$\Theta := M_\alpha \lambda^{\alpha-1} \Gamma(1-\alpha) L_f \left(L_g C_k \eta^{-1} + 1 \right) + 2 M_\alpha N L_1 \left(m^{-\alpha} + (e^\lambda - 1)^{-1} \right),$$

then (4.1) *has a unique classical solution* $u \in WPAP_p(\mathbb{R}, X, \rho)$ *and* $u(t)$ *is an attractor.*

Proof. Let $\mathcal{F} : WPAP_p(\mathbb{R}, X, \rho) \cap \mathcal{U}PC(\mathbb{R}, X) \to PC(\mathbb{R}, X)$ be the operator defined by

$$\begin{aligned} (\mathcal{F}u)(t) = &\int_{-\infty}^{t} A^\alpha T(t-s) f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s))) ds \\ &+ \sum_{t_i < t} A^\alpha T(t-t_i) I_i(A^{-\alpha}u(t_i)). \end{aligned} \tag{4.3}$$

We will show that $\mathcal{F}$ has a fixed point in $WPAP_p(\mathbb{R}, X, \rho) \cap \mathcal{U}PC(\mathbb{R}, X)$ and divide the proof into several steps.

(i) $\mathcal{F}u \in WPAP_p(\mathbb{R}, X, \rho) \cap \mathcal{U}PC(\mathbb{R}, X)$. As in the proof of Lemma 4.3, it is not difficult to see that $\mathcal{F}u \in \mathcal{U}PC(\mathbb{R}, X)$. Next, we show that $\mathcal{F}u \in WPAP_p(\mathbb{R}, X, \rho)$. For $u \in WPAP_p(\mathbb{R}, X, \rho)$, By Lemma 4.3, one has

$$(\Lambda u)(t) = \int_{-\infty}^{t} A^\alpha T(t-s) f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s))) ds \in WPAP_p(\mathbb{R}, X, \rho),$$

It remains to show that

$$\sum_{t_i < t} A^\alpha T(t-t_i) I_i(A^{-\alpha}u(t_i)) \in WPAP_p(\mathbb{R}, X, \rho). \tag{4.4}$$

By Corollary 3.8, $I_i(A^{-\alpha}u(t_i)) \in WPAP(\mathbb{Z}, X, \rho)$, then let $I_i(A^{-\alpha}u(t_i)) = \beta_i + \gamma_i$, where $\beta_i \in AP(\mathbb{Z}, X)$ and $\gamma_i \in WPAP_0(\mathbb{Z}, X, \rho)$, so

$$\begin{aligned} \sum_{t_i < t} A^\alpha T(t-t_i) I_i(A^{-\alpha}u(t_i)) &= \sum_{t_i < t} A^\alpha T(t-t_i)\beta_i + \sum_{t_i < t} A^\alpha T(t-t_i)\gamma_i \\ &:= \Phi_1(t) + \Phi_2(t). \end{aligned}$$

Since $\{t_i^j\}, i, j \in \mathbb{Z}$ are equipotentially almost periodic, then by Lemma 2.9, for any $\varepsilon > 0$, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, such that for $t_i < t \le t_{i+1}$, $\tau \in \Omega_\varepsilon$, $q \in Q_\varepsilon$, $|t - t_i| > \varepsilon$, $|t - t_{i+1}| > \varepsilon$, $j \in \mathbb{Z}$, one has

$$\begin{gathered} t + \tau > t_i + \varepsilon + \tau > t_{i+q}, \\ t_{i+q+1} > t_{i+1} + \tau - \varepsilon > t + \tau, \end{gathered}$$

that is, $t_{i+q} < t + \tau < t_{i+q+1}$; then

$$\begin{aligned} &\|\Phi_1(t+\tau) - \Phi_1(t)\| \\ &= \| \sum_{t_i < t+\tau} A^\alpha T(t+\tau-t_i)\beta_i - \sum_{t_i < t} A^\alpha T(t-t_i)\beta_i \| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{t_i<t} \|A^\alpha T(t-t_i)(\beta_{i+q}-\beta_i)\| \\
&\le \sum_{t_i<t} M_\alpha (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} \|\beta_{i+q}-\beta_i\| \\
&\le M_\alpha \varepsilon \sum_{t_i<t} (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} \\
&\le M_\alpha \varepsilon \Big(\sum_{0<t-t_i\le 1} (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} + \sum_{j=1}^{\infty} \sum_{j<t-t_i\le j+1} (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} \Big) \\
&\le M_\alpha \varepsilon \Big(\sum_{0<t-t_i\le 1} (t-t_i)^{-\alpha} + \sum_{j=1}^{\infty} \sum_{j<t-t_i\le j+1} e^{-\lambda(t-t_i)} \Big) \\
&\le 2M_\alpha N \varepsilon (m^{-\alpha} + (e^\lambda - 1)^{-1}),
\end{aligned}
$$

where $m = \min\{t-t_i, 0<t-t_i\le 1\}$, N is the constant in the Lemma 2.8. Hence $\Phi_1 \in AP_p(\mathbb{R}, X)$.

Next, we show that $\Phi_2 \in WPAP_p^0(\mathbb{R}, X, \rho)$. For a given $i \in \mathbb{Z}$, define the function $\eta(t)$ by

$$
\eta(t) = A^\alpha T(t-t_i)\gamma_i, \quad t_i < t \le t_{i+1},
$$

then

$$
\lim_{t\to\infty} \|\eta(t)\| = \lim_{t\to\infty} \|A^\alpha T(t-t_i)\gamma_i\| \le \lim_{t\to\infty} M_\alpha (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} \|\gamma_i\| = 0,
$$

then $\eta \in PC_p^0(\mathbb{R}, X) \subset WPAP_p^0(\mathbb{R}, X, \rho)$. Define $\eta_n : \mathbb{R} \to X$ by

$$
\eta_n(t) = A^\alpha T(t-t_{i-n})\gamma_{i-n}, \quad t_i < t \le t_{i+1},\ n \in \mathbb{N}^+,
$$

so $\eta_n \in WPAP_p^0(\mathbb{R}, X, \rho)$. Moreover,

$$
\begin{aligned}
\|\eta_n(t)\| &= \|A^\alpha T(t-t_{i-n})\gamma_{i-n}\| \\
&\le M_\alpha \sup_{i\in\mathbb{Z}} \|\gamma_i\| (t-t_{i-n})^{-\alpha} e^{-\lambda(t-t_{i-n})} \\
&\le M_\alpha \sup_{i\in\mathbb{Z}} \|\gamma_i\| (t-t_i+n\kappa)^{-\alpha} e^{-\lambda(t-t_i)} e^{-\lambda\kappa n} \\
&\le M_\alpha \sup_{i\in\mathbb{Z}} \|\gamma_i\| \kappa^{-\alpha} n^{-\alpha} e^{-\lambda\kappa n},
\end{aligned}
$$

therefore, the series $\sum_{n=1}^{\infty} \eta_n$ is uniformly convergent on $\mathbb{R}$. By Lemma 3.6, one has

$$
\Phi_2(t) = \sum_{t_i<t} A^\alpha T(t-t_i)\gamma_i = \sum_{n=0}^{\infty} \eta_n \in WPAP_p^0(\mathbb{R}, X, \rho).
$$

So (4.4) holds.

(ii) $\mathcal{F}$ is a contraction. For $u, v \in WPAP_p(\mathbb{R}, X, \rho) \cap \mathcal{U}PC(\mathbb{R}, X)$,

$$
\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\le \int_{-\infty}^{t} \|A^\alpha T(t-s)[f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s))) - f(s, A^{-\alpha}v(s), K(A^{-\alpha}v(s)))]\| ds \\
&\quad + \sum_{t_i<t} \|A^\alpha T(t-t_i)[I_i(A^{-\alpha}u(t_i)) - I_i(A^{-\alpha}v(t_i))]\|
\end{aligned}
$$

$$\begin{aligned}
&\le \int_{-\infty}^{t} M_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} \big\| f(s, A^{-\alpha}u(s), K(A^{-\alpha}u(s))) \\
&\quad - f(s, A^{-\alpha}v(s), K(A^{-\alpha}v(s)))\big\| ds \\
&\quad + \sum_{t_i<t} M_\alpha (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} \|I_i(A^{-\alpha}u(t_i)) - I_i(A^{-\alpha}v(t_i))\| \\
&\le \Big[M_\alpha \lambda^{\alpha-1}\Gamma(1-\alpha) L_f \left(L_g C_k \eta^{-1} + 1\right) \\
&\quad + 2M_\alpha N L_1 \left(m^{-\alpha} + (e^\lambda - 1)^{-1}\right)\Big] \|u-v\|,
\end{aligned}$$

Since $\Theta < 1$, $\mathcal{F}$ is a contraction.

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point $u_0 \in WPAP_p(\mathbb{R}, X, \rho)$ such that

$$\begin{aligned}
u_0 = &\int_{-\infty}^{t} A^\alpha T(t-s) f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s))) ds \\
&+ \sum_{t_i<t} A^\alpha T(t-t_i) I_i(A^{-\alpha}u_0(t_i)).
\end{aligned} \tag{4.5}$$

Since A^α is closed,

$$A^{-\alpha}u_0 = \int_{-\infty}^{t} T(t-s) f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s))) ds + \sum_{t_i<t} T(t-t_i) I_i(A^{-\alpha}u_0(t_i)).$$

Which implies that $A^{-\alpha}u_0$ is a mild solution of (4.1). Next, we show that it is a classical solution.

(iii) u_0 is Hölder continuous. Note that for every $0 < \beta < 1-\alpha$ and $h \in (0, \kappa)$, $t \in (t_i, t_{i+1} - h]$, by Lemma 2.1, one has

$$\|(T(h) - I)A^\alpha T(t-s)\| \le C_\beta h^\beta \|A^{\alpha+\beta} T(t-s)\|$$

and

$$\begin{aligned}
&\|u_0(t+h) - u_0\| \\
&\le \Big\| \int_{-\infty}^{t} (T(h) - I) A^\alpha T(t-s) f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s))) ds \Big\| \\
&\quad + \Big\| \int_{t}^{t+h} A^\alpha T(t+h-s) f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s))) ds \Big\| \\
&\le M_{\alpha+\beta} M C_\beta h^\beta \int_{-\infty}^{t} (t-s)^{-(\alpha+\beta)} e^{-\lambda(t-s)} ds + M_\alpha M \int_{t}^{t+h} T(t+h-s)^{-\alpha} ds \\
&\le M_{\alpha+\beta} M C_\beta h^\beta \int_{-\infty}^{t} (t-s)^{-(\alpha+\beta)} e^{-\lambda(t-s)} ds + M_\alpha M \frac{h^{1-\alpha}}{1-\alpha},
\end{aligned}$$

where

$$M = \sup_{(t,u,v) \in \mathbb{R} \times X_\alpha \times X_\alpha} \|f(t,u,v)\|.$$

It follows that there is a constant $C > 0$ such that

$$\|u_0(t+h) - u_0\| \le C h^\beta$$

and therefore u_0 is Hölder continuous on $\mathbb{R}$.

Finally, it remains to prove that $t \to f(t, A^{-\alpha}u_0(t), K(A^{-\alpha}u_0(t)))$ is Hölder continuous on $\mathbb{R}$. By (H4), one has

$$\begin{aligned}&\|f(t, A^{-\alpha}u_0(t), K(A^{-\alpha}u_0(t))) - f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s)))\| \\ &\le L_f(|t-s|^\theta + \|u_0(t) - u_0(s)\| + \|K(A^{-\alpha}u_0(t)) - K(A^{-\alpha}u_0(s))\|_\alpha).\end{aligned}$$

Hence $f(t, A^{-\alpha}u_0(t), K(A^{-\alpha}u_0(t)))$ is Hölder continuous on $\mathbb{R}$. Let u_0 be the solution of (4.5) and consider the equation

$$\begin{gathered}\frac{du(t)}{dt} + Au(t) = f(t, A^{-\alpha}u_0(t), K(A^{-\alpha}u_0(t))), \quad t \in \mathbb{R},\ t \neq t_i,\ i \in \mathbb{Z}, \\ \Delta u(t_i) = I_i(A^{-\alpha}u_0(t_i)).\end{gathered}$$

Then this equation has a unique classical solution given by [21]

$$u(t) = \int_{-\infty}^t T(t-s)f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s)))ds + \sum_{t_i<t} T(t-t_i)I_i(A^{-\alpha}u_0(t_i)).$$

Moreover, we have $u(t) \in \mathcal{D}(A)$ and $u(t) \in \mathcal{D}(A^\alpha)$. Therefore, it follows that

$$\begin{aligned}A^\alpha u(t) &= \int_{-\infty}^t A^\alpha T(t-s)f(s, A^{-\alpha}u_0(s), K(A^{-\alpha}u_0(s)))ds \\ &\quad + \sum_{t_i<t} A^\alpha T(t-t_i)I_i(A^{-\alpha}u_0(t_i)) \\ &= u_0(t),\end{aligned}$$

which implies that $u(t) = A^{-\alpha}u_0(t)$ is the classical solution of (4.1), which is a piecewise weighted almost periodic solution.

Next, we show the attractivity of $u(t)$. Since $u(t) \in \mathcal{D}(A^\alpha)$ is the $WPAP_p$ mild solution, so if $t > \sigma$, $\sigma \neq t_i$, $i \in \mathbb{Z}$,

$$u(t) = T(t-\sigma)u(\sigma) + \int_\sigma^t T(t-s)f(s, u(s), (Ku)(s))ds + \sum_{\sigma<t_i<t} T(t-t_i)I_i(u(t_i)).$$

Let $u(t) = u(t, \sigma, \varphi)$ and $v(t) = v(t, \sigma, \psi)$ be two mild solution of (4.1), then

$$\begin{aligned}u(t) &= T(t-\sigma)\varphi + \int_\sigma^t T(t-s)f(s, u(s), (Ku)(s))ds + \sum_{\sigma<t_i<t} T(t-t_i)I_i(u(t_i)), \\ v(t) &= T(t-\sigma)\psi + \int_\sigma^t T(t-s)f(s, v(s), (Kv)(s))ds + \sum_{\sigma<t_i<t} T(t-t_i)I_i(v(t_i)).\end{aligned}$$

So

$$\begin{aligned}\|u(t) - v(t)\|_\alpha &\le \|A^\alpha T(t-\sigma)[\varphi - \psi]\| \\ &\quad + \Big\|\int_\sigma^t A^\alpha T(t-s)[f(s, u(s), (Ku)(s)) - f(s, v(s), (Kv)(s))]ds\Big\| \\ &\quad + \Big\|\sum_{\sigma<t_i<t} A^\alpha T(t-t_i)[I_i(u(t_i)) - I_i(v(t_i))]\Big\| \\ &\le M_\alpha(t-\sigma)^{-\alpha}e^{-\lambda(t-\sigma)}\|\varphi - \psi\| \\ &\quad + \int_\sigma^t M_\alpha(t-s)^{-\alpha}e^{-\lambda(t-s)}L_f(L_gC_k\eta^{-1} + 1)\|u(s) - v(s)\|_\alpha ds\end{aligned}$$

$$+\sum_{\sigma<t_i<t} M_\alpha (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} L_1 \|u(t)-v(t)\|_\alpha .$$

For $M_\alpha \kappa^{-\alpha} L_f (L_g C_k \eta^{-1}+1) < \lambda$, one has $e^{[M_\alpha \kappa^{-\alpha} L_f (L_g C_k \eta^{-1}+1)-\lambda](t-\sigma)} \to 0$, and $(t-\sigma)^{-\alpha} \to 0$, $(t-t_i)^{-\alpha} \to 0$ as $t \to \infty$, hence for $\varepsilon > 0$, there exist $T > \max(0, \sigma+\kappa)$, such that for $t > T$,

$$e^{[M_\alpha \kappa^{-\alpha} L_f (L_g C_k \eta^{-1}+1)-\lambda](t-\sigma)} < \varepsilon, \quad (t-\sigma)^{-\alpha} < \varepsilon, \quad (t-t_i)^{-\alpha} < \varepsilon.$$

Hence for $t > T$, one has

$$\begin{aligned}
\|u(t)-v(t)\|_\alpha &\le M_\alpha (t-\sigma)^{-\alpha} e^{-\lambda(t-\sigma)} \|\varphi-\psi\| \\
&\quad + \int_\sigma^{t-\kappa} M_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} L_f (L_g C_k \eta^{-1}+1) \|u(s)-v(s)\|_\alpha ds \\
&\quad + \int_{t-\kappa}^{t} M_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} L_f (L_g C_k \eta^{-1}+1) \|u(s)-v(s)\|_\alpha ds \\
&\quad + \sum_{\sigma<t_i<t} M_\alpha (t-t_i)^{-\alpha} e^{-\lambda(t-t_i)} L_1 \|u(t)-v(t)\|_\alpha \\
&\le M_\alpha \varepsilon e^{-\lambda(t-\sigma)} \|\varphi-\psi\| \\
&\quad + \int_\sigma^{t-\kappa} M_\alpha \kappa^{-\alpha} e^{-\lambda(t-s)} L_f (L_g C_k \eta^{-1}+1) \|u(s)-v(s)\|_\alpha ds \\
&\quad + M_\alpha L_f (L_g C_k \eta^{-1}+1)(\|u\|_{\alpha,\infty} + \|v\|_{\alpha,\infty}) \int_{t-\kappa}^{t} (t-s)^{-\alpha} ds \\
&\quad + \sum_{\sigma<t_i<t} M_\alpha \varepsilon e^{-\lambda(t-t_i)} L_1 \|u(t)-v(t)\|_\alpha \\
&\le M_\alpha \varepsilon e^{-\lambda(t-\sigma)} \|\varphi-\psi\| \\
&\quad + \int_\sigma^{t} M_\alpha \kappa^{-\alpha} e^{-\lambda(t-s)} L_f (L_g C_k \eta^{-1}+1) \|u(s)-v(s)\|_\alpha ds \\
&\quad + M_\alpha L_f (L_g C_k \eta^{-1}+1)(\|u\|_{\alpha,\infty} + \|v\|_{\alpha,\infty}) \kappa^{1-\alpha} (1-\alpha)^{-1} \\
&\quad + \sum_{\sigma<t_i<t} M_\alpha \varepsilon e^{-\lambda(t-t_i)} L_1 \|u(t)-v(t)\|_\alpha .
\end{aligned}$$

Let $y(t) = e^{\lambda t} \|u(t)-v(t)\|_\alpha$, then

$$\begin{aligned}
y(t) &\le M_\alpha \varepsilon y(\sigma) + e^{\lambda t} M_\alpha L_f (L_g C_k \eta^{-1}+1)(\|u\|_{\alpha,\infty} + \|v\|_{\alpha,\infty}) \kappa^{1-\alpha} (1-\alpha)^{-1} \\
&\quad + \int_\sigma^t M_\alpha \kappa^{-\alpha} L_f (L_g C_k \eta^{-1}+1) y(s) ds + \sum_{\sigma<t_i<t} M_\alpha \varepsilon L_1 y(t_i).
\end{aligned}$$

By the generalized Gronwall-Bellman inequality (Lemma 2.10), one has

$$\begin{aligned}
\|y(t)\| &\le \big[M_\alpha \varepsilon y(\sigma) + e^{\lambda t} M_\alpha L_f (L_g C_k \eta^{-1}+1)(\|u\|_{\alpha,\infty} + \|v\|_{\alpha,\infty}) \kappa^{1-\alpha} (1-\alpha)^{-1}\big] \\
&\quad \times \prod_{\sigma<t_i<t} (1 + M_\alpha L_1 \varepsilon) e^{M_\alpha \kappa^{-\alpha} L_f (L_g C_k \eta^{-1}+1)(t-\sigma)} \\
&= \big[M_\alpha \varepsilon y(\sigma) + e^{\lambda t} M_\alpha L_f (L_g C_k \eta^{-1}+1)(\|u\|_{\alpha,\infty} + \|v\|_{\alpha,\infty}) \kappa^{1-\alpha} (1-\alpha)^{-1}\big] \\
&\quad \times (1 + M_\alpha L_1 \varepsilon)^{i(\sigma,t)} e^{M_\alpha \kappa^{-\alpha} L_f (L_g C_k \eta^{-1}+1)(t-\sigma)},
\end{aligned}$$

where $i(\sigma,t)$ defined in Lemma 2.8. That is

$$\begin{aligned}
&\|u(t)-v(t)\|_\alpha \\
&\le \big[M_\alpha\varepsilon\|\varphi-\psi\| + M_\alpha L_f(L_gC_k\eta^{-1}+1)(\|u\|_{\alpha,\infty}+\|v\|_{\alpha,\infty})\kappa^{1-\alpha}(1-\alpha)^{-1}\big] \\
&\quad \times (1+M_\alpha L_1\varepsilon)^{i(\sigma,t)} e^{[M_\alpha\kappa^{-\alpha}L_f(L_gC_k\eta^{-1}+1)-\lambda](t-\sigma)} \\
&\le \big[M_\alpha\varepsilon\|\varphi-\psi\| + M_\alpha L_f(L_gC_k\eta^{-1}+1)(\|u\|_{\alpha,\infty}+\|v\|_{\alpha,\infty})\kappa^{1-\alpha}(1-\alpha)^{-1}\big] \\
&\quad \times (1+M_\alpha L_1\varepsilon)^{i(\sigma,t)}\varepsilon,
\end{aligned}$$

so $u(t)$ is an attractor. □

Remark 4.5. Consider the nonlinear impulsive integro-differential equations with delay:

$$\begin{gathered}
\frac{du(t)}{dt} + Au(t) = f(t,u(t-\tau),(Ku)(t)), \quad t\in\mathbb{R},\ t\neq t_i,\ i\in\mathbb{Z}, \\
(Ku)(t) = \int_{-\infty}^t k(t-s)g(s,u(s))ds, \\
\Delta u(t_i) := u(t_i^+) - u(t_i^-) = I_i(u(t_i)),
\end{gathered} \tag{4.6}$$

where $\tau\in\mathbb{R}^+$. Note that if $u=u_1+u_2\in WPAP_p(\mathbb{R},X,\rho)$, where $u_1\in AP_p(\mathbb{R},X)$, $u_2\in WPAP_p^0(\mathbb{R},X,\rho)$. For given $\tau\in\mathbb{R}^+$, it is not difficult to see that $u_1(t-\tau)\in AP_p(\mathbb{R},X)$. For $r>0$, we see that

$$\begin{aligned}
&\frac{1}{\mu(r,\rho)}\int_{-r}^r \rho(t)\|u_2(t-\tau)\|dt \\
&= \frac{1}{\mu(r,\rho)}\int_{-r-\tau}^{r-\tau} \rho(t+\tau)\|u_2(t)\|dt \\
&\le \frac{\mu(r+\tau,\rho)}{\mu(r,\rho)} \times \frac{1}{\mu(r+\tau,\rho)}\int_{-r-\tau}^{r+\tau} \frac{\rho(t+\tau)}{\rho(t)}\rho(t)\|u_2(t)\|dt.
\end{aligned}$$

Since $\rho\in U_T$, it implies that there exists $\eta>0$ such that $\frac{\rho(t+\tau)}{\rho(t)}\le\eta$, $\frac{\rho(t-\tau)}{\rho(t)}\le\eta$, for $r>\tau$,

$$\begin{aligned}
\mu(r+\tau,\rho) &= \int_{-r-\tau}^{r-\tau}\rho(t)dt + \int_{r-\tau}^{r+\tau}\rho(t)dt \\
&\le \int_{-r-\tau}^{r-\tau}\rho(t)dt + \int_{-r+\tau}^{r+\tau}\rho(t)dt \\
&= \int_{-r}^{r}\rho(t-\tau)dt + \int_{-r}^{r}\rho(t+\tau)dt \le 2\eta\mu(r,\rho),
\end{aligned}$$

then by $u_2\in WPAP_p^0(\mathbb{R},X,\rho)$, one has

$$\frac{1}{\mu(r,\rho)}\int_{-r}^r \rho(t)\|u_2(t-\tau)\|dt \le \frac{2\eta^2}{\mu(r+\tau,\rho)}\int_{-r-\tau}^{r+\tau}\rho(t)\|u_2(t)\|dt \to 0$$

as $\to\infty$. Hence $u_2(t-\tau)\in WPAP_p^0(\mathbb{R},X,\rho)$, that is $u(t-\tau)\in WPAP_p(\mathbb{R},X,\rho)$ for $\tau\in\mathbb{R}^+$. Thus the conclusion of Theorem 4.4 holds for (4.6).

Remark 4.6. If $(Ku)(t) = 0$, then impulsive integro-differential equations (4.1) become nonlinear impulsive differential equations:

$$\begin{gathered} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in \mathbb{R},\ t \neq t_i,\ i \in \mathbb{Z}, \\ \Delta u(t_i) = I_i(u(t_i)), \end{gathered} \tag{4.7}$$

By Theorem 4.4, one has the following corollary.

Corollary 4.7. *Assume that* (H1), (H5) *hold and satisfy the condition:*

(H4') $f \in WPAP_p(\mathbb{R} \times X_\alpha, X, \rho)$, $\rho \in U_T$ *and there exists constants* $L_f > 0$, $0 < \theta < 1$ *such that*

$$\|f(t_1, u) - f(t_2, v)\| \leq L_f(|t_1 - t_2|^\theta + \|u - v\|_\alpha), \quad t \in \mathbb{R},\ u, v \in X_\alpha,$$

then (4.7) *has a unique classical solution* $u \in WPAP_p(\mathbb{R}, X, \rho)$ *which is an attractor if*

$$\vartheta := M_\alpha \lambda^{\alpha-1}\Gamma(1-\alpha)L_f + 2M_\alpha N L_1 \left(m^{-\alpha} + (e^\lambda - 1)^{-1}\right) < 1.$$

5. Example

Consider the integro-differential equation with impulsive effects

$$\begin{gathered} \frac{\partial w(t,x)}{\partial t} - \frac{\partial^2 w(t,x)}{\partial x^2} = f(t, x, w(t,x), Kw(t,x)), \quad t \in \mathbb{R},\ t \neq t_i,\ i \in \mathbb{Z},\ x \in (0,1), \\ Kw(t,x) = \int_{-\infty}^t e^{-\eta(t-s)} g(s, x, w(s,x))ds, \\ \Delta w(t_i, x) = \beta_i w(t_i, x), \quad i \in \mathbb{Z},\ x \in [0,1], \\ w(t,0) = w(t,1) = 0, \end{gathered} \tag{5.1}$$

where $t_i = i + \frac{1}{4}|\sin i + \sin \sqrt{2} i|$, $\beta_i \in WPAP(\mathbb{Z}, \mathbb{R}, \rho)$, $\rho \in U_T$. Note that $\{t_i^j\}$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}$ are equipotentially almost periodic and $\kappa = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$, one can see [14, 19] for more details.

Define the operator A by

$$Au := -u'', \quad u \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \{u \in H_0^1((0,1), \mathbb{R}) \cap H^2((0,1), \mathbb{R}) :\ u'' \in H\}.$$

The operator A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ and also self adjoint [17]. Let $\alpha = 1/2$, so $\mathcal{D}(A^{1/2})$ is a Banach space endowed with the norm

$$\|u\|_{1/2} = \|A^{1/2}u\|, \quad u \in \mathcal{D}(A^{1/2}).$$

We call this space $X_{1/2}$. For more details about $X_{1/2}$, one can see [2].

Let $u(t)x = w(t,x)$, $t \in \mathbb{R}, x \in [0,1]$ and

$$f(t, u(t), (Ku)(t))(x) = f(t, x, w(t,x), Kw(t,x)).$$

Then (5.1) can be rewritten as the abstract form (4.1). Since $I_i(u) = \beta_i u$ and $\beta_i \in WPAP(\mathbb{Z}, \mathbb{R}, \rho)$, then (H5) hold with $L_1 = \sup_{i\in\mathbb{Z}} \|\beta_i\|$. By Theorem 4.4, one has the following result.

Theorem 5.1. *Assume that*

(A1) *$g \in WPAP_p(\mathbb{R} \times X_{1/2}, X, \rho)$, $\rho \in U_T$ and there exists a constant $L_g > 0$ such that*
$$\|g(t,u) - g(t,v)\| \le L_g \|u - v\|_{1/2}, \quad t \in \mathbb{R},\ u, v \in X_{1/2}.$$
(A2) *$f \in WPAP_p(\mathbb{R} \times X_{1/2} \times X_{1/2}, X, \rho)$, $\rho \in U_T$ and there exists constants $L_f > 0$, $0 < \theta < 1$ such that*
$$\|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)\| \le L_f(|t_1 - t_2|^\theta + \|u_1 - u_2\|_{1/2} + \|v_1 - v_2\|_{1/2}),$$
for each $(t_i, u_i, v_i) \in \mathbb{R} \times X_{1/2} \times X_{1/2}$, $i = 1, 2$.

(A3) *$\beta_i \in WPAP(\mathbb{Z}, \mathbb{R}, \rho)$, $\rho \in U_T$.*

then (5.1) *has a unique $WPAP_p$ solution which is an attractor if $\vartheta < 1$, where*
$$\vartheta := M_\alpha \lambda^{-\frac{1}{2}} \Gamma\big(\frac{1}{2}\big) L_f \left(L_g \eta^{-1} + 1\right) + 2M_\alpha N \Big(m^{-\frac{1}{2}} + (e^\lambda - 1)^{-1}\Big) \sup_{i \in \mathbb{Z}} \|\beta_i\|.$$

Conclusion. The notion of almost periodic function $AP(\mathbb{R}, X)$ introduced by Bohr in 1925. Since then, there have various important generalization of this concept, like:

(i) Asymptotically almost periodic function $AAP(\mathbb{R}, X)$;
(ii) Weakly almost periodic function $WAP(\mathbb{R}, X)$;
(iii) Pseudo almost periodic function $PAP(\mathbb{R}, X)$;
(iv) Weighted pseudo almost periodic function $WPAP(\mathbb{R}, X, \rho)$;

and many more. For origin references, details of these functions, one can see [22] and the relationship between these functions as follows:
$$AP(\mathbb{R}, X) \subset AAP(\mathbb{R}, X) \subset WAP(\mathbb{R}, X) \subset PAP(\mathbb{R}, X) \subset WPAP(\mathbb{R}, X, \rho).$$
The application of these functions in the context of various kinds of abstract differential equations attracted many mathematicians. In this paper, by the fixed point theorem and fractional powers of operators, we investigate the applications of weighted pseudo almost periodic functions to the impulsive integro-differential equations. The existence, uniqueness and attractivity of piecewise $WPAP$ classical solutions of nonlinear impulsive integro-differential equations are given.

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