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THE SPREADING OF CHARGED MICRO-DROPLETS

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ABSTRACT. This article considers the analysis of the Betelu-Fontelos model of the spreading of a charged microdroplet on a flat dielectric surface whose spreading is driven by surface tension and electrostatic repulsion. This model assumes the droplets are circular and spread according to a power law. This leads to a third-order nonlinear ordinary differential equation on $[0,1]$ that gives the evolution of the height profile. We examine existence of solutions for this equation.

1. Introduction

In this article we perform the analysis of the Betelu-Fontelos (BF) model [2] of the spreading of a charged microdroplet. With this model a charged microdroplet spreads over a flat surface. In the absence of charge, it has been shown experimentally [5] that the radius of a circular drop, $a(t)$, spreads according to the law $a(t) = At^{1/10}$ for some constant A. The mathematical analysis of the uncharged case is difficult, the main result [1] being that in the absence of molecular forces, the "paradox of the contact line" arises and the drop does *not* spread. In [2], electric charges were included into the model, and it was shown that the drop *does* spread resolving the paradox of the contact line. Also surprisingly, the presence of the charges does *not* alter the similarity exponent for the spreading of the drop and it, too, spreads according to a $t^{1/10}$ law. This is also surprising since if we add gravity instead then the exponent *does* change [3]. This model has practical implications in physical processes on which electrically charged droplets spread on surfaces such as electro-painting.

This model uses the *lubrication* approximation which assumes that the fluid spreads over a solid surface and that the droplet is thin so that the horizontal component of the velocity is much larger than the vertical component and that the stresses are mostly due to gradients of the velocity in the direction perpendicular to the surface. Using this approximation it is shown in [2] that the height profile $h(r,t)$ of a circular drop satisfies

$$h_t + \frac{1}{r}\frac{\partial}{\partial r}\Big[\frac{r}{3\mu}h^3\frac{\partial}{\partial r}\Big(\frac{Q^2}{2\epsilon_0(4\pi a(t))^2}\frac{1}{a^2(t)-r^2} + \gamma(h_{rr} + \frac{h_r}{r})\Big)\Big] = 0 \tag{1.1}$$

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and

$$\int_0^{a(t)} 2\pi r h(r,t)\,dr = V_0 = \text{constant} \tag{1.2}$$

where $a(t)$ is the radius of the drop and the boundary conditions are

$$h_r(0,t) = h_{rrr}(0,t) = 0 \tag{1.3}$$

(due to the circular symmetry), and

$$h(a(t),t) = 0. \tag{1.4}$$

Here γ is the free surface tension coefficient, ϵ_0 is the permittivity of the gas above the drop, μ is the viscosity, and Q is the total charge. Equation (1.2) states that the volume of the drop remains constant throughout this process.

We seek a self-similar solution such that the radius of the drop $a(t)$ satisfies a power law, i.e. $a(t) = At^\beta$. The height profile will then, by conservation of mass, be of the form

$$h(r,t) = \frac{B}{t^{2\beta}} H\big(\frac{r}{a(t)}\big) = \frac{B}{t^{2\beta}} H\big(\frac{r}{At^\beta}\big)$$

where $\rho = \frac{r}{a(t)}$ and $0 \le \rho \le 1$. In terms of V_0 this gives:

$$V_0 = \int_0^{a(t)} 2\pi r h(r,t)\,dr = A^2 B \int_0^1 2\pi\rho H(\rho)\,d\rho$$

where

$$Y \equiv \int_0^1 2\pi\rho H(\rho)\,d\rho \tag{1.5}$$

denotes the dimensionless "shape factor" of the drop.

Remarkably, with $\beta = \frac{1}{10}$ equation (1.1) becomes

$$\Big[\rho H^3\Big(H_{\rho\rho} + \frac{H_\rho}{\rho} + \frac{XY}{1-\rho^2}\Big)_\rho\Big]_\rho = Z(\rho^2 H_\rho + 2\rho H)$$

where

$$X = \frac{Q^2}{32\pi^2\epsilon_0\gamma V_0}, \quad Z = \frac{3\mu A^4}{10\gamma B^3}.$$

Integrating once, using (1.3), and rewriting yields

$$H''' + \frac{H''}{\rho} - \frac{H'}{\rho^2} = \big(H'' + \frac{H'}{\rho}\big)' = \frac{Z\rho}{H^2} - \frac{2XY\rho}{(1-\rho^2)^2} \quad \text{for } 0 < \rho < 1, \tag{1.6}$$

$$H'(0) = 0, \tag{1.7}$$

$$H(1) = 0. \tag{1.8}$$

Note that X, Y, Z are all positive constants.

Throughout this paper we will also assume:

$$H(0) = 1. \tag{1.9}$$

Note that

$$H(\rho) = 1 - \rho^2 \tag{1.10}$$

is a solution of (1.6)-(1.9) when $Z = 2XY$. A natural question is whether there are other solutions of (1.6)-(1.9). In an earlier paper [7] we showed that (1.10) is the only solution of (1.6)-(1.9) which is differentiable on all of $[0,1]$.

If we weaken the assumption and only look for solutions $H \in C^3[0,1) \cap C[0,1]$, then in an earlier paper [6] we showed that if $0 < Z < 2XY$ then there is a solution of (1.6)-(1.9) with $H \in C^3[0,1) \cap C[0,1]$ such that $\lim_{\rho\to 1^-} H'(\rho) = -\infty$. More specifically we showed that: if $H \in C^3[0,1) \cap C[0,1]$ is a solution of (1.6)-(1.9) which is not differentiable at $x = 1$ then

$$\lim_{\rho\to 1^-} \frac{H'(\rho)}{\ln(1-\rho^2)} = \frac{XY}{4} = \lim_{\rho\to 1^-} -\frac{1}{4}(1-\rho^2)H''(\rho) = \lim_{\rho\to 1^-} -\frac{1}{8}(1-\rho^2)^2 H'''(\rho).$$

In this paper we examine the case when $Z > 2XY$.

In attempting to solve (1.6)-(1.9), we first thought of using the *shooting* method. That is, we would solve (1.6) with

$$H(0) = 1, \tag{1.11}$$

$$H'(0) = 0, \tag{1.12}$$

$$H''(0) = k \tag{1.13}$$

where k is arbitrary and show that if k is sufficiently large then $H > 0$ on $[0,1)$ and if k is sufficiently small then H must have a zero on $[0,1)$. Then making an appropriate choice for k we could show that $H(1) = 0$. Therefore we conjectured that there would be at least one value of k such that H is a solution. However, we discovered that this method does not quite work for this problem. In fact, in [6] we proved the following result.

Theorem 1.1. *Let $H \in C^3(\rho_0,1) \cap C[\rho_0,1)$ be a solution of (1.6) such that $0 \le \rho_0 < 1$ and $H(\rho_0) > 0$. Then $H > 0$ on $(\rho_0,1)$.*

We were able to eventually show that if we look at a slightly different differential equation then it is possible to solve this new problem by the shooting method. The key turned out to be to look at the function:

$$W = H - \sqrt{\frac{Z}{2XY}}(1-\rho^2). \tag{1.14}$$

Using (1.6) it is straightforward to see that

$$\Big(W'' + \frac{W'}{\rho}\Big)' = \Big(H'' + \frac{H'}{\rho}\Big)' = \frac{-2XY\rho}{(1-\rho^2)^2}\,\frac{W\big(H + \sqrt{\frac{Z}{2XY}}(1-\rho^2)\big)}{H^2} \tag{1.15}$$

for $0 < \rho < 1$. The initial conditions for W are related to (1.11)-(1.13) by (1.14):

$$W(0) = 1 - \sqrt{\frac{Z}{2XY}}, \tag{1.16}$$

$$W'(0) = 0, \tag{1.17}$$

$$W''(0) = k + \sqrt{\frac{2Z}{XY}}. \tag{1.18}$$

In [6] we proved the following theorem.

Theorem 1.2. *For each $0 < Z < 2XY$ there is a positive $C^3(0,1)\cap C^1[0,1)\cap C[0,1]$ solution of (1.15) with $W'(0) = 0$, $W(1) = 0$, and $W'(1) = -\infty$. (And thus H solves (1.6) with $H'(0) = 0$, $H(1) = 0$, and $H'(1) = -\infty$). If $Z = 2XY$ then $W \equiv 0$ is a solution of (1.15). (And thus $H = \sqrt{\frac{Z}{2XY}}(1-\rho^2)$ solves (1.6) with $H'(0) = 0$, $H(1) = 0$, and $H'(1) = -\sqrt{\frac{2Z}{XY}}$). Thus we see that there is a solution of (1.6)-(1.9) for $0 < Z \le 2XY$.*

In this paper we prove the following result.

Theorem 1.3. *Let $Z > 2XY$. Then there exist real numbers k^- and k^+ with $k^- \le k^+$such that there are no $C^3(0,1) \cap C^1[0,1) \cap C[0,1]$ solutions of* (1.15) *with* (1.16)-(1.18) *and $W(1) = 0$ if $k > k^+$ or if $k < k^-$. (Thus there are no $C^3(0,1) \cap C^1[0,1) \cap C[0,1]$ solutions of* (1.6) *with $H(0) = 1$, $H'(0) = 0$, $H''(0) = k$, and $H(1) = 0$ if $k > k^+$ or if $k < k^-$).*

In the proof of the Main Theorem (Theorem 1.3) we show that $H(1,k) \to \infty$ as $k \to \infty$ and $k \to -\infty$. We also know from Theorem 1.1 that $H(1,k) \ge 0$ for all $k \in \mathbb{R}$. We then define k_0 to be a value of k so that $H(1,k_0) \le H(1,k)$ for all $k \in \mathbb{R}$. We attempted to prove that either $H(1,k_0) = 0$ or $H(1,k_0) > 0$ but we were not able to prove either of these. However, numerics in [2] strongly suggest that $H(1,k_0) > 0$ and so we conjecture that there are no solutions of (1.6) with $H(0) = 1$, $H'(0) = 0$, and $H(1) = 0$ for $Z > 2XY$.

2. Preliminaries

Rewriting (1.15) we see that

$$W''' + \frac{W''}{\rho} - \frac{W'}{\rho^2} + \frac{2XY\rho}{(1-\rho^2)^2} \frac{W\big(H + \sqrt{\frac{Z}{2XY}}(1-\rho^2)\big)}{H^2} = 0. \tag{2.1}$$

Now we note the following:

Lemma 2.1. *W' does not have a positive local maximum on the set where $W \le 0$ and $0 < \rho < 1$.*

Proof. If there were such a point, p, then at this point we would have $W(p) \le 0$, $W'(p) > 0$, and since W' has a local maximum at p, then from calculus it follows that $W''(p) = 0$ and $W'''(p) \le 0$. This however contradicts (2.1) and Theorem 1.1. □

Lemma 2.2. *W' does not have a negative local minimum on the set where $W \ge 0$ and $0 < \rho < 1$.*

Proof. If there were such a point, p, then at this point we would have $W(p) \ge 0$, $W'(p) < 0$, and since W' has a local minimum at p, then from calculus it follows that $W''(p) = 0$ and $W'''(p) \ge 0$. This again contradicts (2.1) and Theorem 1.1. □

Lemma 2.3. *Suppose $W \in C^3(0,1) \cap C^1[0,1) \cap C[0,1]$ satisfies* (2.1) *with $W(0) < 0$ and $W'(0) = 0$. Then W must get positive somewhere on $(0,1)$.*

Proof. Suppose by way of contradiction that there is a solution, W, with $W \le 0$ on $(0,1)$. Integrating (2.1) on (ρ_0, ρ) where $0 < \rho_0 < \rho < 1$ gives for some constant C_0:

$$W'' + \frac{W'}{\rho} = C_0 - \int_{\rho_0}^{\rho} \frac{2XYtW\big(H + \sqrt{\frac{Z}{2XY}}(1-t^2)\big)}{(1-t^2)^2 H^2}\, dt.$$

Multiplying by ρ and integrating on (ρ_0, ρ) gives

$$\rho W' = \rho_0 W'(\rho_0) + \frac{C_0}{2}(\rho^2 - \rho_0^2) + \int_{\rho_0}^{\rho} t \int_{\rho_0}^{t} \frac{(-2XY)sW\big(H + \sqrt{\frac{Z}{2XY}}(1-s^2)\big)}{(1-s^2)^2 H^2}\, ds\, dt. \tag{2.2}$$

Assuming $W \le 0$ on $(0,1)$ and Theorem 1.1 then the integrand on the right-hand side of (2.2) is nonnegative, and so the integral term is an increasing function. Thus it follows that

$$\lim_{\rho\to 1^-} W'(\rho) \text{ exists (and is possibly } +\infty). \tag{2.3}$$

Since we are also assuming $W \le 0$ on $(0,1)$, it follows by continuity that $W(1) \le 0$. On the other hand, we also know from (1.14) and Theorem 1.1 that $W(1) = H(1) \ge 0$. Thus $W(1) = 0$. Now we know from section 3 in [6] that if the limit in (2.3) is finite then in fact the limit must be zero. Thus, it follows from (2.3) that either: $\lim_{\rho\to 1^-} W'(\rho) = 0$ or $\lim_{\rho\to 1^-} W'(\rho) = +\infty$. Suppose first that $\lim_{\rho\to 1^-} W'(\rho) = 0$. We also know that since $W \le 0$ and $W(0) < 0$ then W has a local and absolute minimum, m, with $0 \le m < 1$ such that $W(m) < 0$ and $W'(m) = 0$. By the mean value theorem, $0 < W(1) - W(m) = W'(c)(1-m)$ for some $c \in (m,1)$ and so we see that W' must get positive somewhere on $(m,1)$. Since $W'(m) = 0 = \lim_{\rho\to 1^-} W'(\rho) = W'(1)$ we see that W' has a positive local maximum on $(m,1)$ with $W \le 0$ contradicting Lemma 2.1. Therefore, the assumption that $\lim_{\rho\to 1^-} W'(\rho) = 0$ must be false. Thus it must be the case that $\lim_{\rho\to 1^-} W'(\rho) = \infty$. Since $W(1) = 0$ then it follows from L'Hopital's rule that

$$\lim_{\rho\to 1^-} \frac{1-\rho}{W(\rho)} = 0. \tag{2.4}$$

Now rewriting (1.15) we see that

$$-\Big(W'' + \frac{W'}{\rho}\Big)' = \frac{2XY\rho}{(1-\rho^2)^2}\Bigg(\frac{1 + 2\sqrt{\frac{Z}{2XY}}\,\frac{1-\rho^2}{W}}{(1+\sqrt{\frac{Z}{2XY}}\,\frac{1-\rho^2}{W})^2}\Bigg). \tag{2.5}$$

By (2.4) the term in parentheses on the right-hand side of (2.5) goes to 1 as $\rho \to 1^-$ and hence is larger than $1/2$ for ρ close to 1. Thus, for ρ close to 1 with $\rho < 1$ we have:

$$-\Big(W'' + \frac{W'}{\rho}\Big)' \ge \frac{XY\rho}{(1-\rho^2)^2}.$$

Integrating this on (ρ_0, ρ) gives for some constant C_1,

$$-\big(W'' + \frac{W'}{\rho}\big) \ge C_1 + \frac{XY}{1-\rho^2}.$$

Multiplying by ρ and integrating on (ρ_0, ρ) gives for some constant C_2,

$$\rho W' \le C_2 - \frac{C_1}{2}\rho^2 + \frac{XY}{2}\ln(1-\rho^2). \tag{2.6}$$

This implies $W' \to -\infty$ as $\rho \to 1^-$ contradicting that $W' \to \infty$. This completes the proof. $\square$

We next show that if k is chosen sufficiently large and $Z > 2XY$ then W has a first positive zero, z, and $W > 0$ on $(z,1]$. This then proves the existence of the number k^+ referred to in Theorem 1.3.

We begin by integrating (1.6) on $(0,\rho)$ and using (1.12)-(1.13) gives

$$H'' + \frac{H'}{\rho} + \frac{XY}{1-\rho^2} = 2k + XY + \int_0^\rho \frac{Zt}{H^2}\,dt.$$

Multiplying by ρ, integrating on $(0,\rho)$, and simplifying gives

$$H' = \frac{XY}{2}\frac{\ln(1-\rho^2)}{\rho} + \big(k + \frac{XY}{2}\big)\rho + \frac{1}{\rho}\int_0^\rho s\int_0^s \frac{Zt}{H^2}\,dt\,ds.$$

Thus by (1.14), we obtain

$$W' = \frac{XY}{2}\frac{\ln(1-\rho^2)}{\rho} + \Big(k + \frac{XY}{2} + \sqrt{\frac{2Z}{XY}}\Big)\rho + \frac{1}{\rho}\int_0^\rho s\int_0^s \frac{Zt}{H^2}\,dt\,ds. \tag{2.7}$$

Thus for k sufficiently large we see by (1.18) and (2.7) that W is increasing on say $[0, 1-\epsilon]$ and since W is bounded below by $-\sqrt{\frac{Z}{2XY}}$ (from Theorem 1.1) we see that there is a z with $0 < z < 1$ such that $W(z) = 0$ and W is increasing on $[0, 1-\epsilon]$ for $\epsilon > 0$ Integrating (2.7) again on $(0,\rho)$ and using (1.11) gives

$$H = 1 + \frac{XY}{2}\int_0^\rho \frac{\ln(1-t^2)}{t}\,dt + \big(\frac{k}{2} + \frac{XY}{4}\big)\rho^2 + \int_0^\rho \frac{1}{x}\int_0^x t\int_0^t \frac{Zs}{H^2}\,ds\,dt\,dx. \tag{2.8}$$

Thus by (1.14) we see that

$$W \ge \big(1 - \sqrt{\frac{Z}{2XY}}\big) + \frac{XY}{2}\int_0^\rho \frac{\ln(1-t^2)}{t}\,dt + \Big(\frac{k}{2} + \frac{XY}{4} + \sqrt{\frac{Z}{2XY}}\Big)\rho^2. \tag{2.9}$$

We note by L'Hopital's rule that

$$\lim_{\rho\to 0^+} \frac{\frac{XY}{2}\int_0^\rho \frac{\ln(1-t^2)}{t}\,dt + (\frac{k}{2} + \frac{XY}{4} + \sqrt{\frac{Z}{2XY}})\rho^2}{\rho^2} = \frac{k}{2} + \sqrt{\frac{Z}{2XY}}.$$

Also, $\frac{\ln(1-t^2)}{t}$ is integrable at $t = 1$ so we see that it follows from (2.9) that W remains positive on all of $[z, 1]$ provided k is chosen large enough.

Thus we see that if k is sufficiently large then W has exactly one zero on $[0, 1]$. Therefore there exists $k^+ > 0$ such that if $k > k^+$ then there are no $C^3(0,1) \cap C^1[0,1) \cap C[0,1]$ solutions of (1.15) with (1.16)-(1.18) and $W(1) = 0$. The rest of the paper is devoted to proving the existence of k^-.

3. Proofs

We now write W_k instead of W to emphasize the dependence of W on k. Throughout this section we assume:

$$W_k \in C^3(0,1) \cap C^1[0,1) \cap C[0,1] \text{ solves (1.15)}, \tag{3.1}$$

$$-\sqrt{\frac{Z}{2XY}} < W_k(0) < 0, \quad \text{and} \quad W_k'(0) = 0. \tag{3.2}$$

Lemma 3.1. *Suppose W_k satisfies (3.1)-(3.2) and $k < -\sqrt{\frac{2Z}{XY}}$. Then there exist points $p_{1,k}$, m_k, z_k, $p_{2,k}$, M_k with $0 < p_{1,k} < m_k < z_k < p_{2,k} < M_k < 1$ such that W_k has a local minimum at m_k, a zero at z_k, a local maximum at M_k, and inflection points at $p_{1,k}$ and $p_{2,k}$. Furthermore, W_k has no other local extrema on $(0, M_k)$ and W_k has no inflection points on (m_k, z_k).*

Proof. By assumption $W_k(0) < 0$ and W_k is continuous so W_k has a zero on $(0,1)$ by Lemma 2.3. Thus there exists a z_k with $0 < z_k < 1$ such that $W_k < 0$ on $[0, z_k)$ and $W_k(z_k) = 0$. Also, for $k < -\sqrt{\frac{2Z}{XY}}$ we see from (1.18) that $W_k' < 0$ for small positive ρ, and so for such k there is an m_k with $0 < m_k < z_k$ such that $W_k' < 0$ on

$(0, m_k)$, $W_k'(m_k) = 0$, and $W_k(m_k) < 0$. It follows from calculus that $W_k''(m_k) \geq 0$. In fact, $W_k''(m_k) > 0$ for if $W_k''(m_k) = 0$ we would have $W_k'(m_k) = W_k''(m_k) = 0$ and from (1.15) it follows that $W_k'''(m_k) > 0$. Thus W_k'' would be increasing in a neighborhood of m_k and since $W_k''(m_k) = 0$, then $W_k'' < 0$ on $(m_k - \delta, m_k)$ for some $\delta > 0$ which implies W_k' is decreasing on $(m_k - \delta, m_k)$, and since $W_k'(m_k) = 0$, it follows that $W_k' > 0$ on $(m_k - \delta, m_k)$ contradicting that $W_k' < 0$ on $(0, m_k)$. Thus $W_k''(m_k) > 0$ and therefore m_k is a local minimum of W_k.

Also since $W_k''(0) < 0$ and $W_k''(m_k) > 0$ it follows that there must be an inflection point, $p_{1,k}$, with $0 < p_{1,k} < m_k$ such that $W_k'' < 0$ on $(0, p_{1,k})$ and $W_k'' \geq 0$ on $(p_{1,k}, p_{1,k} + \delta_2)$ for some $\delta_2 > 0$.

Next, we observe that $W_k'' > 0$ on (m_k, z_k) for if there were an r_k with $m_k < r_k < z_k$ with $W_k'' > 0$ on (m_k, r_k) and $W_k''(r_k) = 0$ then from (2.1) we see that since $W_k(r_k) < 0$ and $W_k'(r_k) > 0$ then $W_k'''(r_k) > 0$. Thus, W_k'' is increasing in a neighborhood of r_k and since $W_k''(r_k) = 0$ then this would imply $W_k'' < 0$ on $(r_k - \delta_3, r_k)$ for some $\delta_3 > 0$ which contradicts that $W_k'' > 0$ on (m_k, r_k). Thus $W_k'' > 0$ on (m_k, z_k) and since $W_k'(m_k) = 0$ it follows that $W_k'(z_k) > 0$. Thus there is a $\delta_4 > 0$ such that $W_k > 0$ on $(z_k, z_k + \delta_4)$.

Now either W_k has a second zero, $z_{2,k}$, on $(z_k, 1)$ or $W_k > 0$ on $(z_k, 1)$.

Case 1: If W_k has another zero, $z_{2,k}$, on $(z_k, 1)$ then there exists M_k with $W_k' > 0$ on $[z_k, M_k)$, $W_k'(M_k) = 0$, and $W_k(M_k) > 0$. This implies $W_k''(M_k) \leq 0$. Now since $W_k'(m_k) = 0$, $W_k''(m_k) > 0$ (shown earlier in this proof), and $W_k'(M_k) = 0$ it follows that W_k' has a positive local maximum, $p_{2,k}$, on (m_k, M_k) with $W_k'' > 0$ on $(m_k, p_{2,k})$ and $m_k < p_{2,k} < M_k$. From Lemma 2.1 it follows that $W_k(p_{2,k}) > 0$ and so $z_k < p_{2,k}$. We also see that $m_k < z_k < p_{2,k} < M_k$. In addition, $W_k''(M_k) < 0$ for if $W_k''(M_k) = 0$ then since $W_k'(M_k) = 0$ it follows from (1.6) that $W_k'''(M_k) < 0$ and so W_k'' is decreasing in a neighborhood of M_k. But since $W_k'' < 0$ on $(p_{2,k}, M_k)$ and W_k'' is decreasing in a neighborhood of M_k then W_k'' could not be zero at M_k. Therefore we see that $W_k''(M_k) < 0$ and so M_k is a local maximum for W_k. This completes the proof of the lemma for Case 1.

Case 2: If $W_k > 0$ on $(z_k, 1]$ then there is a constant $c_0 > 0$ such that

$$\Big(W_k'' + \frac{W_k'}{\rho}\Big)' + \frac{c_0 \rho}{(1-\rho^2)^2} \leq 0$$

near $\rho = 1$. Integrating this on (ρ_0, ρ) gives

$$W_k'' + \frac{W_k'}{\rho} + \frac{c_0}{2(1-\rho^2)} \leq W_k''(\rho_0) + \frac{W_k'(\rho_0)}{\rho_0} + \frac{c_0}{2(1-\rho_0^2)} \equiv b_k \tag{3.3}$$

and thus we see that $W_k'' + \frac{W_k'}{\rho}$ must get negative as $\rho \to 1^-$ since the right-hand side of 3.3 is fixed. Multiplying (3.3) by ρ and integrating on (ρ_0, ρ) gives

$$\rho W_k' \leq \rho_0 W_k'(\rho_0) + b_k(\rho - \rho_0) + \frac{c_0}{4} \ln\Big(\frac{1-\rho^2}{1-\rho_0^2}\Big). \tag{3.4}$$

Thus we see that W_k' becomes negative as $\rho \to 1^-$.

Therefore, we see that there is an M_k with $m_k < z_k < M_k < 1$ such that $W_k' > 0$ on (m_k, M_k), $W_k(M_k) > 0$, and $W_k'(M_k) = 0$. As in the proof of Case 1, it is possible to show that M_k is a local maximum, $W_k''(M_k) < 0$, and there is an inflection point $p_{2,k}$ with $m_k < z_k < p_{2,k} < M_k$ with $W_k(p_{2,k}) > 0$. This completes the proof of the lemma for Case 2. $\square$

Lemma 3.2. *Suppose W_k satisfies* (3.1)-(3.2). *Then $H_k(m_k) \to 0$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. Suppose on the contrary that there exists an $l_0 > 0$ such that $H_k(m_k) \ge l_0 > 0$ for all k sufficiently negative. Then on $[0, m_k]$ we have $0 \ge W_k' = H_k' + \sqrt{\frac{2Z}{XY}}\rho \ge H_k'$. Therefore, H_k is decreasing on $[0, m_k]$ and hence $H_k \ge H_k(m_k) \ge l_0$ on $[0, m_k]$.

Then $\frac{1}{H_k^2} \le \frac{1}{l_0^2}$ on $[0, m_k] \subset [0, 1]$ so we see by integrating (1.15) on $(0, \rho)$, using (1.17)-(1.18), and assuming k is sufficiently negative

$$\begin{aligned} W_k'' + \frac{W_k'}{\rho} + \frac{XY}{1-\rho^2} &= 2\Big(k + \sqrt{\frac{2Z}{XY}}\Big) + XY + \int_0^\rho \frac{Zt}{H_k^2}\,dt \\ &\le 2\Big(k + \sqrt{\frac{2Z}{XY}}\Big) + XY + \frac{Z}{2l_0^2}\rho^2 < 0. \end{aligned} \tag{3.5}$$

Evaluating (3.5) at m_k gives

$$0 < W_k''(m_k) + \frac{XY}{1-m_k^2} \le 2\Big(k + \sqrt{\frac{2Z}{XY}}\Big) + XY + \frac{Z}{2l_0^2}m_k^2 < 0$$

which is a contradiction to (3.5). Thus, the lemma holds. □

Lemma 3.3. *Suppose W_k satisfies* (3.1)-(3.2). *Then $p_{1,k} \to 0$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. Since $W_k'' = H_k'' + \sqrt{\frac{2Z}{XY}}$ we see that $H_k'' \le 0$ when $W_k'' \le 0$. Also, by Lemma 3.1 we note that H_k is concave down on $[0, p_{1,k}]$ and so on this interval the graph of H_k lies above the line through $(0, 1)$ and $(p_{1,k}, H_k(p_{1,k}))$. That is

$$H_k(\rho) \ge 1 - \frac{1 - H_k(p_{1,k})}{p_{1,k}}\rho \text{ on } [0, p_{1,k}).$$

Thus

$$H_k(\rho) \ge 1 - A_k\rho > 0 \quad \text{on } [0, p_{1,k}),$$

where

$$A_k = \frac{1 - H_k(p_{1,k})}{p_{1,k}}. \tag{3.6}$$

Thus

$$\frac{1}{H_k^2} \le \frac{1}{(1 - A_k\rho)^2} \quad \text{on } [0, p_{1,k}). \tag{3.7}$$

And so integrating (1.6) on $(0, \rho) \subset (0, 1)$ and using (3.7) we see

$$\begin{aligned} H_k'' + \frac{H_k'}{\rho} + \frac{XY}{1-\rho^2} &= 2k + XY + \int_0^\rho \frac{Zt}{H_k^2}\,dt \\ &\le 2k + XY + \int_0^\rho \frac{Zt}{(1 - A_k t)^2}\,dt \\ &\le 2k + XY + \frac{\rho Z}{(1 - A_k\rho)}. \end{aligned}$$

Since $\frac{XY}{1-\rho^2} \ge XY$ on $[0, 1]$ this reduces to

$$H_k'' + \frac{H_k'}{\rho} \le 2k + \frac{\rho Z}{d(d - A_k\rho)} \quad \text{on } [0, p_{1,k}).$$

Multiplying by ρ, integrating on $(0,\rho)$, and simplifying gives

$$H_k' \le k\rho - \frac{Z}{A_k\rho}\ln(1-A_k\rho).$$

Integrating on $(0,p_{1,k})$ we obtain

$$0 \le H_k(p_{1,k}) \le 1 + \frac{k}{2}p_{1,k}^2 - \frac{Z}{A_k}\int_0^{p_{1,k}} \frac{\ln(1-A_k t)}{t}\,dt. \tag{3.8}$$

Making the change of variables $u = A_k t$ and using (3.6)-(3.8) we obtain

$$\frac{|k|}{2}p_{1,k}^2 \le 1 + \frac{Zp_{1,k}}{d^2}\frac{1}{\big(1-H_k(p_{1,k})\big)}\int_0^{(1-H_k(p_{1,k}))}\Big(-\frac{\ln(1-u)}{u}\Big)\,du. \tag{3.9}$$

Now let

$$g(x) = \frac{1}{x}\int_0^x \Big(-\frac{\ln(1-u)}{u}\Big)\,du.$$

It follows by using the power series for $\ln(1-u)$ that

$$g(x) = \frac{1}{x}\int_0^x \Big(-\frac{\ln(1-u)}{u}\Big)\,du = 1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \cdots$$

which converges for $0 \le x \le 1$. In addition, for these x we have $1 \le g(x) \le g(1) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} < 2$.

Now notice that since $p_{1,k} \le 1$ then (3.9) can be rewritten as

$$\frac{|k|}{2}p_{1,k}^2 \le 1 + Zp_{1,k}g\left(1-H_k(p_{1,k})\right) \le 1 + Zg\left(1-H_k(p_{1,k})\right).$$

Thus

$$\frac{|k|}{2}p_{1,k}^2 \le 1 + 2Z$$

and so we see that $p_{1,k} \to 0$ as $k \to -\infty$. □

Lemma 3.4. *Suppose W_k satisfies* (3.1)-(3.2). *Then $H_k(p_{1,k}) \to 0$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. Suppose not and so assume there is an $s_0 > 0$ such that $H_k(p_{1,k}) \ge s_0$. Now recall that W_k' has a local minimum at $p_{1,k}$. Therefore from calculus we have $W_k''(p_{1,k}) = 0$ and $W_k'''(p_{1,k}) \ge 0$. Using (1.6) and (1.15) we see that

$$0 \le -\frac{W_k'(p_{1,k})}{p_{1,k}^2} \le \frac{Zp_{1,k}}{H_k^2(p_{1,k})}.$$

Thus, by Lemma 3.3,

$$0 \le -\frac{W_k'(p_{1,k})}{p_{1,k}} \le \frac{Zp_{1,k}^2}{H_k^2(p_{1,k})} \le \frac{Zp_{1,k}^2}{s_0^2} \to 0 \quad \text{as } k \to -\infty. \tag{3.10}$$

On the other hand, since $W_k' = H_k' + \sqrt{\frac{2Z}{XY}}\rho$ and W_k is decreasing on $[0,p_{1,k}]$, it follows that H_k is also decreasing on $[0,p_{1,k}]$ and therefore $H \ge s_0$ on $[0,p_{1,k})$. Using this fact and integrating (1.15) on $(0,\rho)$ gives

$$W_k'' + \frac{W_k'}{\rho} + \frac{XY}{1-\rho^2} = 2k + XY + \int_0^\rho \frac{Zt}{H_k^2}\,dt \le 2k + XY + \frac{Z}{2s_0^2}\rho^2 \text{ on } [0,p_{1,k}].$$

Further since $\frac{XY}{1-\rho^2} \geq XY$ on $[0,1]$ we obtain

$$W_k'' + \frac{W_k'}{\rho} \leq 2k + \frac{Z}{2s_0^2}\rho^2 \quad \text{on } [0, p_{1,k}].$$

Multiplying by ρ and integrating on $(0,\rho)$ gives:

$$W_k' \leq k\rho + \frac{Z}{8s_0^2}\rho^4$$

and so

$$\frac{W_k'(p_{1,k})}{p_{1,k}} \leq k + \frac{Z}{8s_0^2}p_{1,k}^3 \leq k + \frac{Z}{8s_0^2} \to -\infty \quad \text{as } k \to -\infty$$

contradicting (3.10). Thus $H(p_{1,k}) \to 0$ as $k \to -\infty$. $\square$

Lemma 3.5. *Suppose W_k satisfies* (3.1)–(3.2). *Then $m_k \to 0$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. Suppose not. So then there is a t_0 such that $m_k \geq t_0 > 0$. Using Theorem 1.1 and the fact that W_k is decreasing on $[0, m_k]$ we obtain

$$\begin{aligned} H_k(p_{1,k}) - \sqrt{\frac{Z}{2Y}}(1 - p_{1,k}^2) &= W_k(p_{1,k}) \geq W_k(m_k) \\ &= H_k(m_k) - \sqrt{\frac{Z}{2XY}}(1 - m_k^2) \\ &\geq -\sqrt{\frac{Z}{2XY}}(1 - t_0^2). \end{aligned} \tag{3.11}$$

By Lemmas 3.3 and 3.4 we see that the left-hand side of (3.11) goes to $-\sqrt{\frac{Z}{2XY}}$ as $k \to -\infty$ and so this implies $t_0 = 0$ which is a contradiction. $\square$

Now let $0 < \epsilon \leq 1$ and observe that $\frac{W_k'}{\rho^{1-\epsilon}}$ is zero at m_k, M_k, and is positive on (m_k, M_k). Therefore $\frac{W_k'}{\rho^{1-\epsilon}}$ has an absolute and local maximum on (m_k, M_k).

Lemma 3.6. *Let $0 < \epsilon \leq 1$. Suppose W_k satisfies* (3.1)-(3.2). *At an absolute maximum, q_k, of $\frac{W_k'}{\rho^{1-\epsilon}}$ on (m_k, M_k) we have $q_k \leq p_{2,k}$ and*

$$\frac{W_k'(q_k)}{q_k^{1-\epsilon}} \leq \frac{2XYq_k^{2+\epsilon}}{\epsilon(2-\epsilon)(1-q_k^2)^2}.$$

Proof. At q_k we have

$$\Big(\frac{W_k'}{\rho^{1-\epsilon}}\Big)' = 0 \quad \text{and} \quad \Big(\frac{W_k'}{\rho^{1-\epsilon}}\Big)'' \leq 0.$$

Thus, $q_kW''(q_k) - (1-\epsilon)W_k'(q_k) = 0$ and $q_kW'''(q_k) + \epsilon W_k''(q_k) \leq 0$. Therefore by (1.6),

$$\frac{W_k'(q_k)}{q_k^{1-\epsilon}} \leq \frac{2XYq_k^{2+\epsilon}}{\epsilon(2-\epsilon)(1-q_k^2)^2}.$$

In addition, since q_k is the maximum of $\frac{W_k'}{\rho^{1-\epsilon}}$ and $m_k < p_{2,k} < M_k$ then

$$\frac{W_k'(p_{2,k})}{p_{2,k}^{1-\epsilon}} \leq \frac{W_k'(q_k)}{q_k^{1-\epsilon}}.$$

Also, we know that $p_{2,k}$ is the maximum of W_k' on (m_k, M_k) so:

$$\frac{W_k'(q_k)}{q_k^{1-\epsilon}} \le \frac{W_k'(p_{2,k})}{q_k^{1-\epsilon}}.$$

Since $W_k' > 0$ at $p_{2,k}$ and q_k, and also that $0 < \epsilon \le 1$, it then follows from the two previous inequalities that

$$q_k \le p_{2,k}.$$

This completes the proof. $\square$

Lemma 3.7. *Suppose W_k satisfies* (3.1)-(3.2). *Then $p_{2,k} \to 1$ as $k \to -\infty$ (for some subsequence of k's). (And hence $M_k \to 1$ since $p_{2,k} \le M_k$ (for some subsequence of k's)).*

Proof. We first show that $p_{2,k} \not\to 0$. If $p_{2,k} \to 0$ then using Lemma 3.6 with $\epsilon = 1$ we see that

$$0 \le W_k'(p_{2,k}) \le \frac{2XYp_{2,k}^3}{(1-p_{2,k}^2)^2} \to 0 \quad \text{as } k \to -\infty. \tag{3.12}$$

On the other hand, by the mean value theorem we have

$$-W_k(m_k) = W_k(z_k) - W_k(m_k) = W_k'(c_k)(z_k - m_k) \tag{3.13}$$

for some c_k with $m_k < c_k < z_k$. Also, since $p_{2,k}$ is a local maximum for W_k' on (m_k, M_k), it follows that $0 \le W_k'(c_k) \le W_k'(p_{2,k})$. Substituting this into (3.13), using (3.12), and that $0 \le m_k \le z_k \le 1$ gives

$$\begin{aligned}
-W_k(m_k) &= W_k(z_k) - W_k(m_k) \\
&= W_k'(c_k)(z_k - m_k) \\
&\le W_k'(p_{2,k})(z_k - m_k) \\
&\le \frac{2XYp_{2,k}^3}{(1-p_{2,k}^2)^2}.
\end{aligned} \tag{3.14}$$

Next by Lemma 3.2 and (1.14) it follows that $W_k(m_k) \to -\sqrt{\frac{Z}{2XY}}$ as $k \to -\infty$. Therefore, the left-hand side of (3.14) goes to $\sqrt{\frac{Z}{2XY}} > 0$ as $k \to -\infty$ but the right-hand side goes to zero as $k \to -\infty$ by (3.12) which is a contradiction. Thus we see that $p_{2,k} \not\to 0$.

So now suppose that the lemma is not true and that there is a u_0 and v_0 with $0 < u_0 < v_0 < 1$ such that $0 < u_0 < p_{2,k} \le v_0 < 1$. Then we have the following identity which follows from (1.6),

$$\Big(H_k^2\big(H_k'' + \frac{H_k'}{\rho}\big) - H_kH_k'^2\Big)' = Z\rho - \frac{2XY\rho H_k^2}{(1-\rho^2)^2} + \frac{2H_kH_k'^2}{\rho} - H_k'^3.$$

Integrating this on (m_k, ρ), using Lemma 3.1, and that $H_k > 0$, $W_k' > 0$ on (m_k, M_k), as well as $H_k'(m_k) + \sqrt{\frac{2Z}{XY}}m_k = W_k'(m_k) = 0$, $H_k''(m_k) + \sqrt{\frac{2Z}{XY}} = W_k''(m_k) > 0$, and $H_kH_k'^2 \ge 0$ gives

$$\begin{aligned}
H_k^2\left(H_k'' + \frac{H_k'}{\rho}\right) &\ge -2\Big(\sqrt{\frac{2Z}{XY}}H_k^2(m_k) + \frac{Z}{XY}m_k^2H_k(m_k)\Big) \\
&\quad + \frac{Z}{2}(\rho^2 - m_k^2) - \int_{m_k}^{\rho} \frac{2XYtH_k^2}{(1-t^2)^2}\,dt - \int_{m_k}^{\rho} {H_k'}^3\,dt.
\end{aligned}$$

Multiplying by ρ and integrating by parts on (m_k,ρ) gives

$$\begin{aligned}&\rho H_k^2 H_k' - \int_{m_k}^{\rho} 2tH_k H_k'^2\,dt\\ &\geq -\Big(\sqrt{\frac{2Z}{XY}}H_k^2(m_k) + \frac{Z}{XY}m_k^2 H_k(m_k)\Big)(\rho^2-m_k^2) + \int_{m_k}^{\rho}\frac{Z}{2}t(t^2-m_k^2)\,dt\\ &\quad - \int_{m_k}^{\rho} t\int_{m_k}^{t}\frac{2XYsH_k^2}{(1-s^2)^2}\,ds\,dt - \int_{m_k}^{\rho} t\int_{m_k}^{t} H_k'^3\,ds\,dt.\end{aligned}$$

Thus, since $H_k>0$ and $H_kH_k'^2\geq 0$,

$$\begin{aligned}&\rho H_k^2 H_k' + \int_{m_k}^{\rho} t\int_{m_k}^{t}\frac{2XYsH_k^2}{(1-s^2)^2}\,ds\,dt + \int_{m_k}^{\rho} t\int_{m_k}^{t} H_k'^3\,ds\,dt\\ &\geq -\Big(\sqrt{\frac{2Z}{XY}}H_k^2(m_k) + \frac{Z}{XY}m_k^2 H_k(m_k)\Big)(\rho^2-m_k^2)\\ &\quad + \frac{Z}{2}\Big(\frac{(\rho^4-m_k^4)}{4} - \frac{m_k^2(\rho^2-m_k^2)}{2}\Big).\end{aligned}\tag{3.15}$$

Since $p_{2,k}\leq v_0<1$, it follows from Lemma 3.6 and (1.14) that for fixed ϵ with $0<\epsilon<1$,

$$H_k' \leq W_k' \leq C_4\rho^{1-\epsilon}\quad \text{on } [m_k,p_{2,k}] \text{ where } C_4 = \frac{2XYv_0^{2+\epsilon}}{\epsilon(2-\epsilon)(1-v_0^2)^2}.\tag{3.16}$$

Substituting (3.16) into (3.15) gives

$$\begin{aligned}&C_4\rho^{1-\epsilon}H_k^2 + \int_{m_k}^{\rho} t\int_{m_k}^{t}\frac{2XYsH_k^2}{(1-s^2)^2}\,ds\,dt + C_4^3\int_{m_k}^{\rho} t\int_{m_k}^{t} s^{3-3\epsilon}\,ds\,dt\\ &\geq -\Big(\sqrt{\frac{2Z}{XY}}H_k^2(m_k) + \frac{Z}{XY}m_k^2 H_k(m_k)\Big)(\rho^2-m_k^2)\\ &\quad + \frac{Z}{2}\Big(\frac{(\rho^4-m_k^4)}{4} - \frac{m_k^2(\rho^2-m_k^2)}{2}\Big).\end{aligned}\tag{3.17}$$

Also, integrating (3.16) on (m_k,ρ) and using (1.14) as well as Theorem 1.1 gives

$$-\sqrt{\frac{Z}{2XY}}(1-\rho^2) \leq W_k(\rho) \leq W_k(m_k) + \frac{C_4}{2-\epsilon}(\rho^{2-\epsilon}-m_k^{2-\epsilon}) \leq \frac{C_4}{2-\epsilon}(\rho^{2-\epsilon}-m_k^{2-\epsilon}).\tag{3.18}$$

Now we know from Lemma 3.1 that $W_k'\geq 0$ on $[m_k,M_k]$ and so it follows from (3.16)-(3.18) and Lemma 3.6 that W_k and W_k' are uniformly bounded on $[m_k,M_k]$. So since m_k and $p_{2,k}$ are bounded there exists a subsequence (still labeled k) such that $m_k\to m=0$ (by Lemma 3.5) and $p_{2,k}\to p_2$ with $0<u_0\leq p_2\leq v_0<1$. We also know from (3.18) that $-\sqrt{\frac{Z}{2XY}}\leq W_k\leq 0$ and so W_k is bounded on $[0,m_k]$. Thus W_k and hence H_k are uniformly bounded on compact subsets of $[0,p_2)$. Thus, it follows by the Arzela-Ascoli theorem that there is a subsequence (still labeled k) such that $W_k\to W$ uniformly on compact subsets of $(0,p_2)$ as $k\to-\infty$ and hence by (1.14) we have $H_k\to H$ uniformly on compact subsets of $(0,p_2)$ as $k\to-\infty$.

Taking limits in (3.17) on $(0,p_2)$ as $k\to-\infty$ (using Lemmas 3.2, 3.5, and the dominated convergence theorem) gives

$$C_4\rho^{1-\epsilon}H^2 + \int_0^{\rho} t\int_0^{t}\frac{2YsH^2}{(1-s^2)^2}\,ds\,dt + C_4^3\int_0^{\rho} t\int_0^{t} s^{3-3\epsilon}\,ds\,dt \geq \frac{Z}{8}\rho^4.\tag{3.19}$$

Next, it follows from (3.18) and (1.14) that on $(m_k, p_{2,k})$:

$$H_k(\rho) \le H_k(m_k) + \sqrt{\frac{Z}{2XY}}(\rho^2 - m_k^2) + C_5(\rho^{2-\epsilon} - m_k^{2-\epsilon}) \tag{3.20}$$

where $C_5 = \frac{C_4}{2-\epsilon}$. So by taking the limit as $k \to -\infty$ in (3.20) and using Lemmas 3.2, 3.5, and 3.6 we see that

$$0 < H \le C_6 \rho^{2-\epsilon} \quad \text{on } (0, p_2) \tag{3.21}$$

where $C_6 = C_5 + \sqrt{\frac{Z}{2XY}}$. Substituting (3.21) into (3.19) gives

$$C_4 C_6^2 \rho^{5-3\epsilon} + C_7 \rho^{8-2\epsilon} + C_8 \rho^{6-3\epsilon} \ge \frac{Z}{8}\rho^4 \quad \text{on } (0, p_2)$$

where $C_7 = \frac{2XYC_6^2}{(1-p_2)^2(6-2\epsilon)(8-2\epsilon)}$ and $C_8 = \frac{C_4^3}{(4-3\epsilon)(6-3\epsilon)}$. Dividing by ρ^4 we obtain:

$$C_4 C_6^2 \rho^{1-3\epsilon} + C_7 \rho^{4-2\epsilon} + C_8 \rho^{2-3\epsilon} \ge \frac{Z}{8} \quad \text{on } (0, p_2). \tag{3.22}$$

Assuming now that $0 < \epsilon < \frac{1}{3}$ and letting $\rho \to 0^+$ we see that the left-hand side of (3.22) goes to zero but the right-hand side is positive yielding a contradiction. Thus it must be that $p_{2,k} \to 1$ as $k \to -\infty$ for some subsequence of k's. This completes the proof. $\square$

We next observe the following identity which follows from (1.15),

$$\begin{aligned}
&\Big((1-\rho)^2\big(W_k'' + \frac{W_k'}{\rho}\big) + 2(1-\rho)W_k' + \frac{2}{\rho}W_k\Big)' \\
&= (1-\rho)^2\Big(\frac{-2XY\rho}{(1-\rho^2)^2} + \frac{Z\rho}{H_k^2}\Big) - \frac{2}{\rho^2}W_k \\
&= -\frac{2XY\rho W_k\big(H_k + \sqrt{\frac{2Z}{XY}}(1-\rho^2)\big)}{H_k^2(1+\rho)^2} - \frac{2}{\rho^2}W_k.
\end{aligned} \tag{3.23}$$

Note that

$$(1-\rho)^2\Big(W_k'' + \frac{W_k'}{\rho}\Big) + 2(1-\rho)W_k' + \frac{2}{\rho}W_k$$

is decreasing on (z_k, M_k).

Next we observe from the first equality in (3.23) that

$$\Big((1-\rho)^2\big(W_k'' + \frac{W_k'}{\rho}\big) + 2(1-\rho)W_k' + \frac{2}{\rho}W_k\Big)' \ge \frac{-2XY\rho}{(1+\rho)^2} - \frac{2}{\rho^2}W_k.$$

Integrating this on (ρ, M_k), using $W_k'(M_k) = 0$, and also using $W_k''(M_k) \le 0$ gives

$$\begin{aligned}
&(1-\rho)^2\Big(W_k'' + \frac{W_k'}{\rho}\Big) + 2(1-\rho)W_k' + \frac{2}{\rho}W_k \\
&\le \frac{2W_k(M_k)}{M_k} + \int_\rho^{M_k} \frac{2XYt}{(1+t)^2}\,dt + \int_\rho^{M_k} \frac{2}{t^2}W_k\,dt.
\end{aligned} \tag{3.24}$$

Lemma 3.8. *Suppose W_k satisfies* (3.1)-(3.2). *Then $W_k(M_k) \to \infty$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. Suppose the lemma is false and so there is a $C > 0$ such that $W_k(M_k) \le C$ for all k. Now we know from Lemma 3.1 that on $(m_k, p_{2,k})$ we have $W_k' \ge 0$ and $W_k'' \ge 0$, and so it follows from (3.24) and (1.15) that W_k, W_k', and W_k'' are uniformly bounded on compact subsets of $(m_k, p_{2,k})$. It then follows by the Arzela-Ascoli theorem that there is a subsequence (still labeled k) such that $W_k \to W$, $W_k' \to W'$, and $W_k'' \to W''$ uniformly on compact subsets of $(0,1)$ since $m_k \to 0$ by Lemma 3.5 and $p_{2,k} \to 1$ by Lemma 3.7. In addition, $W_k' \ge 0$ on $(m_k, p_{2,k})$ and $W_k'' \ge 0$ on $(m_k, p_{2,k})$. Since $m_k \to 0$ by Lemma 3.5 and $p_{2,k} \to 1$ by Lemma 3.7 it follows that W and W' is increasing on $(0,1)$. Thus we may define $W(1) \equiv \lim_{\rho\to 1^-} W(\rho)$, $W'(1) \equiv \lim_{\rho\to 1^-} W'(\rho)$ and $W(0) \equiv \lim_{\rho\to 0^+} W(\rho)$, $W'(0) \equiv \lim_{\rho\to 0^+} W'(\rho)$. And so there is a corresponding function H such that $H_k \to H$, $H_k' \to H'$, and $H_k'' \to H''$ uniformly on compact subsets of $(0,1)$. We have similar definitions for $H(0)$, $H'(0)$, $H(1)$, and $H'(1)$.

It now follows from (1.15) that W_k''' is uniformly bounded in a neighborhood of any ρ_0 with $0 < \rho_0 < 1$ where $H(\rho_0) > 0$. Along with the boundedness of W_k, W_k', and W_k'' in this neighborhood, it follows that W satisfies (1.15) at any $0 < \rho_0 < 1$ for which $H(\rho_0) > 0$.

Suppose now that there exists ρ_0 with $0 < \rho_0 < 1$ such that $H(\rho_0) = 0$ and $H(\rho) > 0$ for $\rho_0 < \rho < 1$. This would contradict Theorem 1.1 and so it must be the case that either $H > 0$ on $(0,1)$ or $H \equiv 0$ on $(0,1)$.

Case 1: Suppose first that $H > 0$ on $(0,1)$. Next we note that after multiplying (3.24) by ρ and using the fact that $W_k(M_k) \le C$ we see that W_k' and hence H_k' is uniformly bounded, say by T, on (m_k, M_k). Thus, integrating the inequality $H_k' \le T$ on (m_k, ρ) and using Theorem 1.1 gives $0 < H_k(\rho) \le H_k(m_k) + T(\rho - m_k)$. We know $H_k(m_k) \to 0$ by Lemma 3.2 and $m_k \to 0$ by Lemma 3.5, so taking limits as $k \to -\infty$ we see that $0 < H(\rho) \le T\rho$ on $(0,1)$. Thus, we see $H(0) \equiv \lim_{\rho\to 0^+} H(\rho) = 0$.

Now integrating (1.15) on (ρ, ρ_1) where $0 < \rho < \rho_1 < 1$ we see that

$$W''(\rho_1) + \frac{W'(\rho_1)}{\rho_1} + \frac{XY}{1-\rho_1^2} = W''(\rho) + \frac{W'(\rho)}{\rho} + \frac{XY}{1-\rho^2} + \int_\rho^{\rho_1} \frac{Zt}{H^2(t)}\, dt.$$

Therefore,

$$\lim_{\rho\to 0^+} \Big(W''(\rho) + \frac{W'(\rho)}{\rho} + \int_\rho^{\rho_1} \frac{Zt}{H^2(t)}\, dt \Big) \text{ exists and is finite.} \tag{3.25}$$

Now since $0 < H(\rho) \le T\rho$ it follows that

$$\int_\rho^{\rho_1} \frac{Zt}{H^2(t)}\, dt \ge \int_\rho^{\rho_1} \frac{Zt}{T^2 t^2}\, dt = \frac{Z}{T^2} \ln\big(\frac{\rho_1}{\rho}\big) \to \infty \quad \text{as } \rho \to 0^+$$

which along with the fact that $W' \ge 0$ and $W'' \ge 0$ contradicts (3.25). Thus the assumption that $H > 0$ on $(0,1)$ must be false.

Case 2: Next suppose that $H \equiv 0$ on $(0,1)$. We first show that H_k cannot be decreasing on $(m_k, 1)$. Integrating (1.15) on (m_k, ρ) gives

$$H_k'' + \frac{H_k'}{\rho} + \frac{XY}{1-\rho^2} = H_k''(m_k) + \frac{H_k'(m_k)}{m_k} + \frac{XY}{1-m_k^2} + \int_{m_k}^{\rho} \frac{Zt}{H_k^2}\, dt. \tag{3.26}$$

Substituting $H_k'(m_k) + \sqrt{\frac{2Z}{XY}}m_k = W_k'(m_k) = 0$, $H_k''(m_k) + \sqrt{\frac{2Z}{XY}} = W_k''(m_k) \geq 0$, and assuming H_k is decreasing on $(m_k, 1)$ gives

$$H_k'' + \frac{H_k'}{\rho} + \frac{XY}{1-\rho^2} \geq -2\sqrt{\frac{2Z}{XY}} + \frac{Z}{2H_k^2(m_k)}(\rho^2 - m_k^2). \tag{3.27}$$

Multiplying by ρ and integrating again on (m_k, ρ) gives

$$\rho H_k' \geq -\sqrt{\frac{2Z}{XY}}\rho^2 + \frac{XY}{2}\ln\Big(\frac{1-\rho^2}{1-m_k^2}\Big) + \frac{Z}{2H_k^2(m_k)}\int_{m_k}^{\rho} t(t^2 - m_k^2)\,dt. \tag{3.28}$$

Dividing by ρ and integrating again on (m_k, ρ) gives

$$\begin{aligned} H_k \geq H_k(m_k) &- \frac{1}{2}\sqrt{\frac{2Z}{XY}}(\rho^2 - m_k^2) + \int_{m_k}^{\rho}\frac{XY}{2t}\ln\Big(\frac{1-t^2}{1-m_k^2}\Big)\,dt \\ &+ \frac{Z}{2H_k^2(m_k)}\int_{m_k}^{\rho}\frac{1}{s}\int_{m_k}^{s} t(t^2 - m_k^2)\,dt\,ds. \end{aligned} \tag{3.29}$$

Next, making the substitution $u = \frac{1-t}{1-m_k}$ we observe that

$$\begin{aligned} \int_{m_k}^{1}\frac{1}{t}\Big|\ln\Big(\frac{1-t^2}{1-m_k^2}\Big)\Big|\,dt &\leq \int_{m_k}^{1}\frac{1}{t}\Big|\ln\Big[2\Big(\frac{1-t}{1-m_k}\Big)\Big]\Big|\,dt \\ &\leq \int_0^1 \frac{|\ln(2u)|}{1-u}\,du < \infty. \end{aligned} \tag{3.30}$$

Thus we see that $\frac{1}{t}\ln\big(\frac{1-t^2}{1-m_k^2}\big)$ is uniformly integrable on $(m_k, 1)$. Combining this with the fact that $H_k(m_k) \to 0$ by Lemma 3.2 and $m_k \to 0$ by Lemma 3.5 we see that for fixed $\rho > 0$ and k sufficiently negative that the right-hand side of (3.29) goes to $+\infty$. However, the left-hand side of (3.29) is bounded (because by assumption

$$0 < H_k = W_k + \sqrt{\frac{Z}{2XY}}(1-\rho^2) \leq W_k(M_k) + \sqrt{\frac{Z}{2XY}} \leq C + \sqrt{\frac{Z}{2XY}}.$$

This is a contradiction and so the assumption that H_k is decreasing on $(m_k, 1)$ must be false. Thus H_k has a minimum, n_k, with $m_k < n_k < 1$ and $H_k' \leq 0$ on $(0, n_k)$. Also, by Lemma 3.2 we have

$$0 < H_k(n_k) \leq H_k(m_k) \to 0. \tag{3.31}$$

We next claim that $n_k \to 1$ as $k \to -\infty$. Repeating the above argument with m_k being replaced by n_k in (3.26) and using that $H_k'(n_k) = 0$ and $H_k''(n_k) \geq 0$ gives:

$$H_k'' + \frac{H_k'}{\rho} + \frac{XY}{1-\rho^2} \geq \int_{n_k}^{\rho}\frac{Zt}{H_k^2}\,dt.$$

Multiplying by ρ and integrating on (n_k, ρ) gives:

$$\rho H_k' \geq \frac{XY}{2}\ln\Big(\frac{1-\rho^2}{1-n_k^2}\Big) + \int_{n_k}^{\rho} s\int_{n_k}^{s}\frac{Zt}{H_k^2}\,dt\,ds.$$

Dividing by ρ and integrating again gives

$$H_k(\rho) \geq H_k(n_k) + \int_{n_k}^{\rho}\frac{XY}{2\rho}\ln\Big(\frac{1-\rho^2}{1-n_k^2}\Big) + \int_{n_k}^{\rho}\frac{1}{t}\int_{n_k}^{t} s\int_{n_k}^{s}\frac{Zx}{H_k^2}\,dx\,ds\,dt. \tag{3.32}$$

We know $H_k(n_k) \to 0$ by (3.31). Suppose now that $n_k \to n$ where $0 \le n < 1$. Then for fixed ρ with $0 < \rho < 1$ we have $H_k(\rho) \to H(\rho) \equiv 0$ and similarly as in (3.30) the term $\frac{1}{\rho} \ln\Big(\frac{1-\rho^2}{1-n_k^2}\Big)$ is uniformly integrable on $(n_k, 1)$ but the last term on the right in (3.32) goes to infinity since $H_k \to H \equiv 0$ uniformly on compact subsets of (n, ρ) yielding a contradiction. Thus it must be the case that $n_k \to 1$. Next, taking limits in (3.29) for fixed ρ and using that $m_k \to 0$ also yields a contradiction because $\frac{1}{\rho} \ln\big(\frac{1-\rho^2}{1-m_k^2}\big)$ is uniformly integrable on $(m_k, 1)$ and so the right-hand side goes to infinity while the left-hand side is bounded. Thus the assumption that $H \equiv 0$ must also be false and therefore it must be that $W_k(M_k) \to \infty$. This completes the proof. □

Lemma 3.9. *Suppose W_k satisfies* (3.1)-(3.2). *Then either $z_k \to 0$ as $k \to -\infty$ (for some subsequence of k's) or $z_k \to 1$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. Suppose not. Then there is a u_0 and a v_0 with $0 < u_0 \le v_0 < 1$ such that $0 < u_0 \le z_k \le v_0 < 1$ for all k. From Lemma 3.8 we know that $W_k(M_k) \to \infty$ as $k \to -\infty$.

We now define

$$Q_k(\rho) = \frac{W_k(\rho)}{W_k(M_k)}.$$

Note that $Q_k \le 1$ on $[0, M_k]$ and since $W_k(\rho) \ge -\sqrt{\frac{Z}{2XY}}(1-\rho^2) \ge -\sqrt{\frac{Z}{2XY}}$ (by Theorem 1.1) it follows that Q_k is bounded from below independent of k. Thus the Q_k are uniformly bounded on $[0, M_k]$. It follows then from (3.24) that on $[\rho, M_k]$ we have

$$\begin{aligned} &(1-\rho)^2\Big(Q_k'' + \frac{Q_k'}{\rho}\Big) + 2(1-\rho)Q_k' + \frac{2}{\rho}Q_k \\ &\le \frac{2}{M_k} + \frac{1}{W_k(M_k)}\int_\rho^{M_k} \frac{2XYt}{(1+t)^2}\,dt + \int_\rho^{M_k} \frac{2}{t^2}Q_k\,dt. \end{aligned} \tag{3.33}$$

In addition, since $Q_k' \ge 0$ and $Q_k'' \ge 0$ on $(m_k, p_{2,k})$ and $p_{2,k} \to 1$ by Lemma 3.7, it follows from (3.33) that Q_k, Q_k', and Q_k'' are uniformly bounded on compact subsets of $(m_k, v_0]$.

Since $W_k \ge 0$ on $[z_k, M_k]$ we see from (1.15) that $(W_k'' + \frac{W_k'}{\rho})' \le 0$ on $[z_k, M_k]$. In particular, since $z_k \le v_0$ it follows that

$$W_k'' + \frac{W_k'}{\rho} \le W_k''(v_0) + \frac{W_k'(v_0)}{v_0} \quad \text{for } v_0 \le \rho \le M_k.$$

Therefore

$$Q_k'' + \frac{Q_k'}{\rho} \le Q_k''(v_0) + \frac{Q_k'(v_0)}{v_0} \quad \text{for } v_0 \le \rho \le M_k. \tag{3.34}$$

Since $0 \le Q_k \le 1$ on $[z_k, M_k]$ and also $[v_0, M_k] \subset [z_k, M_k]$ it follows from (3.33) and since $v_0 < 1$ that $Q_k''(v_0) + \frac{Q_k'(v_0)}{v_0}$ is bounded independent of k. Since $Q_k' \ge 0$ and $Q_k'' \ge 0$ on $(m_k, p_{2,k})$ then it follows from (3.34) that Q_k, Q_k', and Q_k'' are also uniformly bounded on $[v_0, p_{2,k}]$. In addition, earlier in this proof we showed that Q_k, Q_k', and Q_k'' are uniformly bounded on compact subsets of $(m_k, v_0]$. Combining these results we see that Q_k, Q_k', and Q_k'' are uniformly bounded on compact subsets of $(m_k, p_{2,k}]$.

From Lemmas 3.5 and 3.7 we know that $m_k \to 0$ and $p_{2,k} \to 1$ as $k \to -\infty$ and so by the Arzela-Ascoli theorem there is a further subsequence (again labeled

k) such that $Q_k \to Q$ and $Q_k' \to Q'$ uniformly on compact subsets of $(0,1)$. In particular, Q and Q' are continuous on $(0,1)$. Further, since $Q_k \geq 0$ and $Q_k' \geq 0$ on $(m_k, p_{2,k})$ and $m_k \to 0$ and $p_{2,k} \to 1$, it follows that $Q \geq 0$ and $Q' \geq 0$ on $(0,1)$.

Now let ρ and ρ_0 satisfy $z_k < \rho < \rho_0 < 1$. Then on $[z_k, \rho_0]$ we know that $W_k \geq 0$ so $H_k \geq \sqrt{\frac{Z}{2XY}(1-\rho_0^2)} > 0$ on this set and so we see that $\frac{1}{H_k^2}$ is bounded on $[z_k, \rho_0]$. Also since $u_0 \leq z_k \leq v_0$ there is a subsequence (still labeled k) so that $z_k \to z$ with $0 < u_0 \leq z \leq v_0 < 1$. Therefore it follows from (1.15) that Q_k''' is uniformly bounded on compact subsets of $(z,1)$. Thus Q_k'' is equicontinuous on compact subsets of $(z_k, 1)$ and thus $Q_k'' \to Q''$ uniformly (for some subsequence) on compact subsets of $(z,1)$. From (1.15) it then follows that $Q_k''' \to Q'''$ (for some subsequence) on $(z_k, 1)$. Since

$$\Big(Q_k'' + \frac{Q_k'}{\rho}\Big)' = \frac{1}{W_k(M_k)} \frac{-2XY\rho}{(1-\rho^2)^2} + \frac{1}{W_k(M_k)} \frac{Z\rho}{H_k^2} \quad \text{for } z < \rho < 1,$$

it follows by Lemma 3.8 that

$$\Big(Q'' + \frac{Q'}{\rho}\Big)' = 0 \quad \text{for } z < \rho < 1. \tag{3.35}$$

In addition, since Q_k is increasing on $[m_k, M_k]$, $m_k \to 0$ by Lemma 3.5, $M_k \to 1$ by Lemma 3.7, and $p_{2,k} \leq M_k \leq 1$, it follows that Q is increasing on $(0,1)$. In particular, we define $Q(0) \equiv \lim_{\rho \to 0^+} Q(\rho)$ and $Q(1) \equiv \lim_{\rho \to 1^-} Q(\rho)$. So Q is continuous on $[0,1]$. Similarly, Q_k' is increasing on $(m_k, p_{2,k})$, $m_k \to 0$ by Lemma 3.5, and $p_{2,k} \to 1$ by Lemma 3.7, so Q' is increasing on $(0,1)$. Therefore we may also define $Q'(0) \equiv \lim_{\rho \to 0^+} Q'(\rho)$ and $Q'(1) \equiv \lim_{\rho \to 1^-} Q'(\rho)$. Thus Q_k' is continuous on $[0,1]$.

Also, as mentioned earlier in this proof, $Q_k \to Q$ and $Q_k' \to Q'$ uniformly on compact subsets of $(0,1)$ and since $0 < z < 1$ it follows that $Q(z) = \lim_{k \to -\infty} Q_k(z_k) = 0$. In addition, $Q'(z)$ is defined and in fact $Q'(z) = 0$ for if $Q'(z) \neq 0$ then Q would get negative somewhere in a neighborhood of z contradicting that $Q \geq 0$ on $(0,1)$. If $Q''(z) < 0$ then Q would get negative somewhere in a neighborhood of z contradicting that $Q \geq 0$. If $Q''(z) > 0$ then $Q' < 0$ on $(z - \delta, z)$ for some $\delta > 0$ contradicting $Q' \geq 0$. Thus it must be that $Q''(z) = 0$. Solving (3.35) along with the conditions $Q(z) = Q'(z) = Q''(z) = 0$ implies $Q \equiv 0$.

Next recall from earlier in the proof that Q_k' is uniformly bounded, say by L, on (m_k, M_k). Thus

$$0 \leq 1 - Q_k(\rho) = Q_k(M_k) - Q_k(\rho) = \int_\rho^{M_k} Q_k'(t)\,dt \leq L(M_k - \rho).$$

So taking limits as $k \to -\infty$ gives

$$0 \leq 1 - Q(\rho) \leq L(1 - \rho).$$

Now taking the limit as $\rho \to 1^-$, it follows that $Q(1) = 1$. But we showed earlier that $Q \equiv 0$ so we obtain a contradiction. Therefore, it must be the case that either $z_k \to 0$ or $z_k \to 1$. This completes the proof. □

Lemma 3.10. *Suppose W_k satisfies* (3.1)-(3.2). *Then $z_k \to 0$ as $k \to -\infty$ (for some subsequence of k's).*

Proof. From Lemma 3.9, we know that $z_k \to 0$ or $z_k \to 1$ as $k \to -\infty$. Let us assume that $z_k \to 1$ as $k \to -\infty$.

First, since $m_k \to 0$ by Lemma 3.5 and $z_k \to 1$ by assumption we may choose a $\delta > 0$ such that $m_k \le 1 - 2\delta$ and $1 - z_k \le \frac{\delta}{2}$ for k sufficiently negative. Then by the mean value theorem $-W_k(1-\delta) = W_k(z_k) - W_k(1-\delta) = W_k'(c_k)(z_k - (1-\delta))$ for some c_k with $1 - \delta < c_k < z_k$. Thus, for k sufficiently negative,

$$0 \le W_k'(c_k) = \frac{-W_k(1-\delta)}{z_k - (1-\delta)} \le \frac{\sqrt{\frac{2Z}{XY}}}{\delta}$$

since $0 \le -W_k(1-\delta) \le \sqrt{\frac{Z}{2Y}}$ and $\frac{1}{z_k-(1-\delta)} \le \frac{2}{\delta}$.

Also, since $W_k'' \ge 0$ on (m_k, z_k) we see that for ρ with $m_k \le \rho \le 1 - \delta$ we have

$$0 \le W_k' \le W_k'(c_k) \le \frac{\sqrt{\frac{2Z}{XY}}}{\delta} \quad \text{on } (m_k, 1-\delta). \tag{3.36}$$

Then by the mean value theorem there is an x_k with for $1 - 2\delta < x_k < 1 - \delta$ such that

$$\frac{\sqrt{\frac{2Z}{XY}}}{\delta} \ge W_k'(1-\delta) - W_k'(1-2\delta) = W_k''(x_k)\delta.$$

Thus

$$0 \le W_k''(x_k) \le \frac{\sqrt{\frac{2Z}{XY}}}{\delta^2}.$$

In addition, by (3.36) we have

$$0 \le W_k'(x_k) \le \frac{\sqrt{\frac{2Z}{XY}}}{\delta}, \tag{3.37}$$

$$0 \le \frac{W_k'(x_k)}{x_k} \le \frac{\sqrt{\frac{2Z}{XY}}}{\delta(1-2\delta)}. \tag{3.38}$$

It follows then from (3.23) that for $m_k < x_k < \rho < z_k$,

$$\begin{aligned} &(1-\rho)^2\Big(W_k'' + \frac{W_k'}{\rho}\Big) + 2(1-\rho)W_k' + \frac{2}{\rho}W_k \\ &\le (1-x_k)^2\Big(W_k''(x_k) + \frac{W_k'(x_k)}{x_k}\Big) + 2(1-x_k)W_k'(x_k) + \frac{2}{x_k}W_k(x_k). \end{aligned} \tag{3.39}$$

Multiplying (3.39) by ρ and using (3.37)-(3.38) we see that W_k, $W_k,'$ and W_k'' are uniformly bounded on compact subsets of $(m_k, 1-2\delta]$ and so by the Arzela-Ascoli theorem there is a subsequence (again labeled k) and functions W and H such that $W_k \to W$, $W_k' \to W'$, $H_k \to H$, and $H_k' \to H'$ uniformly on compact subsets of $(0, 1-2\delta]$. Since δ is arbitrary we see that W_k and W_k' converge uniformly on compact subsets of $(0,1)$ to W and W'.

Since $W_k' \ge 0$ on (m_k, M_k) and $m_k \to 0$ by Lemma 3.5 and also $M_k \to 1$ by Lemma 3.7, it follows that $W' \ge 0$ on $(0,1)$. It also follows that $W \le 0$ on $(0,1)$ since $W_k \le 0$ on (m_k, z_k) where $m_k \to 0$ by Lemma 3.5 and $z_k \to 1$ by assumption.

Next it follows from (1.15) that W_k''' is uniformly bounded in a neighborhood of any ρ_0 with $0 < \rho_0 < 1$ such that $H(\rho_0) > 0$. Along with the boundedness of W_k, W_k', and W_k'' in this neighborhood it follows that W solves (1.15) in this neighborhood.

Suppose now that there exists ρ_0 with $0 < \rho_0 < 1$ such that $H(\rho_0) = 0$ and $H(\rho) > 0$ for $\rho_0 < \rho < 1$. This would contradict Theorem 1.1 and so it must be the case that either $H > 0$ on $(0,1)$ or $H \equiv 0$ on $(0,1)$. If $H > 0$ on $(0,1)$ then it actually follows that $W'(0) = 0$. To see this, we first observe that since $W' \geq 0$ and $W'' \geq 0$ on $(0,1)$ it follows that $\lim_{\rho\to 0^+} W'(\rho) = A \geq 0$. We would like to show $A = 0$ so we will assume $A > 0$. Multiplying (1.15) by ρ^2 and taking the limits as $\rho \to 0^+$ gives $\lim_{\rho\to 0^+}(\rho^2 W''' + \rho W') = A$. Thus for small positive ρ we have $(\rho W'')' \geq \frac{A}{2\rho}$. Integrating on (ρ, ρ_0) gives $\rho W'' \leq \frac{A}{2}\ln(\rho) + C_0$ for some constant C_0. Dividing by ρ and integrating again on (ρ, ρ_0) gives $W \geq \frac{A}{4}\frac{\ln^2(\rho)}{\rho} + C_0\ln(\rho) + C_1$ for some constant C_1. This implies $W \to \infty$ as $\rho \to 0^+$ which contradicts that $W(\rho)$ is bounded. Thus we see that $W'(0) \equiv \lim_{\rho\to 0^+} W'(\rho) = 0$. Then by Lemma 2.3 we see that this implies that W must get positive on $(0,1)$ but this contradicts that $W \leq 0$ on $(0,1)$. On the other hand, if $H \equiv 0$ on $(0,1)$ then using a nearly identical argument as in Case 2 of Lemma 3.7, we can arrive at equation (3.29) and from this it follows that H_k must get large and hence a minimum, n_k, with $m_k < n_k$ must exist. The rest of the argument is the same and so again this leads to a contradiction. Thus we see that $z_k \not\to 1$ as $k \to -\infty$ and consequently by Lemma 3.9 we see that $z_k \to 0$ as $k \to -\infty$. This completes the proof. $\square$

Proof Theorem 1.3. If there is such a solution then we know from Lemma 3.10 that there is a subsequence (again labeled k) such that $z_k \to 0$. Integrating (1.15) on (m_k, ρ) gives

$$W_k'' + \frac{W_k'}{\rho} + \frac{XY}{1-\rho^2} = W_k''(m_k) + \frac{XY}{1-m_k^2} + \int_{m_k}^{\rho} \frac{Zt}{H_k^2}\,dt \geq 0.$$

Multiplying by ρ, integrating on (m_k, ρ), and using that m_k is a local minimum so that $W_k''(m_k) \geq 0$ gives

$$0 \geq \rho W_k' \geq \ln\Big(\frac{1-\rho^2}{1-m_k^2}\Big).$$

Suppose now that $W_k(z_{2,k}) = 0$ with $W_k > 0$ on $(z_k, z_{2,k})$. Then we know that $W_k' \leq 0$ on $(M_k, z_{2,k})$ by Lemma 2.2. Thus, on $(M_k, z_{2,k})$ we have

$$W_k'^2 \leq \frac{\ln^2\big(\frac{1-\rho^2}{1-m_k^2}\big)}{\rho^2}.$$

Recall that $m_k \to 0$ by Lemma 3.5 and $M_k \to 1$ by Lemma 3.7. Thus for some positive constant A independent of k we have:

$$\int_{M_k}^{z_{2,k}} W_k'^2\,dt \leq \int_{M_k}^{z_{2,k}} \frac{\ln^2\big(\frac{1-\rho^2}{1-m_k^2}\big)}{\rho^2}\,d\rho \leq \int_{M_k}^{1} \frac{\ln^2\big(\frac{1-\rho^2}{1-m_k^2}\big)}{\rho^2}\,d\rho \leq A < \infty.$$

Thus,

$$\int_{M_k}^{z_{2,k}} Q_k'^2\,dt = \frac{1}{W_k^2(M_k)}\int_{M_k}^{z_{2,k}} W_k'^2\,dt \to 0 \quad \text{as } k \to -\infty. \tag{3.40}$$

Finally by Holder's inequality and (3.40) we have

$$\begin{aligned} 1 = |0-1| = |Q_k(z_{2,k}) - Q_k(M_k)| &= |\int_{M_k}^{z_{2,k}} Q_k'(t)\,dt| \\ &\leq \sqrt{z_{2,k} - M_k}\sqrt{\int_{M_k}^{z_{2,k}} Q_k'^2\,dt} \to 0 \quad \text{as } k \to -\infty \end{aligned}$$

which is impossible. This completes the proof. $\square$

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References

[1] F. Bernis, R. Ferreira; Source type solutions to thin-film equations in higher dimensions, *European Journal of Applied Mathematics*, Vol. 0, Issue 5, (1997) 507-524 .

[2] S. Betelu, M. Fontelos; Spreading of a charged microdroplet, *Physica D* 209 (2005) 28-35.

[3] S. Betelu, J. A. Diez, R. Gratton, B. Marino, L. P. Thomas; Quasi-self-similarity for wetting drops, *Physical Review E* 53, (1996) 3563-3572.

[4] J. Birkhoff, G. C. Rota; *Ordinary Differential Equations*, Ginn and Company, 1959.

[5] J. A. Diez, R. Gratton, B. Marino, L.P. Thomas; Laplace pressure driven drop spreading: quasi-self-similar solution, *Journal of Colloid and Interface Science*, Vol. 168, Issue 1, (1994) 15-20.

[6] J. Iaia; Spreading of Charged Microdroplets, *Electronic Journal of Differential Equations*, Vol. 2013 (2013), No. 202, 1-22.

[7] J. Iaia; Spreading Charged Microdroplets, submitted to *Differential and Integral Equations*, 2014.

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