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TRIGONOMETRIC SERIES ADAPTED FOR THE STUDY OF DIRICHLET BOUNDARY-VALUE PROBLEMS OF LAMÉ SYSTEMS

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Abstract. Several authors have used trigonometric series for describing the solutions to elliptic equations in a plane sector; for example, the study of the biharmonic operator with different boundary conditions, can be found in [2, 9, 10]. The main goal of this article is to adapt those techniques for the study of Lamé systems in a sector.

1. Introduction

Let S be the truncated plane sector of angle $\omega \leq 2\pi$, and positive radius ρ:

$$S = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2,\ 0 < r < \rho,\ 0 < \theta < \omega\}\,. \tag{1.1}$$

Let Σ be the circular boundary part

$$\Sigma = \{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2,\ 0 < \theta < \omega\}. \tag{1.2}$$

We are interested in the study of functions u belonging to the Sobolev space $[H^1(S)]^2$, and that are solutions to the Lamé type system

$$\begin{gathered} Lu = \Delta u + \nu_0 \nabla(\operatorname{div} u) = 0 \quad \text{in } S \\ u = 0 \quad \text{for } \theta = 0,\, \omega\,, \end{gathered} \tag{1.3}$$

where

$$\nu_0 = (1-2\nu)^{-1} = \frac{\lambda+\mu}{\mu},$$

λ, μ are the Lamé constants, with $\lambda \geq 0$, $\mu > 0$ and ν is a real number ($0 < \nu < 1/2$) called Poisson coefficient.

We shall analyze the solutions u of this problem which can be written in series of the form:

$$u(r,\theta) = \sum_{\alpha\in E} c_\alpha r^\alpha v_\alpha(\theta). \tag{1.4}$$

Here E stands for the set of solutions of the equation in a complex variable $\alpha(\nu_0)$,

$$\sin^2 \alpha\omega = \Big(\frac{\nu_0}{\nu_0+2}\ \alpha\Big)^2 \sin^2\omega, \quad \operatorname{Re}\alpha \succ 0 \tag{1.5}$$

For further studies of the set E, see, for example Lozi [5].

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We will adapt the technique used in [2, 9, 10], for the bilaplacian. The novelty is that, here, we treat a system of PDE's instead of the equations studied before. For this purpose we introduce a Betti formula instead of a Green formula. To compute the coefficients of the the singularities which can occur in the solutions, this technique is easier and more direct than the classical one used in [4].

We will focus on the important case of the crack, i.e. $\omega = 2\pi$. The calculations in that case are more explicit and give the known results for the Laplacian as just a particular case.

2. Separation of variables

Replacing u by $r^\alpha v_\alpha(\theta) = r^\alpha(v_{1,\alpha}(\theta), v_{2,\alpha}(\theta))$ in the problem (1.3) and using the change of variables

$$\begin{gathered} w_{1,\alpha}(\theta) = \cos\theta v_{1,\alpha}(\theta) + \sin\theta v_{2,\alpha}(\theta), \\ w_{2,\alpha}(\theta) = -\sin\theta v_{1,\alpha}(\theta) + \cos\theta v_{2,\alpha}(\theta) \end{gathered} \tag{2.1}$$

leads us the system

$$\begin{gathered} w''_{1,\alpha}(\theta) + (\nu_0+1)(\alpha^2-1)w_{1,\alpha}(\theta) + (\nu_0(\alpha-1)-2)w'_{2,\alpha}(\theta) = 0; \\ (\nu_0+1)w''_{2,\alpha}(\theta) + (\alpha^2-1)w_{2,\alpha}(\theta) + (\nu_0(\alpha+1)+2)w'_{1,\alpha}(\theta) = 0. \\ w_{1,\alpha}(0) = w_{2,\alpha}(0) = 0. \\ \cos\omega w_{1,\alpha}(\omega) - \sin\omega w_{2,\alpha}(\omega) = \sin\omega w_{1,\alpha}(\omega) + \cos\omega w_{2,\alpha}(\omega) = 0. \end{gathered} \tag{2.2}$$

By Merouani [7], the solutions of (2.2) are linear combination of the functions

$$\varphi_\alpha(\theta) = \begin{pmatrix} 2\alpha v_0[\cos(\alpha-2)\theta - \cos\alpha\theta] \\ -2\alpha v_0 \sin(\alpha-2)\theta + 2(v_0(\alpha-2)-4)\sin\alpha\theta \end{pmatrix} \tag{2.3}$$

and

$$\psi_\alpha(\theta) = \begin{pmatrix} 2\alpha v_0 \sin(\alpha-2)\theta - 2(v_0(\alpha+2)+4)\sin\alpha\theta \\ 2\alpha v_0[\cos(\alpha-2)\theta - \cos\alpha\theta] \end{pmatrix} \quad 0<\theta<\omega, \tag{2.4}$$

A relationship, similar to classical orthogonality, for this system is given by the following theorem.

Theorem 2.1. *Let $w_\alpha = (w_{1,\alpha}, w_{2,\alpha})$ and $w_\beta = (w_{1,\beta}, w_{2,\beta})$ be solutions of (2.2) with α and β solutions of (1.5). Then, for $\beta \neq \overline{\alpha}$, we have*

$$\begin{aligned} &[w_\alpha, w_\beta] \\ &= \int_0^\omega \Big[\big[\frac{1}{(\overline{\beta}-\alpha)}\nu_0(w'_{2,\alpha}, w'_{1,\alpha}) + ((v_0+1)w_{1,\alpha}, w_{2,\alpha})\big] \begin{pmatrix} \overline{w}_{1,\beta} \\ \overline{w}_{2,\beta} \end{pmatrix} \Big] d\theta = 0 \end{aligned} \tag{2.5}$$

Proof. We shall use Betti formula

$$\int_S (vLu - uLv)dx = \int_\Gamma [v\sigma(u)\cdot\eta - u\sigma(v)\cdot\eta]d\sigma \tag{2.6}$$

where

$$\sigma(u) = \begin{pmatrix} \sigma_{11}(u) & \sigma_{12}(u) \\ \sigma_{12}(u) & \sigma_{22}(u) \end{pmatrix}$$

is the tensor of stress, $\eta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ is the outward unit vector normal to Σ, and Γ is the boundary of S. For two functions u, v which are solutions of (1.3), using

Betti's formula we obtain

$$\int_{\Sigma}[v\sigma(u).\eta - u\sigma(v).\eta]d\sigma = 0 \tag{2.7}$$

on Σ, for the function $u = r^{\alpha}\varphi_{\alpha}$, taking account of the change of variables (2.1), we have

$$\sigma(u)\cdot\eta = \frac{r^{\alpha-1}}{\mu}M_{\alpha,v_0}(w_{\alpha}) \tag{2.8}$$

with $M_{\alpha,v_0}(w_{\alpha})$ being the matrix

$$\begin{pmatrix} ((v_0-1)w'_{2,\alpha} + (\alpha(v_0+1)+(v_0-1))w_{1,\alpha})\cos\theta - (w'_{1,\alpha}+(\alpha-1)w_{2,\alpha})\sin\theta \\ (w'_{1,\alpha}+(\alpha-1)w_{2,\alpha})\cos\theta + ((v_0-1)w'_{2,\alpha}+(\alpha(v_0+1)+(v_0-1))w_{1,\alpha})\sin\theta \end{pmatrix}$$

The results follow from the application of formula (2.7) to the functions $u = r^{\alpha}\varphi_{\alpha}$ and $u = r^{\beta}\psi_{\beta}$, and by using relation (2.8). □

Corollary 2.2. *Let w_{α} and w_{β} be solution of* (2.2) *with α and β solution of* (1.5). *Suppose in addition that*

$$\int_0^{\omega}(w'_{2,\alpha}, w'_{1,\alpha})\begin{pmatrix}\overline{w}_{1,\beta}\\ \overline{w}_{2,\beta}\end{pmatrix} = 0 \tag{2.9}$$

and $\alpha \neq \overline{\beta}$, then

$$[w_{\alpha}, w_{\beta}] = \int_0^{\omega}[((v_0+1)w_{1,\alpha}, w_{2,\alpha})]\begin{pmatrix}\overline{w}_{1,\beta}\\ \overline{w}_{2,\beta}\end{pmatrix}d\theta = 0 \tag{2.10}$$

Proof. Substituting (2.9) in (2.5), we obtain (2.10). □

Remark 2.3. For $w_{\alpha} = r^{\alpha}(w_{1,\alpha}, w_{2,\alpha})$, we define the operator

$$Tw_{\alpha} = r^{\alpha-1}\begin{pmatrix}(v_0+1)w_{1,\alpha}\\ w_{2,\alpha}\end{pmatrix}$$

Corollary 2.4. *From corollary 2.2, if $\alpha \neq \overline{\beta}$, we have*

$$\int_{\Sigma}(Tw_{\alpha}\overline{w_{\beta}} + w_{\alpha}T\overline{w_{\beta}})d\sigma = 0. \tag{2.11}$$

Proof. From the definition of the operator T and Corollary 2.2 we have

$$\int_{\Sigma}(Tw_{\alpha}.\overline{w_{\beta}} + w_{\alpha}T\overline{w_{\beta}})d\sigma = 2r^{\alpha+\beta-1}\int_0^{\omega}((v_0+1)w_{1,\alpha}w_{1,\overline{\beta}} + w_{2,\alpha}w_{2,\overline{\beta}})d\theta = 0.$$

□

Corollary 2.5. *Suppose that $u = \sum_{\alpha\in E}c_{\alpha}r^{\alpha}\varphi_{\alpha}$ is uniformly convergent in $\overline{S}$. If $[\varphi_{\overline{\beta}}, \varphi_{\beta}] \neq 0$ then*

$$c_{\overline{\beta}} = \frac{1}{2}\rho^{-2\overline{\beta}+1}\frac{\int_{\Sigma}(Tu\overline{u_{\beta}} + uT\overline{u_{\beta}})d\sigma}{[\varphi_{\overline{\beta}}, \varphi_{\beta}]}.$$

Proof. For $u = \sum_{\alpha\in E}c_{\alpha}r^{\alpha}\varphi_{\alpha}$ and taking account the definition of the operator T we have

$$\begin{aligned} &\int_{\Sigma}(Tu.\overline{u_{\beta}} + u.T\overline{u_{\beta}})d\sigma \\ &= \int_0^{\omega}\Big(\Big(\sum_{\alpha\in E}c_{\alpha}r^{\alpha-1}\begin{pmatrix}(v_0+1)\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}\Big)r^{\overline{\beta}}\varphi_{\overline{\beta}} \end{aligned}$$

$$+\Big(\sum_{\alpha\in E} c_\alpha r^\alpha \varphi_\alpha\Big) r^{\overline{\beta}-1}\begin{pmatrix}(v_0+1)\varphi_{1,\overline{\beta}}\\ \varphi_{2,\overline{\beta}}\end{pmatrix}\Big)d\theta$$

$$=\sum_{\alpha\in E} c_\alpha r^{\overline{\beta}+\alpha-1}\int_0^\omega \Big(\begin{pmatrix}(v_0+1)\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}\varphi_{\overline{\beta}}+\varphi_\alpha\begin{pmatrix}(v_0+1)\varphi_{1,\overline{\beta}}\\ \varphi_{2,\overline{\beta}}\end{pmatrix}\Big)d\theta.$$

From Corollary 2.2, if $\alpha\neq\overline{\beta}$, then

$$\int_\Sigma (Tu\overline{u_\beta}+uT\overline{u_\beta})d\sigma = 2C_{\overline{\beta}}[\varphi_{\overline{\beta}},\varphi_\beta]\rho^{2\overline{\beta}-1}.$$

Expression $c_{\overline{\beta}}$ of Corollary 2.5 results from this last equality. □

Remark 2.6. The technique we develop for the study of the trigonometric series is based on Theorem 2.1 and Corollary 2.5. To illustrate this, we study the following trigonometric series in the particular case of the crack ($\omega = 2\pi$), which is an important case of singular domains. The explicit knowledge of the roots of (1.5) simplifies the computations.

3. Complete case study of the crack

To simplify the calculations, we decompose every solution u of (1.3) into two parts with respect to θ

$$u=\mathfrak{U}_1+\mathfrak{U}_2.$$

3.1. **Study of first part.** The first part is the expression φ_α and is given by (2.3) where

$$E=\{\frac{k}{2},\, k\in\mathbb{N}^*\}\quad \text{because } \omega=2\pi.$$

After some calculation, we obtain that

$$[\varphi_\beta,\varphi_\beta]=[\beta^2 v_0^2(2v_0+3)+4(v_0(\beta-2)-4)^2]\pi\rho^{2\beta-1}\neq 0\,.$$

Define the sub-sector

$$S_{\rho_0}=S\cap\{(r\cos\theta, r\sin\theta)\in\mathbb{R}^2, r<\rho_0\},\rho_0<\rho.$$

We define the traces on Σ,

$$\mathfrak{U}_1=\xi_1\in\big(\tilde{H}^{1/2}(\Sigma)\big)^2 \quad\text{and}\quad T\mathfrak{U}_1=\phi_1\in\big(\tilde{H}^{1/2}(\Sigma)\big)^2.$$

Let

$$c_\alpha = A_{\alpha,v_0}\int_0^{2\pi}\Big(\xi_1\begin{pmatrix}(v_0+1)\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}+\rho_0\begin{pmatrix}\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}\phi_1\Big)(\rho_0,\theta)d\theta \tag{3.1}$$

with

$$A_{\alpha,v_0}=\frac{\rho_0^{-\alpha}}{2[\alpha^2 v_0^2(2v_0+3)+4(v_0(\alpha-2)-4)^2]\pi}$$

Corollary 3.1. *If $\mathfrak{U}_1$ is solution of (1.3), then*

$$\mathfrak{U}_1=\sum_{\alpha\in E}c_\alpha r^\alpha\varphi_\alpha \tag{3.2}$$

where c_α is given by (3.1). The series converges uniformly in $\overline{S}_{\rho_0}$ for all $\rho_0<\rho$. Moreover (3.2) converges globally in $(H^1(S_\rho))^2$, if $\alpha^{3/2}c_\alpha\rho^\alpha\in l^2$.

Proof. (i) if (3.2) occurs, then c_α is expressed by (3.1) under Corollary 2.5.

(ii) if $\mathfrak{U}_1$ is solution of (1.3) and c_α given by (3.1) then $c_\alpha = \circ(\alpha\rho_0^{-\alpha})$. This implies the uniform convergence of the series in $\overline{S}_{\rho_0}$ towards some W_1 satisfying (1.3).

From Grisvard-Geymonat [3], there exists a positive ε, sufficiently small such that the solution of (1.3) is written as

$$\mathfrak{U}_1 = \sum_{\alpha\in E} K_\alpha r^\alpha \varphi_\alpha$$

which converges for $r < \varepsilon$. Theorem 2.1 implies that $K_\alpha = c_\alpha$ therefore W_1 and $\mathfrak{U}_1$ coincide in S_ε. They coincide in S_{ρ_0} since they are real analytic. □

Remark 3.2. *If ξ_1 belongs to the space $(H^2(]0,2\pi[))^2$ and ϕ_1 to $(H^1(]0,2\pi[))^2$, then $c_\alpha = \circ(\alpha\rho_0^{-\alpha})$ and we have uniform convergence of the series in $\overline{S}_{\rho_0}$ for all $\rho_0 \le \rho$.*

3.2. **Study of the second part.** The second part is the expression ψ_α given by (2.4) where

$$E = \{\frac{k}{2},\, k \in \mathbb{N}^*\}$$

because $\omega = 2\pi$. After some calculations, we obtain

$$[\psi_\alpha, \psi_\alpha] = [(v_0+3)\alpha^2 v_0^2 + 4(v_0+1)(v_0(\alpha+2)+4)^2]\pi\rho^{2\alpha-1} \neq 0\,.$$

We define the following trace on Σ,

$$\mathfrak{U}_2 = \xi_2 \in (\tilde{H}^{1/2}(\Sigma))^2 \quad\text{and}\quad T\mathfrak{U}_2 = \phi_2 \in (\tilde{H}^{1/2}(\Sigma))^2.$$

Let

$$d_\alpha = B\alpha, v_0 \int_0^{2\pi} \Big(\xi_1 \begin{pmatrix} (v_0+1)\psi_{1,\alpha} \\ \psi_{2,\alpha} \end{pmatrix} + \rho_0 \begin{pmatrix} \psi_{1,\alpha} \\ \psi_{2,\alpha} \end{pmatrix} \phi_1)(\rho_0,\theta)d\theta. \tag{3.3}$$

with

$$B_{\alpha,v_0} = \frac{\rho_0^{-\alpha}}{2[(v_0+3)\alpha^2 v_0^2 + 4(v_0+1)(v_0(\alpha+2)+4)^2]\pi}$$

Corollary 3.3. *If $\mathfrak{U}_2$ is solution of (1.3) then*

$$\mathfrak{U}_2 = \sum_{\alpha\in E} d_\alpha r^\alpha \psi_\alpha \tag{3.4}$$

where d_α is given by (3.3). The series converges uniformly in $\overline{S}_{\rho_0}$ for all $\rho_0 < \rho$. Moreover (3.4) converges globally in $(H^1(S_\rho))^2$, if $\alpha^{3/2} d_\alpha \rho^\alpha \in l^2$.

Remark 3.4. For $v_0 = 0$ we obtain the trigonometric series for the Laplace equation in a sector. This is compatible with (1.3) with $v_0 = 0$.

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