Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 183, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ADJOINT SYSTEMS AND GREEN FUNCTIONALS FOR SECOND-ORDER LINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this work, we generalize so called Green's functional concept in literature to second-order linear integro-differential equation with nonlocal conditions. According to this technique, a linear completely nonhomogeneous nonlocal problem for a second-order integro-differential equation is reduced to one and one integral equation to identify the Green's solution. The coefficients of the equation are assumed to be generally nonsmooth functions satisfying some general properties such as p-integrability and boundedness. We obtain new adjoint system and Green's functional for second-order linear integro-differential equation with nonlocal conditions. An application illustrate the adjoint system and the Green's functional. Another application shows when the Green's functional does not exist.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers. Let $G = (x_0, x_1)$ be an open bounded interval in $\mathbb{R}$. Let $L^p(G)$ with $1 \le p < \infty$ be the space of $p-$integrable functions on G and let $W^{2,p}(G)$ with $1 \le p < \infty$ be the space of all classes of functions $u \in L^p(G)$ of x having derivatives $d^k/dx^k \in L^p(G)$, where $k = 1, 2$. The norm on the space $W^{2,p}(G)$ is defined as

$$\|u\|_{W^{2,p}(G)} = \sum_{k=0}^{k=2} \|\frac{d^k u}{dx^k}\|_{L^p(G)}.$$

We consider the second-order integro-differential equation

$$\begin{aligned}(V_2)(x) &\equiv u''(x) + A_1(x)u'(x) + A_0(x)u(x) \\ &\quad + \int_{x_0}^{x_1} [B_1(x,\xi)u'(\xi) + B_0(x,\xi)u(\xi)]d\xi = z_2(x), \quad x \in G \end{aligned} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35A24, 65N80, 34B27.
Key words and phrases. Adjoint system; Green's functional; p-integrability, nonlocal boundary conditions.

Submitted February 12, 2015. Published July 2, 2015.

subject to the nonlocal boundary conditions

$$\begin{aligned} V_1 u &\equiv a_1 u(x_0) + b_1 u'(x_0) + \int_{x_o}^{x_1} g_1(\xi) u''(\xi) d\xi = z_1, \\ V_0 u &\equiv a_0 u(x_0) + b_0 u'(x_0) + \int_{x_o}^{x_1} g_0(\xi) u''(\xi) d\xi = z_0. \end{aligned} \tag{1.2}$$

We investigate for a solution to the problem in the space $W_p = W^{2,p}(G)$. Furthermore, we assume that the following conditions are satisfied: $A_i \in L^p(G)$, $B_i \in L^1(G \times G)$ and $g_i \in L^p(G)$ for $i = 0, 1$ are given functions with $B_i^0 \in L^p(G)$, where $B_i^0(x) = \int_{x_0}^{x_1} |B_i(x,\xi)| d\xi$; a_i, b_i for $i = 0, 1$ are given real numbers; $z_2 \in L^p(G)$ is a given function and z_i for $i = 0, 1$ are given real numbers.

Remark 1.1. In [1], second-order linear integro-differential equation (1.1) is studied with the generally nonlocal multipoint conditions

$$V_i \equiv \sum_{k=0}^{n} [a_{i,k} u(\beta_k) + b_{i,k} u'(\beta_k)] = z_i, \quad i = 0, 1$$

where $a_{i,k}$ and $b_{i,k}$ are given numbers; $\beta_k \in \bar{G}$ are given points with $x_0 = \beta_0 < \dots < \beta_n = x_1$ and z_0 and z_1 are given real numbers.

In the nonlocal boundary conditions (1.2) if we take

$$\begin{gathered} a_i = \sum_{k=0}^{n} a_{i,k}, \quad b_i = \sum_{k=1}^{n} a_{i,k}(\beta_k - x_0) + \sum_{k=0}^{n} b_{i,k}, \\ g_i(\xi) = \sum_{k=1}^{n} a_{i,k}(\beta_k - \xi) H(\beta_k - \xi) + \sum_{k=0}^{n} b_{i,k} H(\beta_k - \xi) \end{gathered}$$

where $H(x)$ is the heaviside function on $\mathbb{R}$, then (1.1)-(1.2) is reduced to the problem studied in [1]. Therefore (1.1)-(1.2) is a generalization of the problem studied in [1].

Remark 1.2. In (1.1) if we take $B_1 = B_2 \equiv 0$, then (1.1)-(1.2) is reduced to the problem studied in [5].

Remark 1.3. In [8], the ordinary differential equation

$$u''(x) + A_0(x) u(x) + A_2(x) u(x_0) = z_2(x), \quad x \in G \tag{1.3}$$

is studied with the nonlocal boundary conditions (1.2). In (1.1) if we take $A_1 \equiv 0$, $B_1(x,\xi) = \frac{A_2(x)(\xi - x_1)}{(x_0 - x_1)}$ and $B_0(x,\xi) = A_2(x)$, then (1.1)-(1.2) is reduced to the problem studied in [8].

So the second-order linear integro-differential equation (1.1) with nonlocal conditions (1.2) is a generalization of the problems studied in [1, 5, 8]. For more information about adjoint system and Green's functional method we refer to the references in this article and the references therein.

2. Adjoint space of the solution space

Problem (1.1)-(1.2) is a linear nonhomogeneous problem which can be considered as an operator equation

$$Vu = z \tag{2.1}$$

with the linear operator $V = (V_2, V_1, V_0)$ and $z = (z_2(x), z_1, z_0)$. In order that the linear operator V defined from the normed space W_p into the Banach space $E_p \equiv L^p(G) \times \mathbb{R}^2$ have an adjoint operator, first of all the linear operator V should be a bounded operator. Since

$$\begin{aligned}
&\|V_2 u\|_{L^p(G)} \\
&= \Big(\int_{x_0}^{x_1} |V_2 u(x)|^p dx \Big)^{1/p} \\
&= \Big(\int_{x_0}^{x_1} \Big| u''(x) + A_1(x) u'(x) + A_0(x) u(x) \\
&\quad + \int_{x_0}^{x_1} [B_1(x,\xi) u'(\xi) + B_0(x,\xi) u(\xi)] d\xi \Big|^p dx \Big)^{1/p} \\
&\le \Big(\int_{x_0}^{x_1} \Big[|u''(x)| + |A_1(x) u'(x)| + |A_0(x) u(x)| \\
&\quad + \int_{x_0}^{x_1} [|B_1(x,\xi) u'(\xi)| + |B_0(x,\xi) u(\xi)|] d\xi \Big]^p dx \Big)^{1/p} \\
&\le \|u\|_{W_p} \Big(\int_{x_0}^{x_1} \Big[1 + |A_1(x)| + |A_0(x)| + \int_{x_0}^{x_1} [|B_1(x,\xi)| + |B_0(x,\xi)|] d\xi \Big]^p dx \Big)^{1/p} \\
&\le \|u\|_{W_p} \Big(\Big(\int_{x_0}^{x_1} |A_1(x)|^p dx \Big)^{1/p} + \Big(\int_{x_0}^{x_1} |A_0(x)|^p dx \Big)^{1/p} \\
&\quad + \Big(\int_{x_0}^{x_1} \Big[\int_{x_0}^{x_1} |B_1(x,\xi)| d\xi \Big]^p dx \Big)^{1/p} + \Big(\int_{x_0}^{x_1} \Big[\int_{x_0}^{x_1} |B_0(x,\xi)| d\xi \Big]^p dx \Big)^{1/p} \Big) \\
&\le \|u\|_{W_p} \Big(\Big(\int_{x_0}^{x_1} |A_1(x)|^p dx \Big)^{1/p} + \Big(\int_{x_0}^{x_1} |A_0(x)|^p dx \Big)^{1/p} \\
&\quad + \Big(\int_{x_0}^{x_1} [B_1^0(x)]^p dx \Big)^{1/p} + \Big(\int_{x_0}^{x_1} [B_0^0(x)]^p dx \Big)^{1/p} \Big)
\end{aligned}$$

and $B_i^0 \in L^p(G)$, $A_i \in L^p(G)$, for $i = 0, 1$ then V_2 is bounded in $L^p(G)$. And, since

$$\|Vu\|_{E_p} = \|V_2 u\|_{L^p(G)} + |V_1 u| + |V_0 u|,$$

then V is bounded from W_p into the Banach space $E_p \equiv L^p(G) \times \mathbb{R}^2$ consisting of elements $z = (z_2(x), z_1, z_0)$ with norm

$$\|z\|_{E_p} = \|z_2\|_{L^p(G)} + |z_1| + |z_0|, \quad 1 \le p < \infty.$$

Problem (1.1)-(1.2) is studied by means of a new concept of the adjoint problem. This concept is introduced in [5, 8] using the adjoint operator V^* of V. Some isomorphic decompositions of the space W_p of solutions and its adjoint space W_p^* are employed. Any function $u \in W_p$ can be represented as

$$u(x) = u(\alpha) + u'(\alpha)(x - \alpha) + \int_\alpha^x (x - \xi) u''(\xi) d\xi \tag{2.2}$$

where α is a given point in $\bar{G}$ which is the set of closure points for G. Furthermore, the trace or value operators $D_0 u = u(\gamma)$, $D_1 u = u'(\gamma)$ are bounded and surjective from W_p onto $\mathbb{R}$ for a point γ of $\bar{G}$. In addition, the values $u(\alpha)$, $u'(\alpha)$ and the second derivative $u''(x)$ are unrelated elements of the function $u \in W_p$ such that for any real numbers ν_0, ν_1 and any function $\nu \in L_p(G)$, there exists one and only one $u \in W_p$ such that $u(\alpha) = \nu_0$, $u'(\alpha) = \nu_1$ and $u''(\alpha) = \nu_2(x)$. Therefore, there exists a linear homeomorphism between W_p and E_P. In other words, the space W_p has the isomorphic decomposition $W_p = L_p(G) \times \mathbb{R} \times \mathbb{R}$.

Theorem 2.1 ([1]). *If $1 \le p < \infty$, then any linear bounded functional $F \in W_p^*$ can be expressed as*

$$F(x) = \int_{x_0}^{x_1} u''(x)\varphi_2(x)dx + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{2.3}$$

with a unique element $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. To give the proof, a bounded linear bijective operator

$$Nu = (u''(x), u'(x_0), u(x_0))$$

is constructed from the space W_p into the space E_p. Since the adjoint operator N^* is also a bounded linear bijective operator from the space E_p^* to the space W_p^* then using the fact that $E_p^* = E_q$ for $\frac{1}{p} + \frac{1}{q} = 1$, the conclusion follows. For the detail of the proof, see [1]. □

3. Adjoint operator and adjoint system of integro-algebraic equations

In this section we consider an explicit form for the adjoint operator V^* of V. To this end, we take any linear bounded functional $f = (f_2(x), f_1, f_0) \in E_q$. We can also assume that

$$f(Vu) \equiv \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + f_1(V_1u) + f_0(V_0u), \quad u \in W_p. \tag{3.1}$$

By substituting expressions (1.1)-(1.2) and expression (2.2) (for $\alpha = x_0$) of $u \in W_p$ into (3.1), we obtain the equation

$$\begin{aligned}
f(Vu) \equiv \int_{x_0}^{x_1} & f_2(x)\Big\{u''(x) + A_1(x)\Big[u'(x_0) + \int_{x_0}^{x} u''(\xi)d\xi\Big] \\
&+ A_0(x)\Big[u(x_0) + u'(x_0)(x - x_0) + \int_{x_0}^{x} (x-\xi)u''(\xi)d\xi\Big] \\
&+ \int_{x_0}^{x_1} B_1(x,s)\Big[u'(x_0) + \int_{x_0}^{s} u''(\xi)d\xi\Big]ds \\
&+ \int_{x_0}^{x_1} B_0(x,s)\Big[u(x_0) + u'(x_0)(s - x_0) + \int_{x_0}^{s} (s-\xi)u''(\xi)d\xi\Big]ds\Big\}dx \\
&+ f_1\Big\{a_1u(x_0) + b_1u'(x_0) + \int_{x_0}^{x_1} g_1(\xi)u''(\xi)d\xi\Big\} \\
&+ f_0\Big\{a_0u(x_0) + b_0u'(x_0) + \int_{x_0}^{x_1} g_0(\xi)u''(\xi)d\xi\Big\}.
\end{aligned}$$

After some calculations, we obtain

$$\begin{aligned}
f(Vu) &\equiv \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + f_1(V_1u) + f_0(V_0u) \\
&= \int_{x_0}^{x_1} (\omega_2 f)(\xi)u''(\xi)d\xi + (\omega_1 f)u'(x_0) + (\omega_0 f)u(x_0) \\
&\equiv (\omega f)(u), \quad \text{for any } f \in E_q \text{ and any } u \in W_p,\ 1 \le p \le \infty,
\end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
(\omega_2 f)(\xi) &= f_2(\xi) + f_1 g_1(\xi) + f_0 g_0(\xi) + \int_{\xi}^{x_1} f_2(s)[A_0(s)(s-\xi) + A_1(s)]ds \\
&\quad + \int_{x_0}^{x_1} f_2(x)\Big[\int_{\xi}^{x_1} B_1(x,s)ds + \int_{\xi}^{x_1} B_0(x,s)(s-\xi)ds\Big]dx, \\
\omega_1 f &= b_1 f_1 + b_0 f_0 + \int_{x_0}^{x_1} f_2(x)[A_0(x)(x-x_0) + A_1(x)]dx \\
&\quad + \int_{x_0}^{x_1}\int_{x_0}^{x_1} f_2(x)[B_0(x,s)(s-x_0) + B_1(x,s)]dsdx, \\
\omega_0 f &= a_1 f_1 + a_0 f_0 + \int_{x_0}^{x_1} f_2(x)A_0(x)dx + \int_{x_0}^{x_1}\int_{x_0}^{x_1} f_2(x)B_0(x,s)dsdx.
\end{aligned}
\tag{3.3}
$$

As shown in the beginning of the second section, the linear operator V defined from the normed space W_p into the Banach space E_p is bounded, its adjoint should be also be linear and bounded. As in the section two, the boundedness of the linear operators ω_2, ω_1, ω_0 from the space E_q of the triples $f = (f_2(x), f_1, f_0)$ into the spaces $L_q(G)$, $\mathbb{R}$, $\mathbb{R}$, respectively, can be shown. Therefore, the operator $\omega = (\omega_2, \omega_1, \omega_0) : E_q \to E_q$ represented by $\omega f = (\omega_2 f, \omega_1 f, \omega_0 f)$ is linear and bounded. By (3.2) and Theorem 2.1, the operator ω is an adjoint operator for the operator V when $1 \le p < \infty$, in other words, $V^* = \omega$.

Following the articles [1, 5, 8], equation (2.1) can be transformed into the equivalent equation

$$
VSh = z, \tag{3.4}
$$

with an unknown $h = (h_2, h_1, h_0) \in E_P$ by the transformation $u = Sh$ where $S = N^{-1}$. If $u = Sh$, then $u''(x) = h_2(x)$, $u'(x_0) = h_1$, $u(x_0) = h_0$. Hence, (3.2) can be written as

$$
\begin{aligned}
f(VSh) &\equiv \int_{x_0}^{x_1} f_2(x)(V_2 Sh)(x)dx + f_1(V_1 Sh) + f_0(V_0 Sh) \\
&= \int_{x_0}^{x_1} (\omega_2 f)(\xi)h_2(\xi)d\xi + (\omega_1 f)h_1 + (\omega_0 f)h_0 \\
&\equiv (\omega f)(h) \quad \text{for any } f \in E_q, \quad \text{for any } u \in W_p, \quad 1 \le p \le \infty.
\end{aligned}
\tag{3.5}
$$

Therefore the operator VS is the adjoint of the operator ω. Consequently, the equation

$$
\omega f = \varphi \tag{3.6}
$$

with an unknown function $f = (f_2(x), f_1, f_0) \in E_q$ and a given function $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ can be considered as an adjoint equation of (2.1) and (3.4) for all $1 \le p \le \infty$. Equation (3.6) can be written in explicit form as the system of equations

$$
\begin{aligned}
(\omega_2 f)(\xi) &= \varphi_2(\xi), \quad \xi \in G, \\
\omega_1 f &= \varphi_1, \\
\omega_0 f &= \varphi_0.
\end{aligned}
\tag{3.7}
$$

4. Solvability conditions for the completely nonhomogeneous problem

Using the argument in the articles [1, 3], we consider the operator $Q = \omega - I_q$, where I_q is the identity operator on E_q. This operator can also be defined as $Q = (Q_2, Q_1, Q_0)$ with

$$\begin{aligned} (Q_2 f)(\xi) &= (\omega_2 f)(\xi) - f_2(\xi), \quad \xi \in G; \\ Q_1 f &= \omega_1 f - f_1, \\ Q_0 f &= \omega_0 f - f_0. \end{aligned} \tag{4.1}$$

The expressions in (3.3) and the conditions imposed on A_i and b_i show that Q_2 is a compact operator from E_q into $L_q(G)$ and also Q_1 and Q_0 are compact operators from E_q into $\mathbb{R}$, where $1 < p < \infty$. Therefore, $Q : E_q \to E_q$ is a compact operator and therefore has a compact adjoint operator $Q^* : E_p \to E_p$. Since $\omega = Q + I_q$ and $VS = Q^* + I_p$, where $I_p = I_q^*$, we have that (3.4) and (3.6) are canonical Fredholm type equations. Consequently, we have the following result.

Theorem 4.1 ([1]). *Assume that $1 < p < \infty$. Then the homogenous equation $Vu = 0$ has either only the trivial solution or a finite number of linearly independent solutions in W_p:*

(1) If $Vu = 0$ has only the trivial solution in W_p then also $\omega f = 0$ has only the trivial solution in E_q. Then the operators $V : W_p \to E_p$ and $\omega : E_q \to E_q$ become linear homeomorphisms.

(2) If $Vu = 0$ has m linear independent solutions $u_1, u_2, \dots, u_m$ in W_p, then the equation $\omega f = 0$ also has m linear independent solutions

$$f^{(1)} = (f_2^{(1)}(x), f_1^{(1)}, f_0^{(1)}), \dots, f^{(m)} = (f_2^{(m)}(x), f_1^{(m)}, f_0^{(m)})$$

in E_q. In this case, (2.1) and (3.6) have solutions $u \in W_p$ and $f \in E_q$ for given $z \in E_p$ and $\varphi \in E_q$ if and only if the conditions

$$\int_{x_0}^{x_1} f_2^{(i)}(\xi) z_2(\xi) d_\xi + f_1^{(i)} z_1 + f_0^{(i)} z_0 = 0, \quad i = 1, 2, \dots, m, \tag{4.2}$$

$$\int_{x_0}^{x_1} \varphi_2(\xi) u_1''(\xi) d\xi + \varphi_1 u_i'(x_0) + \varphi_0 u_i(x_0) = 0, \quad i = 1, 2, \dots, m, \tag{4.3}$$

are satisfied.

5. Green's functional

Consider the equation

$$(\omega f)(u) = u(x), \quad \forall u \in W_p, \tag{5.1}$$

given in the form of a functional identity, where $f = (f_2(\xi), f_1, f_0) \in E_q$ is an unknown triple and $x \in \bar{G}$ is a parameter.

Definition 5.1 ([1]). Suppose that $f(x) = (f_2(\xi, x), f_1(x), f_0(x)) \in E_q$ is a triple with a parameter $x \in \bar{G}$. If for a given $x \in \bar{G}$, $f = f(x)$ is a solution of functional equation (5.1) then $f(x)$ is called a Green's functional of V or a Green's functional of (2.1).

Due to the operator $I_{W_p,C}$ of the imbedding of W_p into the space $C(\bar{G})$ of continuous functions on $\bar{G}$ is bounded, the linear functional $\eta(x)$ defined by $\eta(x)(u) = u(x)$ is bounded on W_p for a given $x \in \bar{G}$. On the other hand, $(\omega f)(u) = (V^* f)(u)$. Thus, (5.1) can also be written as, [2, 3],

$$(V^* f) = \eta(x).$$

In other words, (5.1) can be considered as a special case of the adjoint equation $V^* f = \psi$ for some $\psi = \eta(x)$.

By substituting $\alpha = x_0$ into (2.2) and using (3.2), we can write (5.1) as

$$\begin{aligned} &\int_{x_0}^{x_1} (\omega_2 f)(\xi) u''(\xi) d\xi + (\omega_1 f) u'(x_0) + (\omega_0 f) u(x_0) \\ &= \int_{x_0}^{x} (x-\xi) u''(\xi) d\xi + u'(x_0)(x - x_0) + u(x_0), \end{aligned} \tag{5.2}$$

for any $f \in E_q$ and any $u \in W_p$. The elements $u'' \in L_p(G)$, $u'(x_0) \in \mathbb{R}$ and $u(x_0) \in \mathbb{R}$ of the function $u \in W_p$ are unrelated. Then, we can construct the system

$$\begin{gathered} (\omega_2 f)(\xi) = (x-\xi) H(x-\xi), \quad \xi \in G, \\ (\omega_1 f) = (x - x_0), \\ (w_0 f) = 1, \end{gathered} \tag{5.3}$$

where $H(x-\xi)$ is the Heaviside function on $\mathbb{R}$.

Equation (5.1) is equivalent to the system (5.3) which is a special case for the adjoint system (3.7) when $\varphi_2(\xi) = (x-\xi)H(x-\xi)$, $\varphi_1 = x - x_0$ and $\varphi_0 = 1$. Therefore, $f(x)$ is a Green's functional if and only if $f(x)$ is a solution of the system (5.3) for an arbitrary $x \in \bar{G}$. For a solution $u \in W_p$ of (2.1), we can rewrite (3.2) as

$$\begin{aligned} &\int_{x_0}^{x_1} f_2(\xi, x) z_2(\xi) d\xi + f_1(x) z_1 + f_0(x) z_0 \\ &= \int_{x_0}^{x_1} (x-\xi) H(x-\xi) u''(\xi) d\xi + u'(x_0)(x - x_0) + u(x_0). \end{aligned} \tag{5.4}$$

The right side of (5.4) is equal to $u(x)$. Therefore, we can state the following theorem.

Theorem 5.2 ([1]). *If (2.1) has at least one Green's functional $f(x)$, then any solution $u \in W_p$ of (2.1) can be represented by*

$$u(x) = \int_{x_0}^{x_1} f_2(\xi, x) z_2(\xi) d\xi + f_1(x) z_1 + f_0(x) z_0. \tag{5.5}$$

In particular the homogenous equation $Vu = 0$ has only the trivial solution.

Since one of the operators $V : W_p \to E_p$ and $\omega : E_q \to E_q$ is a homeomorphism, so the other. Therefore, for $1 \le p < \infty$ there exists a unique Green's functional. For $1 < p < \infty$ the necessary and sufficient condition for the existence of a Green's functional can be given in the following theorem.

Theorem 5.3 ([1]). *If there exists a Green's functional, then it is unique. Additionally, a Green's functional exists if and only if $Vu = 0$ has only the trivial solution.*

6. Applications

In this section we present some applications of the theory investigated above.

Example 6.1. First let us consider the problem

$$u''(x) + xu(\frac{1}{2}) = g(x), \quad x \in G = (0,1) \tag{6.1}$$

$$u(0) = \frac{1}{4}u'(\frac{1}{3}), \quad u'(0) = \frac{1}{5}u(\frac{1}{6}) \tag{6.2}$$

where $g \in L_p(G)$. Using the identities

$$\begin{gathered}
u(\alpha) = \int_0^1 \frac{1}{\alpha} H(\alpha - \xi)\xi u'(\xi) d\xi + \int_0^1 \frac{1}{\alpha} H(\alpha - \xi) u(\xi) d\xi, \quad \alpha \in G = (0,1), \\
u(c) = u(0) + cu'(0) + \int_0^1 (c - \xi) H(c - \xi) u''(\xi) d\xi, \quad c \in G = (0,1), \\
u'(c) = u'(0) + \int_0^1 H(c - \xi) u''(\xi) d\xi, \quad c \in G = (0,1),
\end{gathered}$$

for $x \in G = (0,1)$. We can rewrite this problem as

$$\begin{gathered}
(V_2 u)(x) = u''(x) + \int_0^1 [2x\xi u'(\xi) + 2xu(\xi)] H(\frac{1}{2} - \xi) d\xi = g(x) = z_2(x), \\
(V_1 u) = u(0) - \frac{1}{4}u'(0) - \int_0^1 \frac{1}{4} H(\frac{1}{3} - \xi) u''(\xi) d\xi = 0 = z_1, \\
(V_0 u) = -\frac{1}{5}u(0) + \frac{29}{30}u'(0) + \int_0^1 (\frac{1}{6} - \xi) H(\frac{1}{6} - \xi) u'' d\xi = 0 = z_0.
\end{gathered}$$

Therefore, we have

$$\begin{gathered}
A_1(x) = A_0(x) = 0, \quad B_1(x,\xi) = 2x\xi H(\frac{1}{2} - \xi), \\
B_0(x,\xi) = 2xH(\frac{1}{2} - \xi), \quad a_1 = 1, \quad b_1 = -\frac{1}{4}, \\
g_1(\xi) = -\frac{1}{4} H(\frac{1}{3} - \xi), \\
a_0 = -\frac{1}{5}, \quad b_0 = \frac{29}{30}, \quad g_0(\xi) = (\frac{1}{6} - \xi) H(\frac{1}{3} - \xi), \\
z_2(x) = g(x), \quad z_1 = z_0 = 0.
\end{gathered}$$

Thus, the adjoint system corresponding to the problem (6.1)-(6.2) is

$$\begin{aligned}
(\omega_2 f)(\xi) &= f_2(\xi) - f_1 \frac{1}{4} H(\frac{1}{3} - \xi) + f_0(\frac{1}{6} - \xi) H(\frac{1}{3} - \xi) \\
&\quad + \int_0^1 f_2(x) \Big[\int_\xi^1 2xsH(\frac{1}{2} - s) ds + \int_\xi^1 2xH(\frac{1}{2} - s)(s - \xi) ds\Big] dx \\
&= \varphi_2(\xi), \\
\omega_1 f &= -\frac{1}{4} f_1 + \frac{29}{30} f_0 + \int_0^1 \int_0^1 f_2(x) 4xsH(\frac{1}{2} - s) ds dx = \varphi_1(\xi), \\
\omega_0 f &= f_1 - \frac{1}{5} f_0 + \int_0^1 \int_0^1 f_2(x) 2xH(\frac{1}{2} - s) ds dx = \varphi_0(\xi),
\end{aligned} \tag{6.3}$$

where $f = (f_2(x), f_1, f_0) \in E_q$ is unknown function and $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ is a given function. In (6.3), if we take $\varphi_2(x) = (x-\xi)H(x-\xi)$, $\varphi_1 = x$ and $\varphi_0 = 1$ then we can obtain the special adjoint system corresponding to the problem (6.1)-(6.2) as

$$\begin{aligned} &f_2(\xi) - \frac{1}{4}H(\frac{1}{3}-\xi)f_1 + (\frac{1}{6}-\xi)H(\frac{1}{3}-\xi)f_0 \\ &+ \int_0^1\int_\xi^1 f_2(x)[4xs-2x\xi]H(\frac{1}{2}-s)dsdx = (x-\xi)H(x-\xi), \end{aligned} \tag{6.4}$$

$$-\frac{1}{4}f_1 + \frac{29}{30}f_0 + \int_0^1\int_0^1 f_2(x)4xsH(\frac{1}{2}-s)dsdx = x, \tag{6.5}$$

$$f_1 - \frac{1}{5}f_0 + \int_0^1\int_0^1 f_2(x)2xH(\frac{1}{2}-s)dsdx = 1, \tag{6.6}$$

where $\xi \in (0,1)$. To solve the system of equations (6.4), (6.5)), (6.6), first we solve the equations (6.5) and (6.6) to determine f_0 and f_1 with respect to f_2, then we find that

$$\begin{aligned} f_0 &= \frac{3}{11}(4x+1) - \frac{9}{11}K(x), \\ f_1 &= \frac{1}{55}(12x+58) - \frac{64}{55}K(x), \end{aligned}$$

where $K(\alpha) = \int_0^1 xf_2(x,\alpha)dx$. After substituting f_1 and f_0 into the equation (6.4), $f_2(\xi)$ can be found as

$$\begin{aligned} f_2(\xi) = &\Big(-\frac{14}{55} + \frac{19}{110} - \frac{17}{110}K(x) + \frac{3}{11}(4x+1) - \frac{9\xi}{11}K(x)\Big)H(\frac{1}{3}-\xi) \\ &- \int_0^1\int_\xi^1 f_2(x)[4xs-2x\xi]H(\frac{1}{2}-s)dsdx + (x-\xi)H(x-\xi), \end{aligned} \tag{6.7}$$

Thus, the Green's functional $f(x) = (f_2(\xi,x), f_1(x), f_0(x))$ for the problem has been determined. Therefore, by Theorem 5.2, a solution $u \in W_p$ of the problem (6.1)-(6.2) can be represented as

$$u(x) = \int_{x_0}^{x_1} f_2(\xi,x)g(\xi)d\xi.$$

Example 6.2. Now, let us consider the problem

$$u''(x) + u(x) - \frac{1}{2}u(0) = g(x), \quad x \in G = (0,\pi) \tag{6.8}$$

$$u(\pi) = 0 \quad u'(0) = 0 \tag{6.9}$$

where $g \in L_p(G)$. Using the identities given in Example 6.1, for $x \in G = (0,\pi)$ the problem (6.8)-(6.9) can be written as

$$u''(x) + u(x) - \frac{1}{2}\int_0^1 [\xi u'(\xi) + u(\xi)]d\xi = g(x) = z_2(x), \tag{6.10}$$

$$u(0) + \pi u'(0) + \int_0^\pi (\pi-\xi)u''(\xi)d\xi = 0 = z_1, \quad u'(0) = 0 = z_0. \tag{6.11}$$

As is given in Theorem 5.3, in order for problem (6.8)-(6.9) or (6.10)-(6.11) to have a Green's functional, the corresponding homogenous problem should have only the trivial solution. But, the corresponding homogenous problem

$$u''(x) + u(x) - \frac{1}{2}\int_0^1 [\xi u'(\xi) + u(\xi)]d\xi = 0, \quad x \in G = (0, \pi), \tag{6.12}$$

$$u(0) + \pi u'(0) + \int_0^\pi (\pi - \xi)u''(\xi)d\xi = 0, \quad u'(0) = 0, \tag{6.13}$$

has a solution $u(x) = 5\cos x + 5$, other than the trivial solution. So problem (6.8)-(6.9) or problem (6.10)-(6.11) does not have any Green's functionals in accordance with Definition 5.1.

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