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EXISTENCE OF POSITIVE ALMOST PERIODIC SOLUTIONS FOR A NICHOLSON'S BLOWFLIES MODEL

HUI-SHENG DING, JEHAD ALZABUT

Abstract. This article concerns the existence of almost periodic solutions for a Nicholson's blowflies model with a nonlinear density-dependent mortality term. By utilizing a fixed point theorem in cones, we establish the existence of almost periodic solutions. As one will see, our main assumptions here are not the as in [10]. Two examples are given to illustrate our existence result.

1. Introduction and preliminaries

In this article, we study the existence of almost periodic solutions for the Nicholson's blowflies model with a nonlinear density-dependent mortality term,

$$x'(t) = -\alpha + \beta e^{-x(t)} + \gamma(t)x(t-\tau(t))e^{-\delta x(t-\tau(t))}, \quad t \in \mathbb{R}, \tag{1.1}$$

where α, β, δ are positive constants, and $\gamma, \tau : \mathbb{R} \to [0, +\infty)$ are almost periodic functions.

The above model originates from the work of Nicholson [11] and of Gurney, Blythe and Nisbet [8]. They proposed the model

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-\gamma x(t-\tau)}, \tag{1.2}$$

where $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $1/\gamma$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time.

There is a large body of literature on the existence of almost periodic solutions for Nicholson's blowflies model (1.2) and its variants; see, for instance, [1, 3, 10, 7, 12, 13]). Especially, Liu [10] and Xu [13] investigated the existence and stability of almost periodic solutions for Nicholson's blowflies models with a nonlinear density-dependent mortality term.

In fact, as pointed out by Berezansky et al [2], a new study indicates that a linear model of density-dependent mortality will be most accurate for populations at low densities, and marine ecologists are currently in the process of constructing new fishery models with nonlinear density-dependent mortality rates. Therefore, studying the dynamical behavior for Nicholson's blowflies models with a nonlinear density-dependent mortality term is an interesting and important topic.

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Motivated by [10, 13], we further study on the existence of almost periodic solutions for (1.1). As one will see, our main assumptions are different from the assumptions in [10].

Next, let us recall some basic notation and results about almost periodic functions. For more details, we refer the reader to [4, 9].

Definition 1.1. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| < \varepsilon.$$

We denote the set of all such functions by $AP(\mathbb{R})$.

Lemma 1.2. *Let $f, g \in AP(\mathbb{R})$. Then the following assertions hold:*

(a) *$AP(\mathbb{R})$ is a Banach space under the norm $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$.*
(b) *$f + g \in AP(\mathbb{R})$ and $f \cdot g \in AP(\mathbb{R})$.*
(c) *$h \in AP(\mathbb{R})$, where $h(t) = f(t - g(t))$ for all $t \in \mathbb{R}$.*
(d) *$\varphi \circ f \in AP(\mathbb{R})$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function.*
(e) *Let $\omega > 0$ be a fixed constant and $F(t) = \int_{-\infty}^{t} e^{-\omega(t-s)} f(s)ds$ for all $t \in \mathbb{R}$, then $F \in AP(\mathbb{R})$.*

2. Main results

First, let us recall a fixed point theorem in cones. For the basic notations about cone, we refer the reader to [5]. The following theorem can be deduced from [6, Theorem 2.11] (see also [7, Theorem 2.1]).

Theorem 2.1. *Let C be a normal and solid cone in a real Banach space X, and $\Phi : \mathring{C} \to \mathring{C}$ be a nondecreasing operator, where $\mathring{C}$ is the interior of C. Suppose further that there exists a function $\phi : (0,1) \times \mathring{C} \to (0,+\infty)$ such that for each $\lambda \in (0,1)$ and $x \in \mathring{C}$, $\phi(\lambda, x) > \lambda$, $\phi(\lambda, \cdot)$ is nondecreasing in $\mathring{C}$, and*

$$\Phi(\lambda x) \ge \phi(\lambda, x)\Phi(x).$$

Assume, in addition, there exists $z \in \mathring{C}$ such that $\Phi(z) \ge z$. Then Φ has a unique fixed point $\widetilde{x}$ in $\mathring{C}$. Moreover, for any initial $x_0 \in \mathring{C}$, the iterative sequence

$$x_n = \Phi(x_{n-1}), \quad n \in \mathbb{N}, \tag{2.1}$$

satisfies

$$\|x_n - \widetilde{x}\| \to 0 \quad (n \to +\infty). \tag{2.2}$$

In the rest of this article, for each bounded function g on $\mathbb{R}$, we denote

$$g^* = \sup_{t\in\mathbb{R}} g(t), \quad g_* = \inf_{t\in\mathbb{R}} g(t).$$

Now, it is ready to state our main result.

Theorem 2.2. *Let $\beta > \alpha$ and*

$$\beta e^{-1/\delta} + \frac{\gamma^*}{\delta e} \le \alpha \le (1 + 1/\delta)\beta e^{-1/\delta}.$$

Then (1.1) has an almost periodic solution $\widetilde{x}$ with a positive infimum.

Proof. Let
$$C = \{x \in AP(\mathbb{R}) : x(t) \geq 0, \forall t \in \mathbb{R}\}.$$
It is not difficult to verify that C is a normal and solid cone in $AP(\mathbb{R})$, and
$$\mathring{C} = \{x \in AP(\mathbb{R}) : \exists\, \varepsilon > 0 \text{ such that } x(t) > \varepsilon, \forall t \in \mathbb{R}\}.$$
Noting that $\beta > \alpha$, there exists a sufficiently small constant $\varepsilon \in (0, 1/\delta)$ such that
$$\beta e^{-\varepsilon} \geq \alpha. \tag{2.3}$$

For $x > 0$, let $f_1(x) = \beta x - \alpha + \beta e^{-x}$, $f_2(x) = xe^{-\delta x}$, and
$$g_1(x) = \begin{cases} f_1(\varepsilon), & 0 < x < \varepsilon,\\ f_1(x), & \varepsilon \leq x \leq 1/\delta,\\ f_1(1/\delta), & x > 1/\delta, \end{cases} \quad g_2(x) = \begin{cases} f_2(\varepsilon), & 0 < x < \varepsilon,\\ f_2(x), & \varepsilon \leq x \leq 1/\delta,\\ f_2(1/\delta), & x > 1/\delta. \end{cases}$$
We define a nonlinear operator Φ on $\mathring{C}$ by
$$\Phi(x)(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} \big[g_1(x(s)) + \gamma(s) g_2(x(s - \tau(s)))\big] ds, \quad t \in \mathbb{R},\ x \in \mathring{C}.$$
It follows from Lemma 1.2 that Φ maps $AP(\mathbb{R})$ to $AP(\mathbb{R})$. Noting that g_1 and g_2 are both nondecreasing on $(0, +\infty)$, we deduce that Φ is a nondecreasing operator on $\mathring{C}$, and
$$\Phi(x)(t) \geq \frac{g_1(\varepsilon)}{\beta} > 0, \quad t \in \mathbb{R},\ x \in \mathring{C},$$
which means that Φ is from $\mathring{C}$ to $\mathring{C}$. Moreover, by using (2.3), we get
$$\Phi(\varepsilon)(t) \geq \frac{g_1(\varepsilon)}{\beta} = \frac{\beta\varepsilon - \alpha + \beta e^{-\varepsilon}}{\beta} \geq \varepsilon, \quad t \in \mathbb{R},\ x \in \mathring{C}.$$
That is, $\Phi(\varepsilon) \geq \varepsilon$. Letting $\phi_2(\lambda) = \lambda e^{\delta(1-\lambda)\varepsilon}$, we have $\phi_2(\lambda) > \lambda$ for all $\lambda \in (0, 1)$, and
$$f_2(\lambda x) = \lambda e^{\delta(1-\lambda)x} f_2(x) \geq \phi_2(\lambda) f_2(x), \quad \lambda \in (0, 1),\ x \in [\varepsilon, 1/\delta].$$
Combining this with the fact that g_2 is nondecreasing on $(0, +\infty)$, we have
$$g_2(\lambda x) = \begin{cases} f_2(\varepsilon) \geq \phi_2(\lambda) f_2(\varepsilon) = \phi_2(\lambda) g_2(x), & x \in (0, \varepsilon),\\ f_2(\lambda x) \geq \phi_2(\lambda) f_2(x) = \phi_2(\lambda) g_2(x), & x \in [\varepsilon, 1/\delta],\ \lambda x \in [\varepsilon, 1/\delta],\\ f_2(\varepsilon) \geq f_2(\lambda x) \geq \phi_2(\lambda) f_2(x) = \phi_2(\lambda) g_2(x), & x \in [\varepsilon, 1/\delta],\ \lambda x < \varepsilon,\\ f_2(\varepsilon) \geq f_2(\lambda/\delta) \geq \phi_2(\lambda) f_2(1/\delta) = \phi_2(\lambda) g_2(x), & x > 1/\delta,\ \lambda x < \varepsilon,\\ f_2(\lambda x) \geq f_2(\lambda/\delta) \geq \phi_2(\lambda) f_2(1/\delta) = \phi_2(\lambda) g_2(x), & x > 1/\delta,\ \lambda x \in [\varepsilon, 1/\delta],\\ f_2(1/\delta) \geq \phi_2(\lambda) f_2(1/\delta) = \phi_2(\lambda) g_2(x), & \lambda x > 1/\delta, \end{cases}$$
i.e., $g_2(\lambda x) \geq \phi_2(\lambda) g_2(x)$ for all $\lambda \in (0, 1)$ and $x > 0$.

Next, let us show that there exists a function $\phi_1 : (0, 1) \to (0, +\infty)$ such that for all $\lambda \in (0, 1)$ and $x > 0$, we have $\phi_1(\lambda) > \lambda$, and
$$g_1(\lambda x) \geq \phi_1(\lambda) g_1(x).$$
For $\lambda \in (0, 1)$ and $x \in [\varepsilon, 1/\delta]$, define
$$\theta_\lambda(x) = \frac{f_1(\lambda x)}{\lambda f_1(x)} = \frac{\beta\lambda x - \alpha + \beta e^{-\lambda x}}{\lambda\beta x - \lambda\alpha + \lambda\beta e^{-x}}, \quad \text{and} \quad h_\lambda(x) = \beta e^{-\lambda x} - \alpha + \lambda\alpha - \lambda\beta e^{-x}.$$

To show that $\theta_\lambda(x) > 1$ for all $\lambda \in (0,1)$ and $x \in [\varepsilon, 1/\delta]$, we only need to show that $h_\lambda(x) > 0$ for all $\lambda \in (0,1)$ and $x \in [\varepsilon, 1/\delta]$. Since

$$h'_\lambda(x) = -\lambda\beta e^{-\lambda x} + \lambda\beta e^{-x} < 0,$$

we can obtain $h_\lambda(x) > 0$ provided that $h_\lambda(1/\delta) > 0$. Let

$$F(\lambda) = h_\lambda(1/\delta) = \beta e^{-\lambda/\delta} - \lambda\beta e^{-1/\delta} + \alpha\lambda - \alpha.$$

Noting that $F(1) = 0$ and

$$F'(\lambda) = \alpha - \frac{\beta}{\delta} e^{-\lambda/\delta} - \beta e^{-1/\delta} < \alpha - \frac{\beta}{\delta} e^{-1/\delta} - \beta e^{-1/\delta} \le 0, \quad \lambda \in (0,1),$$

we conclude that $F(\lambda) > 0$, i.e., $h_\lambda(1/\delta) > 0$ for all $\lambda \in (0,1)$. Thus, we have proved that $\theta_\lambda(x) > 1$ for all $\lambda \in (0,1)$ and $x \in [\varepsilon, 1/\delta]$. Let

$$\phi_1(\lambda) = \min_{x \in [\varepsilon, 1/\delta]} \lambda\theta_\lambda(x).$$

Then, $\phi_1(\lambda) > \lambda$ for all $\lambda \in (0,1)$, and

$$f_1(\lambda x) \ge \phi_1(\lambda) f_1(x), \quad \lambda \in (0,1),\ x \in [\varepsilon, 1/\delta].$$

By a similar proof to the one of ϕ_2, we obtain

$$g_1(\lambda x) \ge \phi_1(\lambda) g_1(x), \quad \lambda \in (0,1),\ x > 0.$$

Let

$$\phi(\lambda) = \min\{\phi_1(\lambda), \phi_2(\lambda)\}, \quad \lambda \in (0,1).$$

Then, for all $\lambda \in (0,1)$, $x \in \mathring{C}$, and $t \in \mathbb{R}$, there holds

$$\begin{aligned}
\Phi(\lambda x)(t) &= \int_{-\infty}^{t} e^{-\beta(t-s)} [g_1(\lambda x(s)) + \gamma(s) g_2(\lambda x(s - \tau(s)))] ds \\
&\ge \int_{-\infty}^{t} e^{-\beta(t-s)} [\phi_1(\lambda) g_1(x(s)) + \phi_2(\lambda)\gamma(s) g_2(x(s - \tau(s)))] ds \\
&\ge \phi(\lambda) \int_{-\infty}^{t} e^{-\beta(t-s)} [g_1(x(s)) + \gamma(s) g_2(x(s - \tau(s)))] ds \\
&= \phi(\lambda)\Phi(x)(t),
\end{aligned}$$

which yields $\Phi(\lambda x) \ge \phi(\lambda, x)\Phi(x)$.

By applying Theorem 2.1, Φ has a unique fixed point $\widetilde{x} \in \mathring{C}$. Then

$$\widetilde{x}(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} \big[g_1(\widetilde{x}(s)) + \gamma(s) g_2(\widetilde{x}(s - \tau(s)))\big] ds, \quad t \in \mathbb{R}.$$

By using (2.3) and $\beta e^{-1/\delta} + \frac{\gamma^*}{\delta e} \le \alpha$, we have

$$\widetilde{x}(t) \ge \frac{\beta\varepsilon - \alpha + \beta e^{-\varepsilon}}{\beta} \ge \varepsilon, \quad t \in \mathbb{R},$$

$$\widetilde{x}(t) \le \frac{\beta/\delta - \alpha + \beta e^{-1/\delta} + \gamma^*/(\delta e)}{\beta} \le 1/\delta, \quad t \in \mathbb{R}.$$

Thus, we have

$$\widetilde{x}(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} \big[f_1(\widetilde{x}(s)) + \gamma(s) f_2(\widetilde{x}(s - \tau(s)))\big] ds, \quad t \in \mathbb{R},$$

which yields

$$\begin{aligned}\frac{d\widetilde{x}(t)}{dt} &= -\beta\widetilde{x}(t) + f_1(\widetilde{x}(t)) + \gamma(t)f_2(\widetilde{x}(t-\tau(t)))\\ &= -\alpha + \beta e^{-\widetilde{x}(t)} + \gamma(t)\widetilde{x}(t-\tau(t))e^{-\delta\widetilde{x}(t-\tau(t))}, \quad t \in \mathbb{R}.\end{aligned}$$

That is, $\widetilde{x}$ is a solution of equation (1.1). This completes the proof. □

Remark 2.3. Recently, Liu [10] investigated the existence and stability of almost periodic solutions to equation (1.1). Here, we use a different approach from that of [10]. Also, we do not need some restrictive conditions such as $\delta \geq 1$, which is assumed in [10]. But, it seems difficult to obtain the stability of almost periodic solutions to equation (1.1) under the assumptions of Theorem 2.2. We leave this problem for future study. In addition, although we use a similar approach to that of [7], the model in this paper is more difficult and tricky to deal with due to the influence of mortality term.

Example 2.4. Let $\delta = 1$, $\alpha = 2$, $\beta = e$, $\gamma(t) = e - \sin^2 t - \sin^2 \pi t$ and $\tau(t) = \cos^2 t + \cos^2 \sqrt{2}t$. Then, $\gamma^* = e$, and

$$\beta e^{-1/\delta} + \frac{\gamma^*}{\delta e} = 2 \leq \alpha \leq 2 = (1 + 1/\delta)\beta e^{-1/\delta}.$$

By using Theorem 2.2, the equation

$$x'(t) = -2 + e^{1-x(t)} + [e - \sin^2 t - \sin^2 \pi t]x(t - \cos^2 t - \cos^2 \sqrt{2}t)e^{-x(t-\cos^2 t - \cos^2 \sqrt{2}t)} \tag{2.4}$$

admits an almost periodic solution with positive infimum.

By making some modifications on the above example, we can get an example to show that $\delta \geq 1$ is not necessarily needed in our main result.

Example 2.5. Let $\delta = 1/2$, $\alpha = 3$, $\beta = e^2$, $\gamma(t) = e - \sin^2 t - \sin^2 \pi t$ and $\tau(t) = \cos^2 t + \cos^2 \sqrt{2}t$. Then, $\gamma^* = e$, and

$$\beta e^{-1/\delta} + \frac{\gamma^*}{\delta e} = 3 \leq \alpha \leq 3 = (1 + 1/\delta)\beta e^{-1/\delta}.$$

By using Theorem 2.2, equation (1.1) also has an almost periodic solution with positive infimum.

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Hui-Sheng Ding (corresponding author)
College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
E-mail address: dinghs@mail.ustc.edu.cn

Jehad Alzabut
Department of Mathematics and Physical Sciences, Prince Sultan University, P. O. Box 66833, 11586 Riyadh, Saudi Arabia
E-mail address: jalzabut@psu.edu.sa