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SOME RELATIONS BETWEEN THE CAPUTO FRACTIONAL DIFFERENCE OPERATORS AND INTEGER-ORDER DIFFERENCES

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Abstract. In this article, we are concerned with the relationships between the sign of Caputo fractional differences and integer nabla differences. In particular, we show that if $N-1<\nu<N$, $f:\mathbb{N}_{a-N+1}\to\mathbb{R}$, $\nabla_{a*}^{\nu}f(t)\geq 0$, for $t\in\mathbb{N}_{a+1}$ and $\nabla^{N-1}f(a)\geq 0$, then $\nabla^{N-1}f(t)\geq 0$ for $t\in\mathbb{N}_a$. Conversely, if $N-1<\nu<N$, $f:\mathbb{N}_{a-N+1}\to\mathbb{R}$, and $\nabla^N f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$, then $\nabla_{a*}^{\nu}f(t)\geq 0$, for each $t\in\mathbb{N}_{a+1}$. As applications of these two results, we get that if $1<\nu<2$, $f:\mathbb{N}_{a-1}\to\mathbb{R}$, $\nabla_{a*}^{\nu}f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$ and $f(a)\geq f(a-1)$, then $f(t)$ is an increasing function for $t\in\mathbb{N}_{a-1}$. Conversely if $0<\nu<1$, $f:\mathbb{N}_{a-1}\to\mathbb{R}$ and f is an increasing function for $t\in\mathbb{N}_a$, then $\nabla_{a*}^{\nu}f(t)\geq 0$, for each $t\in\mathbb{N}_{a+1}$. We also give a counterexample to show that the above assumption $f(a)\geq f(a-1)$ in the last result is essential. These results demonstrate that, in some sense, the positivity of the ν-th order Caputo fractional difference has a strong connection to the monotonicity of $f(t)$.

1. Introduction

If a is a real number, then we use the notation

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\}.$$

If $f:\mathbb{N}_a\to\mathbb{R}$, then we define the nabla (backwards difference) operator by

$$\nabla f(t) = f(t) - f(t-1), \quad t\in\mathbb{N}_{a+1}.$$

Discrete fractional calculus has generated a lot of interest in recent years. Some of the work has employed the delta (forward) or nabla (forward difference) operator. We refer the readers to [3, 7, 1, 13, 11], for example. It seems, however, that more work has been developed for the backward or nabla difference operator and we refer the readers to [8, 12]. There has also been some work to develop relations between the forward and backward fractional operators, Δ_a^{ν} and ∇_a^{ν} [4] (see also [13]) and fractional calculus on time scales [7]. Anastassiou [2] has introduced the study of nabla fractional calculus in the case of Caputo fractional difference.

This work is motivated by the paper by Dahal and Goodrich [9]. They obtained some interesting monotonicity results for the delta fractional difference operator.

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In this paper, we prove the following corresponding results for Caputo fractional differences.

Theorem 1.1. *Assume that $N-1<\nu<N$, $f:\mathbb{N}_{a-N+1}\to\mathbb{R}$, $\nabla_{a^*}^{\nu}f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$ and $\nabla^{N-1}f(a)\geq 0$. Then $\nabla^{N-1}f(t)\geq 0$ for $t\in\mathbb{N}_a$*

Theorem 1.2. *Assume $N-1<\nu<N$, $f:\mathbb{N}_{a-N+1}\to\mathbb{R}$, and $\nabla^N f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$. Then $\nabla_{a^*}^{\nu}f(t)\geq 0$, for each $t\in\mathbb{N}_{a+1}$.*

As applications, we have:

Corollary 1.3. *Assume that $1<\nu<2$, $f:\mathbb{N}_{a-1}\to\mathbb{R}$, $\nabla_{a^*}^{\nu}f(t)\geq 0$ for $t\in\mathbb{N}_{a+1}$ and $f(a)\geq f(a-1)$, then $f(t)$ is an increasing function for $t\in\mathbb{N}_{a-1}$.*

Corollary 1.4. *Assume $0<\nu<1$, $f:\mathbb{N}_{a-1}\to\mathbb{R}$ and f is an increasing function for $t\in\mathbb{N}_a$. Then $\nabla_{a^*}^{\nu}f(t)\geq 0$, for each $t\in\mathbb{N}_{a+1}$.*

We also give a counterexample to show that the above assumption $f(a)\geq f(a-1)$ in Corollary 1.3 is essential.

2. Caputo fractional difference

The following definitions appear in [13, Chapter 3]. First we define the nabla fractional sum of $f:\mathbb{N}_{a+1}\to\mathbb{R}$ based at a.

Definition 2.1. *Let $f:\mathbb{N}_{a+1}\to\mathbb{R}$ be given and $\nu>0$, then the* nabla fractional sum *of f based at a is defined by*

$$\nabla_a^{-\nu}f(t)=\int_a^t H_{\nu-1}(t,\rho(s))f(s)\nabla s, \tag{2.1}$$

for $t\in\mathbb{N}_a$, where by convention $\nabla_a^{-\nu}f(a)=0$.

Next we define the Caputo fractional difference in terms of the nabla fractional sum.

Definition 2.2. *Assume $f:\mathbb{N}_{a-N+1}\to\mathbb{R}$ and $\mu>0$. Then the μ-th Caputo nabla fractional difference of f based at a is defined by*

$$\nabla_{a^*}^{\mu}f(t)=\nabla_a^{-(N-\mu)}\nabla^N f(t)$$

for $t\in\mathbb{N}_{a+1}$, where $N=\lceil\mu\rceil$, $\lceil\cdot\rceil$ the ceiling of number, $m\in\mathbb{N}$.

Let Γ denote the gamma function, then the *rising function* $t^{\overline{r}}$ is defined by

$$t^{\overline{r}}=\frac{\Gamma(t+r)}{\Gamma(t)},$$

for those values of t and r such that the right hand side of this last equation makes sense. We also use the convention that if the numerator is well defined and the denominator is not well defined, then $t^{\overline{r}}:=0$. We define the μ-th degree Taylor monomial based at a by

$$H_\mu(t,a):=\frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)}.$$

We will use the following power rule (see [13, Chapter 3]):

$$\nabla H_\mu(t_0,t)=-H_{\mu-1}(t_0,\rho(t)), \tag{2.2}$$

where $t_0\in\mathbb{N}_a$.

Then (see [13, Chapter 3]) if $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N-1 < \mu < N$, $N \in \mathbb{N}_1$, then the μ-th nabla fractional difference is given by

$$\nabla_a^{\mu} f(t) = \int_a^t H_{-\mu-1}(t, \rho(s)) f(s) \nabla s, \tag{2.3}$$

for $t \in \mathbb{N}_a$, where by convention $\nabla_a^{\mu} f(a) = 0$.

Theorem 2.3. *Assume that $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$, and $\nabla_{a^*}^{\mu} f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, with $N-1 < \mu < N$. Then*

$$\begin{aligned} \nabla^{N-1} f(a+k) \geq \sum_{i=1}^{k-1} \Big[\frac{(k-i+1)^{\overline{N-\mu-2}}}{\Gamma(N-\mu-1)} \Big] \nabla^{N-1} f(a+i-1) \\ + H_{N-\mu-1}(a+k, a) \nabla^{N-1} f(a), \end{aligned} \tag{2.4}$$

for $k \in \mathbb{N}_1$ (note by our convention on sums the first term on the right hand side is zero when $k = 1$).

Proof. If $t = a+1$ we have

$$\begin{aligned} 0 \leq \nabla_{a^*}^{\mu} f(a+1) &= \nabla_a^{-(N-\mu)} \nabla^N f(t) \\ &= \int_a^{a+1} H_{N-\mu-1}(a+1, \rho(s)) \nabla^N f(s) \nabla s \\ &= H_{N-\mu-1}(a+1, a) \nabla^N f(a+1) \\ &= \nabla^N f(a+1) = \nabla^{N-1} f(a+1) - \nabla^{N-1} f(a), \end{aligned}$$

where we used $H_{N-\mu-1}(a+1, a) = 1$. Solving for $\nabla^{N-1} f(a+1)$ we get the inequality

$$\nabla^{N-1} f(a+1) \geq \nabla^{N-1} f(a)$$

which gives us the inequality (2.4) for $t = a+1$. Hence the inequality (2.4) holds for $t = a+1$.

Next consider the case $t = a+k$ for $k \geq 2$. Taking $t = a+k$, $k \geq 2$ we have from (2.1) we have

$$\begin{aligned} 0 &\leq \nabla_{a^*}^{\mu} f(t) \\ &= \nabla_a^{-(N-\mu)} \nabla^N f(t) \\ &= \int_a^t H_{N-\mu-1}(t, \rho(s)) \nabla^N f(s) \nabla s \\ &= \int_a^{a+k} H_{N-\mu-1}(a+k, \rho(s)) \nabla^N f(s) \nabla s \\ &= \sum_{i=1}^{k} H_{N-\mu-1}(a+k, a+i-1) \nabla^N f(a+i) \\ &= \sum_{i=1}^{k} H_{N-\mu-1}(a+k, a+i-1) \left[\nabla^{N-1} f(a+i) - \nabla^{N-1} f(a+i-1) \right] \\ &= \sum_{i=1}^{k} H_{N-\mu-1}(a+k, a+i-1) \nabla^{N-1} f(a+i) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{k} H_{N-\mu-1}(a+k, a+i-1)\nabla^{N-1} f(a+i-1)\\
&= \nabla^{N-1} f(a+k) + \sum_{i=1}^{k-1} H_{N-\mu-1}(a+k, a+i-1)\nabla^{N-1} f(a+i)\\
&- H_{N-\mu-1}(a+k, a)\nabla^{N-1} f(a)\\
&- \sum_{i=2}^{k} H_{N-\mu-1}(a+k, a+i-1)\nabla^{N-1} f(a+i-1),
\end{aligned}
$$

where we used $H_{N-\mu-1}(a+k, a+k-1) = 1$. It follows that

$$
\begin{aligned}
&0 \le \nabla^{N-1} f(a+k) + \sum_{i=1}^{k-1} H_{N-\mu-1}(a+k, a+i-1)\nabla^{N-1} f(a+i)\\
&- H_{N-\mu-1}(a+k, a)\nabla^{N-1} f(a) - \sum_{i=1}^{k-1} H_{N-\mu-1}(a+k, a+i)\nabla^{N-1} f(a+i)\\
&= \nabla^{N-1} f(a+k) - H_{N-\mu-1}(a+k, a)\nabla^{N-1} f(a+i)\\
&- \sum_{i=1}^{k-1} \Big[H_{N-\mu-1}(a+k, a+i)\\
&\qquad\qquad - H_{N-\mu-1}(a+k, a+i-1)\Big]\nabla^{N-1} f(a+i).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 &\le \nabla^{N-1} f(a+k) - H_{N-\mu-1}(a+k, a)\nabla^{N-1} f(a)\\
&\quad - \sum_{i=1}^{k-1} \nabla_s H_{N-\mu-1}(a+k, s)|_{s=a+i}\nabla^{N-1} f(a+i)\\
&\overset{(2.2)}{=} \nabla^{N-1} f(a+k) - H_{N-\mu-1}(a+k, a)\nabla^{N-1} f(a)\\
&\quad + \sum_{i=1}^{k-1} H_{N-\mu-2}(a+k, a+i-1)\nabla^{N-1} f(a+i)\\
&= \nabla^{N-1} f(a+k) - H_{N-\mu-1}(a+k, a)\nabla^{N-1} f(a)\\
&\quad + \sum_{i=1}^{k-1} \Big[\frac{(k-i+1)^{\overline{N-\mu-2}}}{\Gamma(N-\mu-1)}\Big]\nabla^{N-1} f(a+i).
\end{aligned}
$$

Solving the above inequality for $\nabla^{N-1} f(a+k)$, we obtain the desired inequality (2.4). Next we consider for $1 \le i \le k-1$,

$$
\begin{aligned}
\frac{(k-i+1)^{\overline{N-\mu-2}}}{\Gamma(N-\mu-1)} &= \frac{\Gamma(N-\mu+k-i-1)}{\Gamma(k-i+1)\Gamma(N-\mu-1)}\\
&= \frac{(N-\mu+k-i-2)\dots(N-\mu-1)}{(k-i)!} < 0
\end{aligned}
$$

since $N < \mu + 1$. Also

$$\begin{aligned} H_{N-\mu-1}(a+k,a) &= \frac{k^{\overline{N-\mu-1}}}{\Gamma(N-\mu)} \\ &= \frac{\Gamma(N-\mu+k-1)}{\Gamma(k)\Gamma(N-\mu)} \\ &= \frac{(N-\mu+k-2)\dots(N-\mu)}{(k-1)!} > 0. \end{aligned}$$

□

From Theorem 2.3, we have the following result.

Theorem 2.4. *Assume that $N-1 < \nu < N$, $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$, $\nabla_{a^*}^{\nu} f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$ and $\nabla^{N-1} f(a) \geq 0$. Then $\nabla^{N-1} f(t) \geq 0$ for $t \in \mathbb{N}_a$.*

Proof. By using the principle of strong induction, we prove that the conclusion of the theorem is correct.

By assumption, the result holds for $t = a$. Suppose that $\nabla^{N-1} f(t) \geq 0$, for $t = a, a+1, \dots, a+k-1$. From Theorem 2.3 and (2.4), we have $\nabla^{N-1} f(a+k) \geq 0$ and the proof is complete. □

Taking $N = 2$ and $N = 3$, we can get the following corollaries.

Corollary 2.5. *Assume that $1 < \nu < 2$, $f : N_{a-1} \to \mathbb{R}$, $\nabla_{a^*}^{\nu} f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$ and $f(a) \geq f(a-1)$, then $f(t)$ is increasing for $t \in \mathbb{N}_{a-1}$.*

Corollary 2.6. *Assume that $2 < \nu < 3$, $f : N_{a-2} \to \mathbb{R}$, $\nabla_{a^*}^{\nu} f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$ and $\nabla^2 f(a) \geq 0$, then $\nabla f(t)$ is increasing for $t \in \mathbb{N}_a$.*

In the following, we give the inverse proposition of Theorem 2.4.

Theorem 2.7. *Assume that $N-1 < \nu < N$, $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$, $\nabla^N f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$, then $\nabla_{a^*}^{\nu} f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$.*

Proof. From (2.1), taking $t = a+k$, we have

$$\nabla_{a^*}^{-\mu} f(t) = \nabla_a^{-(N-\mu)} \nabla^N f(t) \tag{2.5}$$

$$= \int_a^t H_{N-\mu-1}(t, \rho(s)) \nabla^N f(s) \nabla s \tag{2.6}$$

$$= \sum_{i=1}^{k} H_{N-\mu-1}(a+k, a+i-1) \nabla^N f(a+i). \tag{2.7}$$

Since

$$\begin{aligned} H_{N-\mu-1}(a+k, a+i-1) &= \frac{(k-i+1)^{\overline{N-\mu-1}}}{\Gamma(N-\mu)} \\ &= \frac{\Gamma(k+N-i-\mu)}{\Gamma(N-\mu)\Gamma(k-i+1)} \\ &= \frac{(-\mu+k+N-i)\dots(N-\mu+1)(N-\mu)}{(k-i)!} > 0, \end{aligned} \tag{2.8}$$

where we used $\mu < N$, from (2.5) and (2.8) we get that $\nabla_{a^*}^{\nu} f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, □

Taking $N = 1$ and $N = 2$, we get the following corollaries.

Corollary 2.8. *Assume that $0 < \nu < 1$, $f : \mathbb{N}_{a-1} \to \mathbb{R}$ and f is an increasing function for $t \in \mathbb{N}a$. Then $\nabla^{\nu}_{a^*} f(t) \geq 0$, for $t \in \mathbb{N}_{a+1}$.*

Corollary 2.9. *Assume that $1 < \nu < 2$, $f : \mathbb{N}_{a-1} \to \mathbb{R}$ and $\nabla^2 f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$. Then $\nabla^{\nu}_{a^*} f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$.*

In the following, we will give a counterexample to show that the assumption in Corollary 2.5 "$f(a) \geq f(a-1)$" is essential. To verify this example we will use the following simple lemma.

Lemma 2.10. *Assume that $f''(t) \geq 0$ on $[a, \infty)$. Then $\nabla^{\nu}_{a^*} f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, with $1 < \nu < 2$.*

Proof. By Taylor's Theorem,

$$f(a+i+1) = f(a+i) + f'(a+i) + \frac{f''(\xi^i)}{2}, \quad \xi^i \in [a+i, a+i+1], \tag{2.9}$$

$$f(a+i-1) = f(a+i) - f'(a+i) + \frac{f''(\eta^i)}{2}, \quad \eta^i \in [a+i-1, a+i] \tag{2.10}$$

for $i = 0, 1, \dots, k-1$. Using (2.9) and (2.10), we have

$$\begin{aligned} \nabla^2 f(a+i+1) &= f(a+i+1) - 2f(a+i) + f(a+i-1) \\ &= \frac{f''(\xi^i) + f''(\eta^i)}{2} \geq 0. \end{aligned} \tag{2.11}$$

From (2.11) and Corollary 2.9, we obtain $\nabla^{\nu}_{a^*} f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, with $1 < \nu < 2$. ☐

Example 2.11. Let $f(t) = -\sqrt{t}$, $a = 2$. We have $f''(t) \geq 0$, for $t \geq 1$. By Corollary 2.9, we have $\nabla^{\nu}_{a^*} f(t) \geq 0$

Note that $f(a-1) = f(1) = -1 > f(a) = -\sqrt{2}$. Therefore $f(x)$ does not satisfy the assumptions of Corollary 2.5. In fact, $f(t)$ is decreasing, for $t \geq 1$.

We conclude this note by mentioning a representative consequence of Corollary 2.5.

Corollary 2.12. *Let $h : \mathbb{N}_{a+1} \times \mathbb{R} \to \mathbb{R}$ be a nonnegative, continuous function. Then any solution of the Caputo nabla fractional difference equation*

$$\nabla^{\nu}_{a^*} y(t) = h(t, y(t)), \quad t \in \mathbb{N}_{a+1}, \quad 1 < \nu < 2 \tag{2.12}$$

satisfying $\nabla y(a) = A \geq 0$ is increasing on $\mathbb{N}_{a-1}$.

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