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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR RADIAL p-LAPLACIAN EQUATIONS

SYRINE MASMOUDI, SAMIA ZERMANI

Abstract. This article concerns the existence, uniqueness and boundary behavior of positive solutions to the nonlinear problem

$$\frac{1}{A}(A\Phi_p(u'))' + a_1(x)u^{\alpha_1} + a_2(x)u^{\alpha_2} = 0, \quad \text{in } (0,1),$$
$$\lim_{x\to 0} A\Phi_p(u')(x) = 0, \quad u(1) = 0,$$

where $p > 1$, $\alpha_1, \alpha_2 \in (1-p, p-1)$, $\Phi_p(t) = t|t|^{p-2}$, $t \in \mathbb{R}$, A is a positive differentiable function and a_1, a_2 are two positive measurable functions in $(0,1)$ satisfying some assumptions related to Karamata regular variation theory.

1. Introduction

In recent years, the existence of positive solutions for elliptic problems involving the p-Laplacian has found considerable interest and different approaches have been developed. This is due to their signification in various areas of pure and applied mathematics including geological sciences, fluid dynamics, electrostatics, cosmology (see [1, 2, 6, 10, 12]), as well as in relation to inequalities of Poincaré, Writinger, Sobolev type and isoperimetric inequalities (see [3, 8, 9, 13, 15]). Motivations for studying radial solutions can be found in [4, 5, 11, 13, 19] and references therein.

Mâagli et al [15] considered the quasilinear elliptic problem

$$\begin{gathered} -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = q(x)u^{\alpha} \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a C^2 bounded domain of $\mathbb{R}^n$ $(n \geq 2)$, $p > 1$, the exponent $\alpha \in (-1, p-1)$ and $q \in C(\Omega)$ is a positive function having singular behavior near the boundary $\partial\Omega$. More precisely, let $d(x)$ be the Euclidean distance of $x \in \Omega$ to $\partial\Omega$ then $q(x) = d(x)^{-\beta}L(d(x))$, with $0 < \beta < p$ and L belongs to a functional class $\mathcal{K}$ called Karamata class and defined on $(0,\eta], (\eta > \operatorname{diam}(\Omega))$ by

$$\mathcal{K} := \Big\{t \to L(t) := c\exp\Big(\int_t^{\eta} \frac{z(s)}{s}ds\Big) : c > 0 \text{ and } z \in C([0,\eta]), \ \ z(0) = 0\Big\}.$$

For the convenience of the readers, we briefly describe the result proved in [15].

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Theorem 1.1 ([15]). *Let $L \in \mathcal{K} \cap \mathcal{C}^2((0,\eta])$ such that $\int_0^\eta t^{\frac{1-\beta}{p-1}} L(t)^{\frac{1}{p-1}} dt < \infty$. Then problem* (1.1) *has a unique positive and continuous solution u satisfying, for $x \in (0,1)$,*

$$u(x) \approx \begin{cases} \Big(\int_0^{d(x)} s^{-1} L(s)^{\frac{1}{p-1}} ds \Big)^{\frac{p-1}{p-1-\alpha}} & \text{if } \beta = p, \\ d(x)^{\frac{p-\beta}{p-1-\alpha}} L(d(x))^{\frac{1}{p-1-\alpha}} & \text{if } 1+\alpha \le \beta < p, \\ d(x) \Big(\int_{d(x)}^{\eta} s^{-1} L(s) ds \Big)^{\frac{1}{p-1-\alpha}} & \text{if } \beta = 1+\alpha, \\ d(x) & \text{if } \beta < 1+\alpha. \end{cases} \tag{1.2}$$

Here and throughout this paper, the notation $f(x) \approx g(x)$, $x \in S$ for f and g nonnegative functions defined on a set S, means that there exists $c > 0$ such that $\frac{1}{c} g(x) \le f(x) \le c g(x)$, for each $x \in S$.

If Ω is the unit ball, similar result was shown in [4] for radial solution of problem (1.1) which becomes in the radial form

$$\begin{gathered} \frac{1}{A}(A\Phi_p(u'))' + q(x) u^\alpha = 0, \quad \text{in } (0,1), \\ \lim_{x \to 0} A\Phi_p(u')(x) = 0, \quad u(1) = 0, \end{gathered} \tag{1.3}$$

where $\Phi_p(t) = t|t|^{p-2}$, $t \in \mathbb{R}$ and $A(t) = t^{n-1}$. Indeed, by using Karamata variation theory, the authors in [4] established existence and asymptotic behavior of a unique positive continuous solution to (1.3) for $\alpha < p-1$ and for a large class of functions A including the example $A(t) = t^{n-1}$. More precisely, they proved Theorem 1.3 below under the following assumptions:

- (H0) A is a continuous function in $[0,1)$, positive and differentiable in $(0,1)$ such that
$$A(x) \approx x^\lambda (1-x)^\mu,$$
where $\lambda \ge 0$ and $\mu < p-1$.
- (H1) q is a positive measurable function on $(0,1)$ such that
$$q(x) \approx (1-x)^{-\beta} L(1-x)$$
with $\beta \le p$ and $L \in \mathcal{K}$ defined on $(0,\eta]$ $(\eta > 1)$ such that
$$\int_0^\eta t^{\frac{1-\beta}{p-1}} L(t)^{\frac{1}{p-1}} dt < \infty.$$

Remark 1.2. We need to verify condition $\int_0^\eta t^{\frac{1-\beta}{p-1}} L(t)^{\frac{1}{p-1}} dt < \infty$ in hypothesis (H1), only for $\beta = p$. This is due to Karamata's theorem which we recall in Lemma 2.2 below.

Theorem 1.3 ([4]). *Assume* (H0)–(H1) *hold. Then problem* (1.3) *has a unique positive and continuous solution u satisfying, for $x \in (0,1)$,*

$$u(x) \approx (1-x)^{\min(\frac{p-\beta}{p-1-\alpha}, \frac{p-1-\mu}{p-1})} \Psi_{L,\beta,\alpha}(x), \tag{1.4}$$

where $\Psi_{L,\beta,\alpha}$ *is the function defined on* $(0,1)$ *by*

$$\Psi_{L,\beta,\alpha}(x) := \begin{cases} \Big(\int_0^{1-x} s^{-1}L(s)^{\frac{1}{p-1}}\,ds\Big)^{\frac{p-1}{p-1-\alpha}} & \text{if } \beta = p,\\ L(1-x)^{\frac{1}{p-1-\alpha}} & \text{if } \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1} < \beta < p,\\ \Big(\int_{1-x}^{\eta} s^{-1}L(s)ds\Big)^{\frac{1}{p-1-\alpha}} & \text{if } \beta = \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1},\\ 1 & \text{if } \beta < \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}. \end{cases} \tag{1.5}$$

For the special case when $A(t) = t^{n-1}$ the estimates (1.2) and (1.4) are the same.

In this article, we study the boundary-value problem

$$\begin{gathered} \frac{1}{A}(A\Phi_p(u'))' + a_1(x)u^{\alpha_1} + a_2(x)u^{\alpha_2} = 0, \quad \text{in } (0,1),\\ \lim_{x\to 0} A\Phi_p(u')(x) = 0, \quad u(1) = 0, \end{gathered} \tag{1.6}$$

where $\alpha_1, \alpha_2 \in (1-p, p-1)$ and A satisfies (H0). Our purpose is to establish an existence and a uniqueness of a continuous solution to (1.6) and to give estimates on such solution, where appear the combined effects of singular and sublinear terms in the nonlinearity.

The pure elliptic semilinear problem corresponding to (1.6) is

$$\begin{gathered} \Delta u + a_1(x)u^{\alpha_1} + a_2(x)u^{\alpha_2} = 0, \quad \text{in } \Omega,\\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

which has been studied by several authors on smooth domains, see for example [7, 14, 16, 17, 20] and references therein.

Let us introduce our conditions on the functions a_i:

(H2) For $i \in \{1,2\}$, a_i is a positive measurable function and satisfies for each $x \in (0,1)$
$$a_i(x) \approx (1-x)^{-\beta_i}L_i(1-x),$$
where $\beta_i \le p$ and $L_i \in \mathcal{K}$ defined on $(0,\eta]$, $(\eta > 1)$ such that
$$\int_0^{\eta} t^{\frac{1-\beta_i}{p-1}} L_i(t)^{\frac{1}{p-1}}\,dt < \infty.$$

As it turns out, estimates (1.4) depend closely on $\min\big(\frac{p-\beta}{p-1-\alpha}, \frac{p-1-\mu}{p-1}\big)$. Also, as it will be seen, the numbers

$$\begin{gathered} \delta_1 = \min\Big(\frac{p-\beta_1}{p-1-\alpha_1}, \frac{p-1-\mu}{p-1}\Big),\\ \delta_2 = \min\Big(\frac{p-\beta_2}{p-1-\alpha_2}, \frac{p-1-\mu}{p-1}\Big) \end{gathered}$$

play a crucial role in the combined effects of singular and sublinear nonlinearities in problem (1.6) and lead to a competition. However, without loss of generality, we can suppose that $\frac{p-\beta_1}{p-1-\alpha_1} \le \frac{p-\beta_2}{p-1-\alpha_2}$ and we introduce the function θ defined on $(0,1)$ by

$$\theta(x) := \begin{cases} (1-x)^{\delta_1}\Psi_{L_1,\beta_1,\alpha_1}(x) & \text{if } \delta_1 < \delta_2,\\ (1-x)^{\delta_1}(\Psi_{L_1,\beta_1,\alpha_1}(x) + \Psi_{L_2,\beta_2,\alpha_2}(x)) & \text{if } \delta_1 = \delta_2, \end{cases} \tag{1.8}$$

where for $i \in \{1,2\}$, $\Psi_{L_i,\beta_i,\alpha_i}$ is the function defined by (1.5).

Now, we are ready to state our main result.

Theorem 1.4. *Assume* (H0) *and* (H2) *hold and suppose that* $\alpha_1, \alpha_2 \in (1-p, p-1)$. *Then problem* (1.6) *has a unique positive and continuous solution* u *satisfying, for each* $x \in (0,1)$,

$$u(x) \approx \theta(x), \tag{1.9}$$

where θ *is the function defined by* (1.8).

The outline of this article is as follows. In Section 2, we give some already known results on functions in $\mathcal{K}$ useful for our study and we give estimates of some potential functions. In Section 3, we prove our main result.

2. Preliminary results

Our arguments combine a method of fixed point theorem with Karamata regular variation theory. So, we are quoting some properties of functions in $\mathcal{K}$ useful for our study.

2.1. **The Karamata class** $\mathcal{K}$**.** It is obvious to see that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$ such that

$$\lim_{t\to 0} \frac{tL'(t)}{L(t)} = 0.$$

A standard function belonging to the class $\mathcal{K}$ is given by

$$L(t) := \prod_{k=1}^{p} (\log_k(\frac{\omega}{t}))^{\lambda_k},$$

where $p \in \mathbb{N}^*$, $(\lambda_1, \lambda_2, \dots, \lambda_p) \in \mathbb{R}^p$, ω is a positive real number sufficiently large and $\log_k(x) = \log o \log o \dots o \log(x)$ (k times).

Lemma 2.1 ([18])**.**

(i) *Let* $L \in \mathcal{K}$ *and* $\varepsilon > 0$. *Then* $\lim_{t\to 0} t^{\varepsilon} L(t) = 0$.

(ii) *Let* $L_1, L_2 \in \mathcal{K}, p \in \mathbb{R}$. *Then* $L_1 + L_2 \in \mathcal{K}, L_1 L_2 \in \mathcal{K}$ *and* $L_1^p \in \mathcal{K}$.

Lemma 2.2 (Karamata's Theorem)**.** *Let* $L \in \mathcal{K}$ *be defined on* $(0,\eta]$ *and* $\sigma \in \mathbb{R}$.

(i) *If* $\sigma > -1$, *then* $\int_0^\eta t^\sigma L(t)dt$ *converges and*

$$\int_0^t s^\sigma L(s)ds \sim \frac{t^{1+\sigma}L(t)}{\sigma+1} \quad \text{as } t \to 0^+.$$

(ii) *If* $\sigma < -1$, *then* $\int_0^\eta t^\sigma L(t)dt$ *diverges and*

$$\int_t^\eta s^\sigma L(s)ds \sim -\frac{t^{1+\sigma}L(t)}{\sigma+1} \quad \text{as } t \to 0^+.$$

Lemma 2.3 ([7])**.** *Let* $L \in \mathcal{K}$ *be defined on* $(0,\eta]$. *Then*

$$t \to \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(t)}{t} dt$ *converges, then*

$$t \to \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}.$$

Lemma 2.4 ([7]). *For $i \in \{1,2\}$, let $\eta_i < 1$ and $L_i \in \mathcal{K}$. For $t \in (0,\eta)$, put*
$$J(t) = \Big(\int_t^\eta \frac{L_1(s)}{s}ds\Big)^{\frac{1}{1-\eta_1}} + \Big(\int_t^\eta \frac{L_2(s)}{s}ds\Big)^{\frac{1}{1-\eta_2}}.$$
Then, for $t \in (0,\eta)$, we have
$$\int_t^\eta \frac{(J^{\eta_1}L_1 + J^{\eta_2}L_2)(s)}{s}ds \approx J(t).$$

Lemma 2.5 ([7]). *For $i \in \{1,2\}$, let $\eta_i < 1$ and $L_i \in \mathcal{K}$ such that $\int_0^\eta \frac{L_i(s)}{s}ds < \infty$. For $t \in (0,\eta)$, put*
$$I(t) = \Big(\int_0^t \frac{L_1(s)}{s}ds\Big)^{\frac{1}{1-\eta_1}} + \Big(\int_0^t \frac{L_2(s)}{s}ds\Big)^{\frac{1}{1-\eta_2}}.$$
Then, for $t \in (0,\eta)$, we have
$$\int_0^t \frac{(I^{\eta_1}L_1 + I^{\eta_2}L_2)(s)}{s}ds \approx I(t).$$

2.2. **Potential estimates.** For a nonnegative measurable function f in $(0,1)$, let
$$G_p f(x) := \int_x^1 (\frac{1}{A(t)}\int_0^t A(s)f(s)ds)^{\frac{1}{p-1}}dt.$$
We point out that if f is a nonnegative measurable function such that the mapping $x \to A(x)f(x)$ is integrable in $[0,1]$, then $G_p f$ is the solution of the problem
$$\begin{gathered} \frac{1}{A}(A\Phi_p(u'))' + f = 0, \quad \text{in } (0,1),\\ \lim_{x\to 0} A\Phi_p(u')(x) = 0, \quad u(1) = 0. \end{gathered} \tag{2.1}$$
In what follows, we aim to prove Proposition 2.8. To this end, we need the following two lemmas which are proved in [4] and [7].

Lemma 2.6 ([4]). *Assume* (H0) *and* (H1) *hold. Then for $x \in (0,1)$, we have*
$$G_p(q)(x) \approx (1-x)^{\min(\frac{p-\beta}{p-1},\frac{p-1-\mu}{p-1})}\begin{cases} \int_0^{1-x}\frac{L(s)^{\frac{1}{p-1}}}{s}ds & \text{if } \beta = p,\\ L(1-x)^{\frac{1}{p-1}} & \text{if } \mu+1<\beta<p,\\ (\int_{1-x}^\eta \frac{L(s)}{s}ds)^{\frac{1}{p-1}} & \text{if } \beta = \mu+1,\\ 1 & \text{if } \beta<\mu+1. \end{cases}$$

Lemma 2.7 ([7]). *For $s,t>0$, $\eta_1<1$ and $\eta_2<1$, we have*
$$2^{-max(1-\eta_1,1-\eta_2)}(t+s) \le t^{1-\eta_1}(t+s)^{\eta_1} + s^{1-\eta_2}(t+s)^{\eta_2} \le 2(t+s).$$

Proposition 2.8. *Assume* (H0) *and* (H2) *hold. Let θ be the function given by* (1.8). *Then for $x \in (0,1)$,*
$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx \theta(x).$$

Proof. For $t \in (0,1)$, let
$$K(t) := \Big(L_1^{\frac{1}{p-1-\alpha_1}} + L_2^{\frac{1}{p-1-\alpha_2}}\Big)(t),$$
$$N(t) := \Big(\int_t^\eta \frac{L_1(s)}{s}ds\Big)^{\frac{1}{p-1-\alpha_1}} + \Big(\int_t^\eta \frac{L_2(s)}{s}ds\Big)^{\frac{1}{p-1-\alpha_2}},$$

$$M(t) := \Big(\int_0^t \frac{(L_1(s))^{\frac{1}{p-1}}}{s}ds\Big)^{\frac{p-1}{p-1-\alpha_1}} + \Big(\int_0^t \frac{(L_2(s))^{\frac{1}{p-1}}}{s}ds\Big)^{\frac{p-1}{p-1-\alpha_2}}.$$

Since $\delta_1 < \delta_2$ is equivalent to $\frac{p-\beta_1}{p-1-\alpha_1} < \frac{p-\beta_2}{p-1-\alpha_2}$ and $\frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p$, we can write

$$\theta(x) = \begin{cases} \Big(\int_0^{1-x} \frac{L_1(s)^{\frac{1}{p-1}}}{s}ds\Big)^{\frac{p-1}{p-1-\alpha_1}} \quad \text{if } \beta_1 = p,\ \beta_2 < p, \\ (1-x)^{\frac{p-\beta_1}{p-1-\alpha_1}} L_1(1-x)^{\frac{1}{p-1-\alpha_1}} \\ \quad \text{if } \frac{p-\beta_1}{p-1-\alpha_1} < \frac{p-\beta_2}{p-1-\alpha_2},\ \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p, \\ (1-x)^{\frac{p-\beta_1}{p-1-\alpha_1}} K(1-x) \\ \quad \text{if } \frac{p-\beta_1}{p-1-\alpha_1} = \frac{p-\beta_2}{p-1-\alpha_2},\ \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p, \\ (1-x)^{\frac{p-1-\mu}{p-1}} N(1-x) \\ \quad \text{if } \beta_1 = \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1},\ \beta_2 = \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}, \\ (1-x)^{\frac{p-1-\mu}{p-1}} (\int_{1-x}^{\eta} \frac{L_1(s)}{s}ds)^{\frac{1}{p-1-\alpha_1}} \\ \quad \text{if } \beta_1 = \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1},\ \beta_2 < \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}, \\ (1-x)^{\frac{p-1-\mu}{p-1}} \quad \text{if } \beta_1 < \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1}, \\ M(1-x) \quad \text{if } \beta_1 = \beta_2 = p. \end{cases}$$

The main idea is to prove that the function $a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2}$ satisfies (H1) and then to apply Lemma 2.6.

Throughout the proof, we use Lemma 2.1 and Lemma 2.3 to verify that some functions are in $\mathcal{K}$. We distinguish the following cases.

Case 1. $\beta_1 = p$ and $\beta_2 < p$. Using (H2) we have

$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{-p}L_1(1-x)\theta^{\alpha_1}(x) + (1-x)^{-\beta_2}L_2(1-x)\theta^{\alpha_2}(x).$$

Since $\theta(x) = (\int_0^{1-x} s^{-1}L_1(s)^{\frac{1}{p-1}}ds)^{\frac{p-1}{p-1-\alpha_1}} < \infty$, by Lemmas 2.1 and 2.3, the function $x \to L_i(1-x)\theta^{\alpha_i}(x) \in \mathcal{K}$, for $i \in \{1,2\}$.

Now, using $\beta_2 < p$ we deduce by Lemma 2.1(i) that

$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{-p}L_1(1-x)\theta^{\alpha_1}(x).$$

Moreover, by calculus we have

$$\begin{aligned} \int_0^{\eta} \frac{(L_1(t)\theta^{\alpha_1}(1-t))^{\frac{1}{p-1}}}{t}dt &= \int_0^{\eta} \frac{L_1(t)^{\frac{1}{p-1}}}{t}(\int_0^t \frac{L_1(s)^{\frac{1}{p-1}}}{s}ds)^{\frac{\alpha_1}{p-1-\alpha_1}}dt \\ &= \frac{p-1-\alpha_1}{p-1}(\int_0^{\eta} \frac{L_1(t)^{\frac{1}{p-1}}}{t}dt)^{\frac{p-1}{p-1-\alpha_1}} < \infty\,. \end{aligned}$$

So applying Lemma 2.6, for $\beta = p$ and $L(t) = L_1(t)\theta^{\alpha_1}(1-t)$, we obtain

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx \Big(\int_0^{1-x} \frac{L_1(s)^{\frac{1}{p-1}}}{s}ds\Big)^{\frac{p-1}{p-1-\alpha_1}} = \theta(x).$$

Case 2. $\frac{p-\beta_1}{p-1-\alpha_1} < \frac{p-\beta_2}{p-1-\alpha_2}$ and $\frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p$. Using (H2) we have

$$a_1(x)\theta^{\alpha_1}(x) \approx (1-x)^{\frac{p\alpha_1+\beta_1(1-p)}{p-1-\alpha_1}} L_1(1-x)^{\frac{p-1}{p-1-\alpha_1}}$$

and
$$a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{\alpha_2 \frac{p-\beta_1}{p-1-\alpha_1} - \beta_2}(L_2 L_1^{\frac{\alpha_2}{p-1-\alpha_1}})(1-x).$$
Now, since $\frac{p-\beta_1}{p-1-\alpha_1} < \frac{p-\beta_2}{p-1-\alpha_2}$, we have
$$(p-\beta_1)(1+\frac{\alpha_1-\alpha_2}{p-1-\alpha_1}) < p-\beta_2,$$
and so
$$\frac{p\alpha_1+\beta_1(1-p)}{p-1-\alpha_1} < \alpha_2\frac{p-\beta_1}{p-1-\alpha_1} - \beta_2.$$
We deduce by Lemma 2.1(i) that
$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{\frac{p\alpha_1+\beta_1(1-p)}{p-1-\alpha_1}} L_1(1-x)^{\frac{p-1}{p-1-\alpha_1}}.$$
Moreover, since $\frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p$, we deduce by a simple calculus that
$$\frac{-p\alpha_1+\beta_1(p-1)}{p-1-\alpha_1} \in (\mu+1, p).$$
So applying Lemma 2.6, for $\beta = \frac{-p\alpha_1+\beta_1(p-1)}{p-1-\alpha_1}$ and $L(t) = (L_1(t))^{\frac{p-1}{p-1-\alpha_1}}$ we obtain that
$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-\beta_1}{p-1-\alpha_1}} L_1(1-x)^{\frac{1}{p-1-\alpha_1}} = \theta(x).$$

Case 3. $\frac{p-\beta_1}{p-1-\alpha_1} = \frac{p-\beta_2}{p-1-\alpha_2}$ and $\frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p$. Using (H2) we have
$$a_1(x)\theta^{\alpha_1}(x) \approx (1-x)^{\frac{p\alpha_1+\beta_1(1-p)}{p-1-\alpha_1}}(L_1K^{\alpha_1})(1-x)$$
and
$$a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{\alpha_2\frac{p-\beta_1}{p-1-\alpha_1}-\beta_2}(L_2K^{\alpha_2})(1-x).$$
Now, $\frac{p-\beta_1}{p-1-\alpha_1} = \frac{p-\beta_2}{p-1-\alpha_2}$ is equivalent to
$$(p-\beta_1)(1+\frac{\alpha_1-\alpha_2}{p-1-\alpha_1}) = p-\beta_2;$$
that is,
$$\frac{p\alpha_1+\beta_1(1-p)}{p-1-\alpha_1} = \alpha_2\frac{p-\beta_1}{p-1-\alpha_1} - \beta_2.$$
Then
$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{\frac{p\alpha_1+\beta_1(1-p)}{p-1-\alpha_1}}(L_1K^{\alpha_1} + L_2K^{\alpha_2})(1-x).$$
Furthermore, the function $x \to (L_1K^{\alpha_1} + L_2K^{\alpha_2})(1-x) \in \mathcal{K}$. Now using that $\frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p$, we deduce that
$$\frac{-p\alpha_1+\beta_1(p-1)}{p-1-\alpha_1} \in (\mu+1, p).$$
So applying Lemma 2.6, for $\beta = \frac{-p\alpha_1+\beta_1(p-1)}{p-1-\alpha_1}$ and $L = L_1K^{\alpha_1} + L_2K^{\alpha_2}$, we obtain that
$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-\beta_1}{p-1-\alpha_1}}(L_1K^{\alpha_1} + L_2K^{\alpha_2})^{\frac{1}{p-1}}(1-x).$$
Hence, since $\frac{\alpha_i}{p-1} < 1$ for $i \in \{1,2\}$ and
$$(s+t)^{\frac{1}{p-1}} \approx s^{\frac{1}{p-1}} + t^{\frac{1}{p-1}}, \quad \text{for } s,t > 0, \tag{2.2}$$

applying Lemma 2.7 for $t = L_1(x)^{\frac{1}{p-1-\alpha_1}}$, $s = L_2(x)^{\frac{1}{p-1-\alpha_2}}$ and $\eta_i = \frac{\alpha_i}{p-1}$, ($i \in \{1,2\}$), we obtain that

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-\beta_1}{p-1-\alpha_1}} K(1-x) = \theta(x).$$

Case 4. $\beta_1 = \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1}$ and $\beta_2 = \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}$. Using (H2) we have

$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{-(1+\mu)}(L_1N^{\alpha_1} + L_2N^{\alpha_2})(1-x).$$

Furthermore, the function $x \to (L_1N^{\alpha_1} + L_2N^{\alpha_2})(1-x) \in \mathcal{K}$. So applying Lemma 2.6 for $\beta = 1+\mu$ and $L = L_1N^{\alpha_1} + L_2N^{\alpha_2}$, we obtain that

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}} \Big(\int_{1-x}^{\eta} \frac{(L_1N^{\alpha_1} + L_2N^{\alpha_2})(t)}{t} dt\Big)^{\frac{1}{p-1}}.$$

Since

$$N^{p-1}(t) \approx \Big(\int_t^{\eta} \frac{L_1(s)}{s} ds\Big)^{\frac{p-1}{p-1-\alpha_1}} + \Big(\int_t^{\eta} \frac{L_2(s)}{s} ds\Big)^{\frac{p-1}{p-1-\alpha_2}},$$

it follows that $N^{p-1} \approx J$, where J is the function given in Lemma 2.4, for $\eta_i = \frac{\alpha_i}{p-1}, i \in \{1,2\}$. So, we have

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}} \Big(\int_{1-x}^{\eta} \frac{(L_1J^{\eta_1} + L_2J^{\eta_2})(t)}{t} dt\Big)^{\frac{1}{p-1}}.$$

Hence, since $\eta_i < 1$ for $i \in \{1,2\}$, by Lemma 2.4, it follows that

$$\begin{aligned} G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) &\approx (1-x)^{\frac{p-1-\mu}{p-1}} J^{\frac{1}{p-1}}(1-x) \\ &\approx (1-x)^{\frac{p-1-\mu}{p-1}} N(1-x) = \theta(x). \end{aligned}$$

Case 5. $\beta_1 = \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1}$ and $\beta_2 < \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}$. Let $b(1-x) = \Big(\int_{1-x}^{\eta} \frac{L_1(s)}{s} ds\Big)^{\frac{1}{p-1-\alpha_1}}$. Using (H2), we have

$$\begin{aligned} &a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \\ &\approx (1-x)^{-(1+\mu)}(L_1b^{\alpha_1})(1-x) + (1-x)^{-\beta_2+\alpha_2\frac{p-1-\mu}{p-1}}(L_2b^{\alpha_2})(1-x). \end{aligned}$$

Now, since for $i \in \{1,2\}$, the function $x \to (L_ib^{\alpha_i})(1-x) \in \mathcal{K}$ and $-(1+\mu) < -\beta_2 + \alpha_2\frac{p-1-\mu}{p-1}$, it follows from Lemma 2.1(i) that

$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{-(1+\mu)}(L_1b^{\alpha_1})(1-x).$$

Applying again Lemma 2.6, for $\beta = 1+\mu$ and $L = L_1b^{\alpha_1}$, we obtain

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}} \Big(\int_{1-x}^{\eta} \frac{(L_1b^{\alpha_1})(s)}{s} ds\Big)^{\frac{1}{p-1-\alpha_1}} \approx \theta(x).$$

Case 6. $\beta_1 < \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1}$. Using (H2), we have

$$a_1(x)\theta^{\alpha_1}(x) \approx (1-x)^{-\beta_1+\alpha_1\frac{p-1-\mu}{p-1}} L_1(1-x).$$

Applying Lemma 2.6 for $\beta = \beta_1 - \alpha_1\frac{p-1-\mu}{p-1} < 1+\mu$ and $L = L_1$, we deduce that

$$G_p(a_1\theta^{\alpha_1})(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}.$$

Moreover, since $\frac{p-1-\mu}{p-1} < \frac{p-\beta_1}{p-1-\alpha_1} \leq \frac{p-\beta_2}{p-1-\alpha_2}$, it follows that $\beta_2 < \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}$.

So, in the same manner we obtain

$$G_p(a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}.$$

Then, we conclude that

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}} = \theta(x).$$

Case 7. $\beta_1 = \beta_2 = p$. Using (H2), we have

$$a_1(x)\theta^{\alpha_1}(x) + a_2(x)\theta^{\alpha_2}(x) \approx (1-x)^{-p}(L_1M^{\alpha_1} + L_2M^{\alpha_2})(1-x).$$

Furthermore, the function $x \to (L_1M^{\alpha_1} + L_2M^{\alpha_2})(1-x) \in \mathcal{K}$. Now, since $\frac{\alpha_i}{p-1} < 1$ and $\int_0^\eta \frac{(L_i(s))^{\frac{1}{p-1}}}{s} ds < \infty$, for $i \in \{1,2\}$, applying (2.2) and Lemma 2.5 for $\eta_i = \frac{\alpha_i}{p-1}$, we obtain

$$\begin{aligned}\int_0^\eta \frac{(L_1M^{\alpha_1} + L_2M^{\alpha_2})^{\frac{1}{p-1}}(t)}{t} dt &\approx \int_0^\eta \frac{(L_1^{\frac{1}{p-1}} M^{\frac{\alpha_1}{p-1}} + L_2^{\frac{1}{p-1}} M^{\frac{\alpha_2}{p-1}})(t)}{t} dt \\ &\approx M(\eta) < \infty.\end{aligned}$$

So applying Lemma 2.6 for $\beta = p$ and $L = L_1M^{\alpha_1} + L_2M^{\alpha_2}$, by (2.2) we deduce that

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx \int_0^{1-x} \frac{((L_1M^{\alpha_1})^{\frac{1}{p-1}} + (L_2M^{\alpha_2})^{\frac{1}{p-1}})(t)}{t} dt.$$

Then, using again Lemma 2.5 we conclude that

$$G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \approx M(1-x) = \theta(x).$$

□

Proposition 2.9. *Assume* (H0)–(H1) *hold. Then the family of functions*

$$\mathcal{F}_q = \{x \to G_pf(x); f \in B((0,1)), |f| \leq q\}$$

is uniformly bounded and equicontinuous in $[0,1]$. *Consequently* $\mathcal{F}_q$ *is relatively compact in* $C([0,1])$.

Proof. Let f be a measurable function such that $|f(x)| \leq q(x), x \in (0,1)$. By Proposition 2.6, we have

$$|G_pf(x)| \leq G_pq(x) \approx (1-x)^{\min(\frac{p-\beta}{p-1}, \frac{p-1-\mu}{p-1})}\Psi_{L,\beta,0}(x).$$

From Lemma 2.1 and Lemma 2.3, the function $x \to (1-x)^{\min(\frac{p-\beta}{p-1}, \frac{p-1-\mu}{p-1})}\Psi_{L,\beta,0}(x)$ is continuous on $[0,1)$ and tends to zero as $x \to 1$. Then, we prove that $\mathcal{F}_q$ is uniformly bounded and $\lim_{x\to 1} G_p(f)(x) = 0$, uniformly on f.

Moreover, let $x, x' \in [0,1]$ such that $x < x'$. Then

$$|G_pf(x) - G_pf(x')| \leq \int_x^{x'} \Big(\frac{1}{A(t)} \int_0^t A(s)q(s)ds\Big)^{\frac{1}{p-1}} dt.$$

Since $G_pq(0) < \infty$, it follows by the dominated convergence theorem, the equicontinuity of $\mathcal{F}_q$ in $[0,1]$. Hence, by Ascoli's theorem, we deduce that $\mathcal{F}_q$ is relatively compact in $C([0,1])$. □

3. Proof of main result

3.1. **Existence and boundary behavior.** Assume (H0) and (H2) hold. Let θ be the function given by (1.8). By Proposition 2.8, there exists a constant $m \geq 1$ such that for each $x \in [0,1)$,

$$\frac{1}{m}\theta(x) \leq G_p(a_1\theta^{\alpha_1} + a_2\theta^{\alpha_2})(x) \leq m\theta(x). \tag{3.1}$$

Now, we look at the existence of positive solution of problem (1.6) satisfying $u(x) \approx \theta(x)$. We will use a fixed-point argument. We consider the set

$$\Gamma := \{u \in C([0,1]) : \frac{1}{c_0}\theta \leq u \leq c_0\theta\},$$

where $c_0 = m^{\frac{p-1}{p-1-\max(|\alpha_1|,|\alpha_2|)}}$. Obviously, the set Γ is not empty. We consider the integral operator T on Γ defined by

$$Tu(x) := G_p(a_1u^{\alpha_1} + a_2u^{\alpha_2})(x), \quad x \in [0,1].$$

First, we observe that $T\Gamma \subset \Gamma$. Indeed, let $u \in \Gamma$, then for $i \in \{1,2\}$, we have

$$c_0^{-|\alpha_i|}a_i(x)\theta^{\alpha_i}(x) \leq a_i(x)u^{\alpha_i}(x) \leq c_0^{|\alpha_i|}a_i(x)\theta^{\alpha_i}(x), \quad x \in [0,1).$$

This and (3.1) imply

$$\frac{1}{mc_o^{\frac{max(|\alpha_1|,|\alpha_2|)}{p-1}}}\theta(x) \leq Tu(x) \leq mc_o^{\frac{max(|\alpha_1|,|\alpha_2|)}{p-1}}\theta(x).$$

Since $mc_o^{\frac{max(|\alpha_1|,|\alpha_2|)}{p-1}} = c_0$ and $T\Gamma \subset C([0,1])$, it follows that T leaves invariant the convex Γ.

Next, let $q = c_0^{|\alpha_1|}a_1\theta^{\alpha_1} + c_0^{|\alpha_2|}a_2\theta^{\alpha_2}$. By the proof of Proposition 2.8, the function q satisfies (H1). Since $T\Gamma$ is a closed set of $\mathcal{F}_q$, it follows by Proposition 2.9, that $T\Gamma$ is relatively compact in $C([0,1])$.

Now, we shall prove the continuity of operator T in Γ. Let (u_n) be a sequence in Γ converging uniformly to u in Γ. Then, by applying the dominated convergence theorem, we conclude that for each $x \in [0,1], Tu_n(x) \to Tu(x)$ as $n \to +\infty$. Consequently, as $T\Gamma$ is relatively compact in $C([0,1])$, we deduce that the pointwise convergence implies the uniform convergence. Namely, $\|Tu_n - Tu\|_\infty \to 0$ as $n \to +\infty$.

Therefore, T is a compact operator from Γ into itself. So the Schauder fixed-point theorem implies the existence of $u \in \Gamma$ such that

$$u = G_p(a_1u^{\alpha_1} + a_2u^{\alpha_2})(x), \ x \in [0,1].$$

We conclude that u is a positive continuous solution of problem (1.6) which satisfies (1.9).

3.2. **Uniqueness.** Let u and v be two solutions of (1.6) in Γ. Then, there exists a constant $M > 1$ such that

$$\frac{1}{M} \leq \frac{u}{v} \leq M.$$

This implies that the set

$$J = \{t \in (1,\infty) : \frac{1}{t}u \leq v \leq tu\}$$

is not empty. Now, put $c := \inf J$. We aim to show that $c = 1$. Suppose that $c > 1$ and let $\alpha = max(|\alpha_1|, |\alpha_2|)$, then we have

$$\begin{gathered}
-\frac{1}{A}(A\Phi_p(v'))' + \frac{1}{A}(A\Phi_p(c^{\frac{-\alpha}{p-1}}u'))' \\
= a_1(x)(v^{\alpha_1} - c^{-\alpha}u^{\alpha_1}) + a_2(x)(v^{\alpha_2} - c^{-\alpha}u^{\alpha_2}) \geq 0, \quad \text{in } (0,1), \\
\lim_{x\to 0}(A\Phi_p(v') - A\Phi_p(c^{\frac{-\alpha}{p-1}}u'))(x) = 0, \\
u(1) = v(1) = 0.
\end{gathered}$$

This implies that the function

$$\psi(x) := A\Phi_p(c^{\frac{-\alpha}{p-1}}u') - A\Phi_p(v')(x)$$

is nondecreasing on $(0,1)$ with $\lim_{x\to 0}\psi(x) = 0$. Hence from the monotonicity of Φ_p, we obtain that the function $c^{\frac{-\alpha}{p-1}}u - v$ is nondecreasing on $(0,1)$ satisfying $(c^{\frac{-\alpha}{p-1}}u - v)(1) = 0$. This implies that $c^{\frac{-\alpha}{p-1}}u \leq v$. On the other hand, we deduce by symmetry that $v \leq c^{\frac{\alpha}{p-1}}u$. Then, we have $c^{\frac{\alpha}{p-1}} \in J$. Now, since $\alpha \in (0, p-1)$, it follows that $c^{\frac{\alpha}{p-1}} < c$ and this yields to a contradiction with the definition of c. Hence, $c = 1$ and so $u = v$.

Example 3.1. Let $1-p < \alpha_1 < 0 < \alpha_2 < p-1$ and $\beta_1, \beta_2 < p$ such that $\frac{p-\beta_1}{p-1-\alpha_1} \leq \frac{p-\beta_2}{p-1-\alpha_2}$. Let a_1, a_2 be continuous functions on $(0,1)$ such that $a_i(x) \approx (1-x)^{-\beta_i}$, for $i \in \{1,2\}$. Then problem (1.6) has a unique continuous solution u satisfying for $x \in (0,1)$,

$$u(x) \approx \begin{cases}
(1-x)^{\frac{p-\beta_1}{p-1-\alpha_1}} \quad \text{if } \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1} < \beta_1 < p, \\
(1-x)^{\frac{p-1-\mu}{p-1}}(\log(\frac{\eta}{1-x}))^{\frac{1}{p-1-\alpha_2}} \\
\quad \text{if } \beta_1 = \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1},\ \beta_2 = \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}, \\
(1-x)^{\frac{p-1-\mu}{p-1}}(\log(\frac{\eta}{1-x}))^{\frac{1}{p-1-\alpha_1}} \\
\quad \text{if } \beta_1 = \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1},\ \beta_2 < \frac{(\mu+1)(p-1-\alpha_2)+p\alpha_2}{p-1}, \\
(1-x)^{\frac{p-1-\mu}{p-1}} \quad \text{if } \beta_1 < \frac{(\mu+1)(p-1-\alpha_1)+p\alpha_1}{p-1}.
\end{cases}$$

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Syrine Masmoudi
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: syrine.sassi@fst.rnu.tn

Samia Zermani
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: zermani.samia@yahoo.fr