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BLOW-UP OF SOLUTIONS TO PARABOLIC INEQUALITIES IN THE HEISENBERG GROUP

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Abstract. We establish a Fujita-type theorem for the blow-up of nonnegative solutions to a certain class of parabolic inequalities in the Heisenberg group. Our proof is based on a duality argument.

1. Introduction

In this article, we establish a Fujita-type theorem for parabolic inequality

$$\begin{gathered} u_t - \operatorname{div}_{\mathbb{H}} A(\vartheta, u, \nabla_{\mathbb{H}} u) + f(\vartheta, u, \nabla_{\mathbb{H}} u) \geq u^q, \quad \text{in } \mathcal{H}, \\ u \geq 0, \quad \text{a.e. in } \mathcal{H}, \\ u(\vartheta, 0) = u_0(\vartheta), \quad \text{in } \mathbb{H}. \end{gathered} \tag{1.1}$$

Here, $\mathbb{H}$ is the $(2N+1)$-dimensional Heisenberg group, $\mathcal{H} = \mathbb{H} \times (0, \infty)$ and $u_0 \in L^1_{\rm loc}(\mathbb{H})$. The operator $A : \mathbb{H} \times \mathbb{R} \times \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$ is a Carathéodory function satisfying

$$(A(\vartheta, \xi, v), v) \geq c_A |A(\vartheta, \xi, v)|^{m'}, \tag{1.2}$$

where $c_A > 0$, $(\cdot, \cdot)$ is the standard scalar product in $\mathbb{R}^{2N+1}$, $|\cdot| = \sqrt{(\cdot, \cdot)}$, and $m' > 1$. The function $f : \mathbb{H} \times \mathbb{R} \times \mathbb{R}^{2N+1} \to \mathbb{R}$ is continuous and satisfies

$$f(\vartheta, \xi, v) \leq \lambda |A(\vartheta, \xi, v)|^{\sigma}, \tag{1.3}$$

where $\lambda \geq 0$, $\sigma = \frac{m'q}{q+1}$, and $q > \max\{1, m-1\}$, with $m = \frac{m'}{m'-1}$. The proof of our main result is based on a duality argument [4, 5, 6].

First, let us recall some background facts that will be used in this article. The $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$ is the space $\mathbb{R}^{2N+1}$ endowed with the group operation

$$\vartheta \diamond \vartheta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),$$

for all $\vartheta = (x, y, \tau), \vartheta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

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The distance from an element $\vartheta = (x, y, \tau) \in \mathbb{H}$ to the origin is given by

$$|\vartheta|_{\mathbb{H}} = \Big(\tau^2 + \Big(\sum_{i=1}^{N} x_i^2 + y_i^2\Big)^2\Big)^{1/4},$$

where $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$.

The Gradient $\nabla_{\mathbb{H}}$ over $\mathbb{H}$ is defined by

$$\nabla_{\mathbb{H}} = (X_1, .., X_N, Y_1, .., Y_N),$$

where for $i = 1, \dots, N$,

$$X_i = \partial_{x_i} + 2y_i\partial_\tau \quad \text{and} \quad Y_i = \partial_{y_i} - 2x_i\partial_\tau.$$

Let

$$M = \begin{pmatrix} I_N & 0 & 2y \\ 0 & I_N & -2x \end{pmatrix},$$

where I_N is the identity matrix of size N. Then

$$\nabla_{\mathbb{H}} = M\nabla_{\mathbb{R}^{2N+1}}.$$

A simple computation gives the expression

$$|\nabla_{\mathbb{H}} u|^2 = 4(|x|^2 + |y|^2)(\partial_\tau u)^2 + \sum_{i=1}^{N}\Big((\partial_{x_i} u)^2 + (\partial_{y_i} u)^2 + 4\partial_\tau u(y_i\partial_{x_i} u - x_i\partial_{y_i} u)\Big).$$

The divergence operator in $\mathbb{H}$ is defined by

$$\operatorname{div}_{\mathbb{H}}(u) = \operatorname{div}_{\mathbb{R}^{2N+1}}(Mu).$$

For more details on Heisenberg groups and partial differential equations in Heisenberg groups, we refer to [1, 2, 3, 8, 9] and references therein.

For the proof of our main result, the following inequality will be used several times.

Lemma 1.1. *Let $a, b, \varepsilon > 0$. Then*

$$ab \le \varepsilon a^p + c_\varepsilon b^{p'},$$

where $p > 1$, p' is its corresponding conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$; and $c_\varepsilon = \big(\frac{1}{\varepsilon p}\big)^{p'/p}\frac{1}{p'}$.

2. Main result

Definition 2.1. Let $u \in W^{1,m}_{\rm loc}(\mathcal{H}; \mathbb{R}_+) \cap L^q_{\rm loc}(\mathcal{H}; \mathbb{R}_+)$ and $u_0 \in L^1_{\rm loc}(\mathbb{H}; \mathbb{R}_+)$. We say that u is a global weak solution of (1.1) if the following conditions are satisfied:

(i) $A(\vartheta, u, \nabla_{\mathbb{H}} u) \in L^{m'}_{\rm loc}(\mathcal{H}; \mathbb{R}^{2N+1})$;
(ii) For any $\varphi \in W^{1,m}_{\rm loc}(\mathcal{H}; \mathbb{R}_+)$ with a compact support,

$$\begin{aligned} \int_{\mathcal{H}} u^q\varphi\, d\mathcal{H} \le \int_{\mathcal{H}} (A(\vartheta, u, \nabla_{\mathbb{H}} u), \nabla_{\mathbb{H}}\varphi)\, d\mathcal{H} + \int_{\mathcal{H}} f(\vartheta, u, \nabla_{\mathbb{H}} u)\varphi\, d\mathcal{H} \\ - \int_{\mathcal{H}} u\varphi_t\, d\mathcal{H} - \int_{\mathbb{H}} u_0(\vartheta)\varphi(\vartheta, 0)\, d\vartheta. \end{aligned} \tag{2.1}$$

Observe that all the integrals in (2.1) are well defined. Our main result is given in the following theorem.

Theorem 2.2. *Assume that conditions* (1.2) *and* (1.3) *are satisfied. Let us consider* $\alpha \in (\alpha_0, 0)$*, where* $\alpha_0 = \max\{-1, 1-m\} < 0$*. If*

$$0 \le \lambda < \lambda^* = (q+1)\Big(\frac{|\alpha| c_A}{q}\Big)^{\frac{q}{q+1}} \tag{2.2}$$

and

$$\max\{1, m-1\} < q < m - 1 + \frac{m}{Q}, \tag{2.3}$$

where $Q = 2N+2$ *is the homogeneous dimension of* $\mathbb{H}$*, then* (1.1) *has no global nontrivial weak solutions.*

The following lemma provides a preliminary estimate of solutions.

Lemma 2.3. *Suppose that all the assumptions of Theorem 2.2 are satisfied. Let* u *be a global weak solution to* (1.1). *Then for any* $\alpha \in (\alpha_0, 0)$ *and any* $\varphi \in W^{1,\infty}(\mathcal{H}; \mathbb{R}_+)$*, we have*

$$\begin{aligned} &\int_{\mathcal{H}} u^{q+\alpha}\varphi \, d\mathcal{H} + \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u^{\alpha-1}\varphi \, d\mathcal{H} + \int_{\mathbb{H}} u_0(\vartheta)^{\alpha+1}\varphi(\vartheta, 0)\, d\vartheta \\ &\le C\Big(\int_{\mathcal{H}} \Big(\frac{|\varphi_t|^r}{\varphi}\Big)^{\frac{1}{r-1}} d\mathcal{H} + \int_{\mathcal{H}} \varphi^{1-ms}|\nabla_{\mathbb{H}}\varphi|^{ms}\, d\mathcal{H}\Big), \end{aligned} \tag{2.4}$$

for some constant $C > 0$*, where* $r = \frac{q+\alpha}{1+\alpha}$*, and* $s = \frac{q+\alpha}{q-m+1}$*.*

Proof. Let $\varepsilon > 0$ be fixed and $\alpha \in (\alpha_0, 0)$. Suppose that u is a global weak solution to (1.1). Let

$$u_\varepsilon(\vartheta, t) = u(\vartheta, t) + \varepsilon, \quad (\vartheta, t) \in \mathcal{H}.$$

Define φ_ε as

$$\varphi_\varepsilon(\vartheta, t) = u_\varepsilon^\alpha(\vartheta, t)\varphi(\vartheta, t),$$

where $\varphi \in W^{1,\infty}(\mathcal{H}; \mathbb{R}_+)$ has a compact support. Observe that φ_ε belongs to the set of admissible test functions in the sense of Definition 2.1. By (2.1), we have

$$\begin{aligned} &\int_{\mathcal{H}} u^q u_\varepsilon^\alpha \varphi \, d\mathcal{H} \\ &\le \alpha \int_{\mathcal{H}} (A(\vartheta, u, \nabla_{\mathbb{H}} u), \nabla_{\mathbb{H}} u) u_\varepsilon^{\alpha-1}\varphi \, d\mathcal{H} + \int_{\mathcal{H}} (A(\vartheta, u, \nabla_{\mathbb{H}} u), \nabla_{\mathbb{H}}\varphi) u_\varepsilon^\alpha \, d\mathcal{H} \\ &\quad + \int_{\mathcal{H}} f(\vartheta, u, \nabla_{\mathbb{H}} u) u_\varepsilon^\alpha \varphi \, d\mathcal{H} - \frac{1}{\alpha+1}\int_{\mathcal{H}} u_\varepsilon^{\alpha+1}\varphi_t \, d\mathcal{H} \\ &\quad - \frac{1}{\alpha+1}\int_{\mathbb{H}} (u_0(\vartheta) + \varepsilon)^{\alpha+1}\varphi(\vartheta, 0)\, d\vartheta. \end{aligned} \tag{2.5}$$

Using the condition (1.2), we obtain

$$\int_{\mathcal{H}} (A(\vartheta, u, \nabla_{\mathbb{H}} u), \nabla_{\mathbb{H}} u) u_\varepsilon^{\alpha-1}\varphi \, d\mathcal{H} \ge c_A \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1}\varphi \, d\mathcal{H}.$$

Since $\alpha < 0$, we have

$$\alpha \int_{\mathcal{H}} (A(\vartheta, u, \nabla_{\mathbb{H}} u), \nabla_{\mathbb{H}} u) u_\varepsilon^{\alpha-1}\varphi \, d\mathcal{H} \le c_A \alpha \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1}\varphi \, d\mathcal{H}. \tag{2.6}$$

The Cauchy-Schwarz inequality yields

$$\int_{\mathcal{H}} (A(\vartheta, u, \nabla_{\mathbb{H}} u), \nabla_{\mathbb{H}}\varphi) u_\varepsilon^\alpha \, d\mathcal{H} \le \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)||\nabla_{\mathbb{H}}\varphi| u_\varepsilon^\alpha \, d\mathcal{H}. \tag{2.7}$$

Using the condition (1.3), we obtain

$$\int_{\mathcal{H}} f(\vartheta, u, \nabla_{\mathbb{H}} u) u_\varepsilon^\alpha \varphi \, d\mathcal{H} \le \lambda \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^\sigma u_\varepsilon^\alpha \varphi \, d\mathcal{H}. \tag{2.8}$$

Now, (2.5), (2.6), (2.7) and (2.8) yield

$$\begin{aligned} &\int_{\mathcal{H}} u^q u_\varepsilon^\alpha \varphi \, d\mathcal{H} + c_A |\alpha| \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} \\ &+ \frac{1}{\alpha+1} \int_{\mathbb{H}} (u_0(\vartheta) + \varepsilon)^{\alpha+1} \varphi(\vartheta, 0) \, d\vartheta \\ &\le \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)| |\nabla_{\mathbb{H}} \varphi| u_\varepsilon^\alpha \, d\mathcal{H} + \lambda \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^\sigma u_\varepsilon^\alpha \varphi \, d\mathcal{H} \\ &\quad + \frac{1}{\alpha+1} \int_{\mathcal{H}} u_\varepsilon^{\alpha+1} |\varphi_t| \, d\mathcal{H}. \end{aligned} \tag{2.9}$$

Now, using Lemma 1.1, we estimate the individual terms on the right hand side of (2.9).

• Estimation of $\int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)| |\nabla_{\mathbb{H}} \varphi| u_\varepsilon^\alpha \, d\mathcal{H}$. We have

$$|A(\vartheta, u, \nabla_{\mathbb{H}} u)| |\nabla_{\mathbb{H}} \varphi| u_\varepsilon^\alpha = \Big(|A(\vartheta, u, \nabla_{\mathbb{H}} u)| u_\varepsilon^{\frac{\alpha-1}{m'}} \varphi^{\frac{1}{m'}} \Big) \Big(u_\varepsilon^{\frac{\alpha+m-1}{m}} \varphi^{\frac{-1}{m'}} |\nabla_{\mathbb{H}} \varphi| \Big).$$

Applying Lemma 1.1 with parameters m' and m, we obtain

$$\begin{aligned} &\int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)| |\nabla_{\mathbb{H}} \varphi| u_\varepsilon^\alpha \, d\mathcal{H} \\ &\le \varepsilon_1 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_1} \int_{\mathcal{H}} u_\varepsilon^{\alpha+m-1} \varphi^{1-m} |\nabla_{\mathbb{H}} \varphi|^m \, d\mathcal{H}, \end{aligned}$$

for some $\varepsilon_1 > 0$. Again, writing

$$u_\varepsilon^{\alpha+m-1} \varphi^{1-m} |\nabla_{\mathbb{H}} \varphi|^m = \Big(\varphi^{\frac{1-ms}{s}} |\nabla_{\mathbb{H}} \varphi|^m \Big) \Big(\varphi^{\frac{s-1}{s}} u_\varepsilon^{\alpha+m-1} \Big)$$

and using Lemma 1.1 with parameters s and s', we obtain

$$\int_{\mathcal{H}} u_\varepsilon^{\alpha+m-1} \varphi^{1-m} |\nabla_{\mathbb{H}} \varphi|^m \, d\mathcal{H} \le \varepsilon_2 \int_{\mathcal{H}} \varphi^{1-ms} |\nabla_{\mathbb{H}} \varphi|^{ms} \, d\mathcal{H} + c_{\varepsilon_2} \int_{\mathcal{H}} \varphi u_\varepsilon^{(\alpha+m-1)s'} \, d\mathcal{H},$$

for some $\varepsilon_2 > 0$. As consequence, we have

$$\begin{aligned} &\int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)| |\nabla_{\mathbb{H}} \varphi| u_\varepsilon^\alpha \, d\mathcal{H} \\ &\le \varepsilon_1 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} \\ &\quad + c_{\varepsilon_1} \varepsilon_2 \int_{\mathcal{H}} \varphi^{1-ms} |\nabla_{\mathbb{H}} \varphi|^{ms} \, d\mathcal{H} + c_{\varepsilon_1} c_{\varepsilon_2} \int_{\mathcal{H}} \varphi u_\varepsilon^{(\alpha+m-1)s'} \, d\mathcal{H}. \end{aligned} \tag{2.10}$$

• Estimation of $\int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^\sigma u_\varepsilon^\alpha \varphi \, d\mathcal{H}$. We write

$$|A(\vartheta, u, \nabla_{\mathbb{H}} u)|^\sigma u_\varepsilon^\alpha \varphi = \Big(|A(\vartheta, u, \nabla_{\mathbb{H}} u)|^\sigma u_\varepsilon^{\frac{(\alpha-1)\sigma}{m'}} \varphi^{\frac{\sigma}{m'}} \Big) \Big(\varphi^{\frac{m'-\sigma}{m'}} u_\varepsilon^{\frac{\alpha m' - (\alpha-1)\sigma}{m'}} \Big).$$

We apply Lemma 1.1 with parameters $\frac{m'}{\sigma}$ and $\frac{m'}{m'-\sigma}$ to obtain

$$\int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^\sigma u_\varepsilon^\alpha \varphi \, d\mathcal{H}$$

$$\leq \varepsilon_3 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_3} \int_{\mathcal{H}} \varphi u_\varepsilon^{\alpha - 1 + \frac{m'}{m'-\sigma}} \, d\mathcal{H},$$

for some $\varepsilon_3 > 0$. Since $\sigma = m'q/(q+1)$, we obtain

$$\begin{aligned} &\int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{\sigma} u_\varepsilon^{\alpha} \varphi \, d\mathcal{H} \\ &\leq \varepsilon_3 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_3} \int_{\mathcal{H}} \varphi u_\varepsilon^{\alpha+q} \, d\mathcal{H}. \end{aligned} \tag{2.11}$$

• Estimation of $\int_{\mathcal{H}} u_\varepsilon^{\alpha+1} |\varphi_t| \, d\mathcal{H}$. Similarly, we write

$$u_\varepsilon^{\alpha+1} |\varphi_t| = \left(u_\varepsilon^{\alpha+1} \varphi^{\frac{1}{r}} \right) \left(\varphi^{-\frac{1}{r}} |\varphi_t| \right).$$

Lemma 1.1 with parameters r and r' yields

$$\int_{\mathcal{H}} u_\varepsilon^{\alpha+1} |\varphi_t| \, d\mathcal{H} \leq \varepsilon_4 \int_{\mathcal{H}} u_\varepsilon^{(\alpha+1)r} \varphi \, d\mathcal{H} + c_{\varepsilon_4} \int_{\mathcal{H}} \Big(\frac{|\varphi_t|^r}{\varphi} \Big)^{\frac{1}{r-1}} \, d\mathcal{H}, \tag{2.12}$$

for some $\varepsilon_4 > 0$. Now, substituting (2.10), (2.11) and (2.12) in (2.9), we obtain

$$\begin{aligned} &\int_{\mathcal{H}} u^q u_\varepsilon^{\alpha} \varphi \, d\mathcal{H} + c_A |\alpha| \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} \\ &+ \frac{1}{\alpha+1} \int_{\mathbb{H}} (u_0(\vartheta) + \varepsilon)^{\alpha+1} \varphi(\vartheta, 0) \, d\vartheta \\ &\leq \varepsilon_1 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_1} \varepsilon_2 \int_{\mathcal{H}} \varphi^{1-ms} |\nabla_{\mathbb{H}} \varphi|^{ms} \, d\mathcal{H} \\ &\quad + c_{\varepsilon_1} c_{\varepsilon_2} \int_{\mathcal{H}} \varphi u_\varepsilon^{(\alpha+m-1)s'} \, d\mathcal{H} + \lambda \varepsilon_3 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u_\varepsilon^{\alpha-1} \varphi \, d\mathcal{H} \\ &\quad + \lambda c_{\varepsilon_3} \int_{\mathcal{H}} \varphi u_\varepsilon^{\alpha+q} \, d\mathcal{H} + \frac{\varepsilon_4}{\alpha+1} \int_{\mathcal{H}} u_\varepsilon^{(\alpha+1)r} \varphi \, d\mathcal{H} + \frac{c_{\varepsilon_4}}{\alpha+1} \int_{\mathcal{H}} \Big(\frac{|\varphi_t|^r}{\varphi} \Big)^{\frac{1}{r-1}} \, d\mathcal{H}. \end{aligned}$$

Now, we let $\varepsilon \to 0$ in the obtained inequality, we use Fatou's lemma and the dominated convergence theorem to obtain

$$\begin{aligned} &\int_{\mathcal{H}} u^{q+\alpha} \varphi \, d\mathcal{H} + c_A |\alpha| \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u^{\alpha-1} \varphi \, d\mathcal{H} + \frac{1}{\alpha+1} \int_{\mathbb{H}} u_0^{\alpha+1}(\vartheta) \varphi(\vartheta, 0) \, d\vartheta \\ &\leq \varepsilon_1 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_1} \varepsilon_2 \int_{\mathcal{H}} \varphi^{1-ms} |\nabla_{\mathbb{H}} \varphi|^{ms} \, d\mathcal{H} \\ &\quad + c_{\varepsilon_1} c_{\varepsilon_2} \int_{\mathcal{H}} \varphi u^{q+\alpha} \, d\mathcal{H} + \lambda \varepsilon_3 \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u^{\alpha-1} \varphi \, d\mathcal{H} \\ &\quad + \lambda c_{\varepsilon_3} \int_{\mathcal{H}} \varphi u^{\alpha+q} \, d\mathcal{H} + \frac{\varepsilon_4}{\alpha+1} \int_{\mathcal{H}} u^{q+\alpha} \varphi \, d\mathcal{H} + \frac{c_{\varepsilon_4}}{\alpha+1} \int_{\mathcal{H}} \Big(\frac{|\varphi_t|^r}{\varphi} \Big)^{\frac{1}{r-1}} \, d\mathcal{H}, \end{aligned}$$

i.e.,

$$\begin{aligned} &\mathcal{A} \int_{\mathcal{H}} u^{q+\alpha} \varphi \, d\mathcal{H} + \mathcal{B} \int_{\mathcal{H}} |A(\vartheta, u, \nabla_{\mathbb{H}} u)|^{m'} u^{\alpha-1} \varphi \, d\mathcal{H} \\ &+ \frac{1}{\alpha+1} \int_{\mathbb{H}} u_0^{\alpha+1}(\vartheta) \varphi(\vartheta, 0) \, d\vartheta \\ &\leq \max\big\{ c_{\varepsilon_1} \varepsilon_2, \frac{c_{\varepsilon_4}}{\alpha+1} \big\} \Big(\int_{\mathcal{H}} \varphi^{1-ms} |\nabla_{\mathbb{H}} \varphi|^{ms} \, d\mathcal{H} + \int_{\mathcal{H}} \Big(\frac{|\varphi_t|^r}{\varphi} \Big)^{\frac{1}{r-1}} \, d\mathcal{H} \Big), \end{aligned} \tag{2.13}$$

where

$$\mathcal{A} = 1 - \lambda c_{\varepsilon_3} - \frac{\varepsilon_4}{\alpha+1} - c_{\varepsilon_1} c_{\varepsilon_2} \quad \text{and} \quad \mathcal{B} = c_A|\alpha| - \varepsilon_1 - \lambda\varepsilon_3.$$

From Lemma 1.1, we have

$$c_{\varepsilon_1} = \frac{1}{m}\Big(\frac{1}{\varepsilon_1 m'}\Big)^{m'/m}, \quad c_{\varepsilon_2} = \frac{1}{s'}\Big(\frac{1}{\varepsilon_2 s}\Big)^{s'/s},$$
$$c_{\varepsilon_3} = \frac{1}{q+1}\Big(\frac{q}{\varepsilon_3(q+1)}\Big)^{q}, \quad c_{\varepsilon_4} = \frac{1}{r'}\Big(\frac{1}{\varepsilon_4 r}\Big)^{r'/r}.$$

For $\varepsilon_1 > 0$ small enough, taking

$$0 \le \lambda < \frac{c_A|\alpha|}{\varepsilon_3}, \tag{2.14}$$

we obtain $\mathcal{B} > 0$. For $\varepsilon_i > 0$ small enough ($i = 1, 2, 4$), taking

$$0 \le \lambda < \frac{1}{c_{\varepsilon_3}}, \tag{2.15}$$

we obtain $\mathcal{A} > 0$. Now, we choose $\varepsilon_3 > 0$ such that

$$\frac{c_A|\alpha|}{\varepsilon_3} = \frac{1}{c_{\varepsilon_3}},$$

i.e.,

$$\frac{c_A|\alpha|}{\varepsilon_3} = (q+1)\Big(\frac{q}{\varepsilon_3(q+1)}\Big)^{-q}.$$

A simple computation yields

$$\varepsilon_3 = (c_A|\alpha|)^{\frac{1}{q+1}}\Big(\frac{q}{q+1}\Big)^{\frac{q}{q+1}}\Big(\frac{1}{q+1}\Big)^{\frac{1}{q+1}}.$$

We substitute ε_3 into (2.14) (or (2.15)) to get

$$0 \le \lambda < \lambda^* = (q+1)\Big(\frac{|\alpha| c_A}{q}\Big)^{\frac{q}{q+1}}.$$

Thus, for $0 \le \lambda < \lambda^*$ and $\varepsilon_i > 0$ small enough ($i = 1, 2, 4$), we have

$$\mathcal{A} > 0 \quad \text{and} \quad \mathcal{B} > 0. \tag{2.16}$$

Finally, the desired result follows from (2.13) and (2.16) with

$$C = \frac{\max\{c_{\varepsilon_1}\varepsilon_2, \frac{c_{\varepsilon_4}}{\alpha+1}\}}{\min\{\mathcal{A}, \mathcal{B}, \frac{1}{\alpha+1}\}}.$$

The lemma is proved. □

Proof of Theorem 2.2. Suppose that u is a nontrivial global weak solution to (1.1). Let us consider the test function

$$\varphi_R(\vartheta, t) = \varphi_R(x, y, \tau, t) = \phi^{\omega}\Big(\frac{t^{2\theta_1} + |x|^{4\theta_2} + |y|^{4\theta_2} + \tau^{2\theta_2}}{R^{4\theta_2}}\Big), \quad R > 0,\ \omega \gg 1,$$

where $\phi \in C_0^{\infty}(\mathbb{R}^+)$ is a decreasing function satisfying

$$\phi(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1 \\ 0 & \text{if } z \ge 2, \end{cases}$$

and θ_j, $j=1,2$ are positive parameters, whose exact values will be specified later. Let

$$\rho = \frac{t^{2\theta_1} + |x|^{4\theta_2} + |y|^{4\theta_2} + \tau^{2\theta_2}}{R^{4\theta_2}}\,.$$

Clearly φ_R is supported on

$$\Omega_R = \{(\vartheta,t) \in \mathcal{H} :\, 0 \le \rho \le 2\},$$

while $(\varphi_R)_t$ and $\nabla_{\mathbb{H}}\varphi_R$ are supported on

$$\Theta_R = \{(\vartheta,t) \in \mathcal{H} :\, 1 \le \rho \le 2\}.$$

A simple computation yields

$$\partial_t \varphi_R(\vartheta,t) = 2\theta_1 \omega t^{2\theta_1 - 1} R^{-4\theta_2} \phi^{\omega-1}(\rho)\phi'(\rho),$$

while

$$\begin{aligned}\nabla_{\mathbb{H}}\varphi_R(t,\vartheta)|^2 &= 16\theta_2^2\omega^2 R^{-8\theta_2}(\phi'(\rho))^2\phi^{2\omega-2}(\rho)\Big((|x|^2+|y|^2)\tau^{4\theta_2-2} \\ &\quad + (|x|^{8\theta_2-2} + |y|^{8\theta_2-2}) + 2\tau^{2\theta_2-1}\sum_{i=1}^{N} x_i y_i (|x|^{4\theta_2-2} - |y|^{4\theta_2-2})\Big).\end{aligned}$$

Then, for all $(\vartheta,t) \in \Omega_R$, we have

$$R|\nabla_{\mathbb{H}}\varphi_R| + R^{2\theta_2/\theta_1}|\partial_t\varphi_R| \le C|\phi'(\rho)|\phi^{\omega-1}(\rho). \tag{2.17}$$

For simplicity, in the sequel, we will write φ in the place of φ_R. Let us consider now the change of variables

$$(x,y,\tau,t) = (\vartheta,t) \mapsto (\widetilde{x},\widetilde{y},\widetilde{\tau},\widetilde{t}) = (\widetilde{\vartheta},\widetilde{t}),$$

where

$$\widetilde{t} = R^{-2\theta_2/\theta_1}t, \quad \widetilde{x} = R^{-1}x, \quad \widetilde{y} = R^{-1}y, \quad \widetilde{\tau} = R^{-2}\tau.$$

In the same way, let

$$\begin{gathered}\widetilde{\rho} = \widetilde{t}^{2\theta_1} + |\widetilde{x}|^{4\theta_2} + |\widetilde{y}|^{4\theta_2} + \widetilde{\tau}^{2\theta_2},\\ \widetilde{\Omega} = \{(\widetilde{x},\widetilde{y},\widetilde{\tau},\widetilde{t}) \in \mathcal{H} : 0 \le \widetilde{\rho} \le 2\},\\ \widetilde{\Theta} = \{(\widetilde{x},\widetilde{y},\widetilde{\tau},\widetilde{t}) \in \mathcal{H} : 1 \le \widetilde{\rho} \le 2\}.\end{gathered}$$

Using the above change of variables together with (2.17), we obtained

$$\int_{\mathcal{H}} \Big(\frac{|\varphi_t|^r}{\varphi}\Big)^{\frac{1}{r-1}}\, d\mathcal{H} \le CR^{Q+2\frac{\theta_2}{\theta_1}(1-\frac{r}{r-1})} \int_{\mathcal{H}} \phi^{\omega-\frac{r}{r-1}}|\phi'|^{\frac{r}{r-1}}\, d\widetilde{\mathcal{H}} \tag{2.18}$$

and

$$\int_{\mathcal{H}} \varphi^{1-ms}|\nabla_{\mathbb{H}}\varphi|^{ms}\, d\mathcal{H} \le CR^{Q+2\frac{\theta_2}{\theta_1}-ms} \int_{\mathcal{H}} \phi^{\omega-ms}|\phi'|^{ms}\, d\widetilde{\mathcal{H}}. \tag{2.19}$$

Setting

$$\frac{\theta_2}{\theta_1} = \frac{ms(r-1)}{2r},$$

we have

$$Q + 2\frac{\theta_2}{\theta_1}(1-\frac{r}{r-1}) = Q + 2\frac{\theta_2}{\theta_1} - ms = Q - \frac{m(q+\alpha)}{q-m+1} + \frac{m(q-1)}{q-m+1}. \tag{2.20}$$

Using (2.4), (2.18)-(2.20), we obtain

$$\int_{\mathcal{H}} u^{q+\alpha}\varphi\, d\mathcal{H} \le CR^{Q-\frac{m(q+\alpha)}{q-m+1}+\frac{m(q-1)k}{q-m+1}}. \tag{2.21}$$

Furthermore, noting that

$$Q - \frac{m(q+\alpha)}{q-m+1} + \frac{m(q-1)}{q-m+1} < 0$$

for $q < m - 1 + \frac{m}{Q}$ and some $\alpha \in (\alpha_0, 0)$ small enough. Under the above condition, letting $R \to \infty$ in (2.21) and using the monotone convergence theorem, we obtain

$$\int_{\mathcal{H}} u^{q+\alpha}\, d\mathcal{H} \le 0,$$

which contradicts our assumption about u. This completes the proof. □

Let us consider now some examples where Theorem 2.2 can be applied.

Corollary 2.4. *If* $\max\{1, m-1\} < q < m - 1 + \frac{m}{Q}$, *then the problem*

$$\begin{gathered} u_t - \operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} u|^{m-2} \nabla_{\mathbb{H}} u\right) \ge u^q \quad \text{in } \mathcal{H}, \\ u \ge 0, \quad \text{a.e. in } \mathcal{H}, \\ u(\vartheta, 0) = u_0(\vartheta), \quad \text{in } \mathbb{H}, \end{gathered}$$

where $u_0 \in L^1_{\rm loc}(\mathbb{H}; \mathbb{R}_+)$, *has no nontrivial global weak solution.*

Proof. The result follows from Theorem 2.2 with $\lambda = 0$ and

$$A(\vartheta, u, \nabla_{\mathbb{H}}) = |\nabla_{\mathbb{H}} u|^{m-2} \nabla_{\mathbb{H}} u.$$

Observe that condition (1.2) is satisfied with $c_A = 1$. □

Take $m = 2$ in Corollary 2.4, we obtain the following Heisenberg version of Fujita theorem (see [7]).

Corollary 2.5. *If* $1 < q < 1 + \frac{2}{Q}$, *then the problem*

$$\begin{gathered} u_t - \Delta_{\mathbb{H}} u \ge u^q \quad \text{in } \mathcal{H}, \\ u \ge 0, \quad \text{a.e. in } \mathcal{H}, \\ u(\vartheta, 0) = u_0(\vartheta), \quad \text{in } \mathbb{H}, \end{gathered}$$

where $u_0 \in L^1_{\rm loc}(\mathbb{H}; \mathbb{R}_+)$, *has no nontrivial global weak solution.*

Corollary 2.6. *If* $1 < q < 1 + \frac{2}{Q}$, *then the problem*

$$\begin{gathered} u_t - \operatorname{div}_{\mathbb{H}}\Big(\frac{|\nabla_{\mathbb{H}} u|}{\sqrt{1 + |\nabla_{\mathbb{H}} u|^2}}\Big) \ge u^q \quad \text{in } \mathcal{H}, \\ u \ge 0, \quad \text{a.e. in } \mathcal{H}, \\ u(\vartheta, 0) = u_0(\vartheta), \quad \text{in } \mathbb{H}, \end{gathered}$$

where $u_0 \in L^1_{\rm loc}(\mathbb{H}; \mathbb{R}_+)$, *has no nontrivial global weak solution.*

Proof. The result follows from Theorem 2.2 with $\lambda = 0$ and

$$A(\vartheta, u, \nabla_{\mathbb{H}}) = \frac{|\nabla_{\mathbb{H}} u|}{\sqrt{1 + |\nabla_{\mathbb{H}} u|^2}}.$$

Observe that condition (1.2) is satisfied with $c_A = 1$. □

Note that Corollary 2.6 is a Heisenberg version of [5, Corollary 33.3].

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