Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 161, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXTREMAL POINTS FOR A HIGHER-ORDER FRACTIONAL BOUNDARY-VALUE PROBLEM

AIJUN YANG, JOHNNY HENDERSON, CHARLES NELMS JR.

Abstract. The Krein-Rutman theorem is applied to establish the extremal point, b_0, for a higher-order Riemann-Liouville fractional equation, $D_{0+}^{\alpha} y + p(t)y = 0$, $0 < t < b$, $n < \alpha \leq n+1$, $n \geq 2$, under the boundary conditions $y^{(i)}(0) = 0$, $y^{(n-1)}(b) = 0$, $i = 0, 1, 2, \dots, n-1$. The key argument is that a mapping, which maps a linear, compact operator, depending on b to its spectral radius, is continuous and strictly increasing as a function of b. Furthermore, we also treat a nonlinear problem as an application of the result for the extremal point for the linear case.

1. Introduction

Differential equations of fractional order have proved to be valuable tools in modeling many physical phenomena [12, 21, 22]. Also, there has been a significant development in the theory for fractional differential equations; we refer the readers to the monographs by Kilbas et al [17], Miller and Ross [23], Podlubny [25] and Samko et al [26].

The Krein-Rutman theorem [19], a generalization of the Perron-Frobenius theorem for compact linear operators in infinite-dimensional Banach spaces, has been applied extensively to establish the existence of extremal points for second order differential equations, higher order differential equations, and m-dimensional systems of differential equations; we refer the reader to the monograph of Coppel [1] or to the landmark papers of Hartman [15], Levin [20] or Schmitt and Smith [27]. A standard approach for the description the extremal point of boundary value problems involves discussion of the existence of a nontrivial solution that lies in a cone; see [3, 5, 9, 10, 16]. Cone theoretic arguments are applied to linear, monotone, compact operators, which are constructed to complement the usual Green's function approach. The u_0-positivity of these operators is obtained by showing the operator maps nonzero elements of a cone into the interior of that cone. Sign properties of a Green's function, which serve as the kernel of the operators, are employed to prove the mapping preserves the cone. The theory of u_0-positive operators, as developed by Krein and Rutman, gives the existence of largest eigenvalues of the operator,

2010 *Mathematics Subject Classification.* 26A33, 34B08, 34B40.
Key words and phrases. u_0-positive operator; fractional boundary value problem; spectral radius.

Submitted May 19, 2015. Published June 16, 2015.

with the corresponding eigenfunction existing in a cone. This methods were extended in works by Eloe et al [3, 4, 8], Eloe and Henderson [6, 7], and Hankerson and Henderson [14] for a range of boundary-value problems for n^{th}-order differential equations.

Inspired by above works, in this article, for $b > 0$, we investigate the following family of higher-order fractional boundary value problems (BVPs):

$$D_{0+}^{\alpha} y + p(t)y = 0, \quad 0 < t < b, \tag{1.1}$$

$$y^{(i)}(0) = 0, \quad y^{(n-1)}(b) = 0, \quad i = 0, 1, 2, \dots, n-1, \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative with $n < \alpha \le n+1$ for $n \ge 2$, and $p(t)$ is a nonnegative continuous function on $[0, \infty)$ which does not vanish identically on any compact subinterval of $[0, \infty)$.

The purpose of this article is to establish the existence of a largest interval, $[0, b_0)$, such that on any subinterval $[\gamma_1, \gamma_2]$ of $[0, b_0)$, there is only the trivial solution of(1.1) satisfying(1.2). In particular, we define the first extremal point of(1.1) corresponding to the boundary conditions(1.2), to be this value b_0. Since b is a variable in this article, we shall refer to the BVP (b), (1.1)-(1.2).

In Section 2, we give some preliminary definitions and theorems from the theory of cones in Banach spaces that are employed to obtain the characterization of the first extremal point. In Section 3, we first give some sign properties of Green's function for $-D_{0+}^{\alpha} y = 0$ under the boundary conditions(1.2), and construct suitable cones in Banach spaces, and then we apply preliminary results to characterize the first extremal point. Finally, in Section 4, we establish sufficient conditions for the existence of nontrivial solutions of a nonlinear fractional differential equation.

2. Preliminaries

We will state some definitions and theorems on which the paper's main results depend.

Definition 2.1. The (left-sided) α-th fractional integral of a function $u : [0, b] \to \mathbb{R}$, denoted $I_{0+}^{\alpha} u$, is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided the right-hand side is pointwise defined on $[0, b]$, where $\Gamma(\alpha)$ is the Euler gamma function.

Definition 2.2. Let $n < \alpha \le n+1$. The α-th Riemann-Liouville fractional derivative of the function $u : [0, b] \to \mathbb{R}$, denoted $D_{0+}^{\alpha} u$, is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n+1-\alpha)} \Big(\frac{d}{dt}\Big)^{n+1} \int_0^t \frac{u(s) ds}{(t-s)^{\alpha-n}},$$

provided the right-hand side exists.

Definition 2.3. We say b_0 is the first extremal point of the BVP(b), (1.1)-(1.2), if

$$b_0 = \inf\{b > 0 : (b), (1.1)\text{-}(1.2) \text{ has a nontrivial solution}\}.$$

A cone $\mathcal{P}$ is *solid* if the interior, $\mathcal{P}^{\circ}$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is *reproducing* if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.4. Krasnosel'skii [18] showed that every solid cone is reproducing.

Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. For $u, v \in \mathcal{B}$, $u \preceq v$ with respect to $\mathcal{P}$, if $u - v \in \mathcal{P}$. A bounded linear operator $L : \mathcal{B} \to \mathcal{B}$ is said to be *positive* with respect to the cone $\mathcal{P}$ if $L : \mathcal{P} \to \mathcal{P}$. $L : \mathcal{B} \to \mathcal{B}$ is *u_0-positive with respect to $\mathcal{P}$* if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \preceq Lu \preceq k_2 u_0$ with respect to $\mathcal{P}$.

Remark 2.5. Throughout this article, let $\mathcal{B}$ be a partially ordered Banach space over $\mathbb{R}$ and $\mathcal{P}$ a cone in the Banach space $\mathcal{B}$. Let $\preceq$ be the partial ordering on the Banach space $\mathcal{B}$ induced by the cone $\mathcal{P}$, and $\leq$, the usual partial ordering on $\mathbb{R}$ induced by $\mathbb{R}^+$. Also, $u \preceq v$ will be used in the same way as $v \succeq u$. In addition, We will denote the *spectral radius* of the bounded linear operator L by $r(L)$.

The following five results are fundamental to our extremal point results. The first two results are found in Krasnosel'skii's book [18]. The third one is proved in Nussbaum [24]. The last two results are found in [18, 19]. In each of the following theorems, assume that $\mathcal{B}$ is a Banach space and $\mathcal{P}$ is a reproducing cone, and that $L : \mathcal{B} \to \mathcal{B}$ is a compact, linear, and positive operator with respect to $\mathcal{P}$.

Theorem 2.6. *Let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $L : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $L : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$, then L is u_0-positive.*

Theorem 2.7. *Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.8. *Let L_b, $\eta \leq b \leq \beta$ be a family of compact, linear operators on Banach space such that the mapping $b \mapsto L_b$ is continuous in the uniform operator topology. Then the mapping $b \mapsto r(L_b)$ is continuous.*

Theorem 2.9. *Assume $r(L) > 0$. Then $r(L)$ is an eigenvalue of L, and there is a corresponding eigenvalue in $\mathcal{P}$.*

Theorem 2.10. *Suppose there exists $\gamma > 0$, $u \in \mathcal{B}$, $-u \notin \mathcal{P}$, such that $\gamma u \preceq Lu$ with respect to $\mathcal{P}$. Then L has an eigenvector in $\mathcal{P}$ which corresponding to an eigenvalue λ with $\lambda \geq \gamma$.*

3. Criteria for extremal points

First, we introduce a family of Green's functions for $-D_{0+}^{\alpha} y = 0$ with $n < \alpha \leq n + 1$, $n \geq 2$, under the boundary conditions(1.2), can be calculated as

$$G(b; t, s) = \frac{1}{\Gamma(\alpha) b^{\alpha-n}} \begin{cases} t^{\alpha-1}(b-s)^{\alpha-n}, & 0 \leq t \leq s \leq b, \\ t^{\alpha-1}(b-s)^{\alpha-n} - b^{\alpha-n}(t-s)^{\alpha-1}, & 0 \leq s < t \leq b. \end{cases}$$

Obviously, $G(b; t, s) > 0$ and

$$\frac{\partial G(b; t, s)}{\partial b} = \frac{(\alpha - n) t^{\alpha-1} s}{\Gamma(\alpha) b^{\alpha+1-n} (b-s)^{n+1-\alpha}} > 0$$

on $(0, b) \times (0, b)$. In particular, we note that $G(b; t, s) = t^{\alpha-n} K(b; t, s)$, where

$$K(b; t, s) = \frac{1}{\Gamma(\alpha) b^{\alpha-n}} \begin{cases} t^{n-1}(b-s)^{\alpha-n}, & 0 \leq t \leq s \leq b, \\ t^{n-1}(b-s)^{\alpha-n} - b^{\alpha-n} t^{n-\alpha} (t-s)^{\alpha-1}, & 0 \leq s < t \leq b. \end{cases}$$

It is easy to deduce the sign properties of K as:

(1) $K(b; t, s) > 0$ for $(t, s) \in (0, b] \times (0, b)$.

(2) $K(b;0,s)=0$ for $s\in(0,b)$.
(3) $\frac{\partial^i K(b;0,s)}{\partial t^i}=0$, $i=1,2,\dots,n-2$.
(4) $\frac{\partial^{n-1}K(b;0,s)}{\partial t^{n-1}}=\frac{(n-1)!(b-s)^{\alpha-n}}{\Gamma(\alpha)b^{\alpha-n}}>0$ for $s\in(0,b)$.
(5) $\frac{\partial K(b;t,s)}{\partial b}=\frac{\alpha-n}{\Gamma(\alpha)}b^{n-\alpha-1}(b-s)^{\alpha-n-1}st^{n-1}>0$ for $(t,s)\in(0,b)\times(0,b)$.
(6) $\frac{\partial}{\partial b}(\frac{\partial^{n-1}K(b;0,s)}{\partial t^{n-1}})=\frac{(\alpha-n)(n-1)!}{\Gamma(\alpha)}b^{n-\alpha-1}(b-s)^{\alpha-n-1}s>0$ for $s\in(0,b)$.

Next, we consider the Banach space $(\mathcal{B},\|\cdot\|)$ defined by

$$\mathcal{B}:=\{y:[0,b]\to\mathbb{R}:y=t^{\alpha-n}z,\ z\in C[0,b]\},\quad \|y\|:=\sup_{0\le t\le b}|z(t)|=|z|_0.$$

Also, we define a cone $\mathcal{P}\subset\mathcal{B}$ by

$$\mathcal{P}:=\{y\in\mathcal{B}:y(t)\ge 0 \text{ on } [0,b]\}.$$

The cone $\mathcal{P}$ is a reproducing cone since if $y\in\mathcal{B}$,

$$y_1(t)=\max\{0,y(t)\},\quad y_2(t)=\max\{0,-y(t)\},$$

are in $\mathcal{P}$ and $y=y_1-y_2$.

For each $\beta>0$, define the Banach space

$$\mathcal{B}_\beta:=\{y:[0,\beta]\to\mathbb{R}:y=t^{\alpha-n}z,\ z\in C^{n-1}[0,\beta],\ z^{(i)}(0)=0,\ i=0,1,2,\dots,n-2\}$$

with the norm

$$\|y\|_\beta:=\sup_{0\le t\le\beta}|z^{n-1}(t)|=|z^{n-1}|_0.$$

By this norm, for $y\in\mathcal{B}_\beta$, we have

$$\begin{aligned}|z(t)|&=\Big|\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-2}}z^{(n-1)}(s)ds\cdots dt_2dt_1\Big|\\&\le\frac{t^{n-1}}{(n-1)!}|z^{(n-1)}|_0\\&=\frac{t^{n-1}}{(n-1)!}\|y\|_\beta,\quad t\in[0,\beta].\end{aligned}$$

Then

$$|y(t)|=|t^{\alpha-n}z(t)|\le\frac{t^{\alpha-1}}{(n-1)!}\|y\|_\beta,\quad t\in[0,\beta].$$

For each $\beta>0$, define the cone $\mathcal{P}_\beta\subset\mathcal{B}_\beta$ to be

$$\mathcal{P}_\beta:=\{y\in\mathcal{B}_\beta:y(t)\ge 0 \text{ on } [0,\beta]\}.$$

Lemma 3.1. *The cone $\mathcal{P}_\beta$ is solid in $\mathcal{B}_\beta$ and hence reproducing.*

Proof. Define

$$\begin{aligned}\Omega_\beta=\Big\{&y\in\mathcal{B}_\beta:y(t)>0 \text{ for } t\in(0,\beta),\ z^{(n-1)}(0)>0,\\&z(\beta)>0,\ \text{where } y=t^{\alpha-n}z\Big\}.\end{aligned}$$

We will show $\Omega_\beta\subset\mathcal{P}_\beta^\circ$. Let $y\in\Omega_\beta$. Since $z^{(n-1)}(0)>0$, there exists an $\varepsilon_1>0$ such that $z^{(n-1)}(0)-\varepsilon_1>0$. Since $z\in C^{(n-1)}[0,\beta]$, there exists a $\gamma_1\in(0,\beta)$ such that $z^{(n-1)}(t)>\varepsilon_1$ for $t\in(0,\gamma_1)$. So,

$$y(t)=t^{\alpha-n}z(t)$$

$$
\begin{aligned}
&= t^{\alpha-n}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-2}} z^{(n-1)}(s)ds\cdots dt_2dt_1\\
&> \frac{t^{\alpha-1}}{(n-1)!}\varepsilon_1, \quad t\in(0,\gamma_1).
\end{aligned}
$$

Now, since $z(\beta)>0$, there exists an $\varepsilon_2>0$ such that $z(\beta)-\varepsilon_2>0$. Since $z\in C^{n-1}[0,\beta]$, there exists a $\gamma_2\in(0,\beta)$ such that $z(t)>\varepsilon_2$ for $t\in(\gamma_2,\beta)$. Thus $y(t)=t^{\alpha-n}z(t)>\varepsilon_2 t^{\alpha-n}$ for all $t\in(\gamma_2,\beta)$. Also, since $y(t)>0$ on $[\gamma_1,\gamma_2]$, there exists an $\varepsilon_3>0$ such that $y(t)>\varepsilon_3$ for all $t\in[\gamma_1,\gamma_2]$.

Let $\varepsilon=\min\big\{\frac{\varepsilon_1}{2},\ \frac{(n-1)!\varepsilon_2}{2\beta^{n-1}},\ \frac{(n-1)!\varepsilon_3}{2\beta^{\alpha-1}}\big\}$. Define $B_\varepsilon(y)=\{\hat y\in\mathcal{B}_\beta:\|y-\hat y\|_\beta<\varepsilon\}$. Let $\hat y\in B_\varepsilon(y)$, then $\hat y=t^{\alpha-n}\hat z$, where $\hat z\in C^{n-1}[0,\beta]$ with $\hat z^{(i)}(0)=0$, $i=0,1,2,\dots,n-2$. Now,

$$
|\hat y(t)-y(t)|\le \frac{t^{\alpha-1}}{(n-1)!}\|\hat y-y\|_\beta<\frac{t^{\alpha-1}}{(n-1)!}\varepsilon,\quad t\in[0,\beta].
$$

So for $t\in(0,\gamma_1)$,

$$
\hat y(t)>y(t)-\frac{t^{\alpha-1}}{(n-1)!}\varepsilon>\frac{t^{\alpha-1}}{(n-1)!}\varepsilon_1-\frac{t^{\alpha-1}}{(n-1)!}\varepsilon>\frac{t^{\alpha-1}}{2(n-1)!}\varepsilon_1>0.
$$

For $t\in(\gamma_2,\beta)$,

$$
\hat y(t)>\varepsilon_2t^{\alpha-n}-\frac{t^{\alpha-1}}{(n-1)!}\varepsilon>\Big(\varepsilon_2-\frac{\beta^{n-1}}{(n-1)!}\varepsilon\Big)t^{\alpha-n}>\frac{\varepsilon_2}{2}t^{\alpha-n}>0.
$$

Also,

$$
\hat y(t)>y(t)-\frac{t^{\alpha-1}}{(n-1)!}\varepsilon>\varepsilon_3-\frac{\beta^{\alpha-1}}{(n-1)!}\varepsilon>0
$$

for $t\in[\gamma_1,\gamma_2]$. So $\hat y\in\mathcal{P}_\beta$ and thus $B_\varepsilon(y)\subset\mathcal{P}_\beta$. Then $\Omega_\beta\subset\mathcal{P}_\beta^\circ$. □

Let $N_0y(t)\equiv0$, $t\in[0,b]$, and for each $\beta>0$, define $N_\beta:\mathcal{B}\to\mathcal{B}$ by

$$
N_\beta y(t)=\begin{cases}\int_0^\beta G(\beta;t,s)p(s)y(s)ds, & 0\le t\le\beta,\\ \int_0^\beta G(\beta;\beta,s)p(s)y(s)ds, & \beta\le t\le b.\end{cases}\tag{3.1}
$$

We shall refer to $N_\beta:\mathcal{B}_\beta\to\mathcal{B}_\beta$, where N_β is defined by

$$
\begin{aligned}
N_\beta y(t)&=\int_0^\beta G(\beta;t,s)p(s)y(s)ds\\
&=t^{\alpha-n}\int_0^\beta K(\beta;t,s)p(s)y(s)ds,\quad 0\le t\le\beta.
\end{aligned}
$$

By employing the methods used in [10], the existence of the extremal point b_0 for BVP (b),(1.1)-(1.2), is positive, can be seen from the following theorem.

Theorem 3.2. *Let $\delta>0$ be such that*

$$
\Big(\frac{1}{\Gamma(\alpha)(\alpha-n+1)}+\frac{2^n}{\Gamma(\alpha-n+2)(n-1)!}\Big)P\delta^\alpha=1,
$$

where $P=\max_{0\le t\le\beta}|p(t)|$. Then the BVP($\beta$), (1.1)-(1.2) has a unique solution for $\beta\in(0,\delta)$; in particular, if $\beta\ge\delta$, then $u\equiv0$ is the only solution of BVP (β), (1.1)-(1.2).

Proof. We shall show there exists $\delta > 0$ such that for $\beta \in (0, \delta)$, $N_\beta : \mathcal{B}_\beta \to \mathcal{B}_\beta$ is a contraction map. Let $y_1, y_2 \in \mathcal{B}_\beta$ and consider

$$\begin{aligned}(N_\beta y_2 - N_\beta y_1)(t) = t^{\alpha-n}\Big(&\int_0^\beta \frac{t^{n-1}(\beta-s)^{\alpha-n}}{\Gamma(\alpha)\beta^{\alpha-n}} p(s)(y_2-y_1)(s)ds \\ &\times \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)t^{\alpha-n}} p(s)(y_2-y_1)(s)ds\Big).\end{aligned}$$

Set

$$z(t) = \int_0^\beta \frac{t^{n-1}(\beta-s)^{\alpha-n}}{\Gamma(\alpha)\beta^{\alpha-n}} p(s)(y_2-y_1)(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)t^{\alpha-n}} p(s)(y_2-y_1)(s)ds.$$

Then, $\|N_\beta y_2 - N_\beta y_1\|_\beta = |z^{(n-1)}|_0$. For $t \in (0,\beta)$,

$$\begin{aligned}
&|z^{(n-1)}(t)| \\
&= \Big| \int_0^\beta \frac{(n-1)!(\beta-s)^{\alpha-n}}{\Gamma(\alpha)\beta^{\alpha-n}} p(s)(y_2-y_1)(s)ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\partial^{n-1}((t-s)^{\alpha-1}t^{n-\alpha})}{\partial t^{n-1}} p(s)(y_2-y_1)(s)ds\Big| \\
&\le \frac{(n-1)!}{\Gamma(\alpha)\beta^{\alpha-n}} \cdot P \cdot |y_2-y_1|_0 \int_0^\beta (\beta-s)^{\alpha-n}ds + \frac{1}{\Gamma(\alpha)} \cdot P \cdot |y_2-y_1|_0 \\
&\quad \times \int_0^t \sum_{k=0}^{n-1} C_{n-1}^k \frac{\Gamma(\alpha)}{\Gamma(\alpha-k)} \frac{\Gamma(\alpha-n+(n-k-1))}{\Gamma(\alpha-n)} (t-s)^{\alpha-k-1} t^{n-\alpha-(n-k-1)} ds \\
&\le \Big(\frac{(n-1)!}{\Gamma(\alpha)(\alpha-n+1)} + \frac{2^{n-1}}{\Gamma(\alpha-n+2)}\Big) P\beta |y_2-y_1|_0 \\
&\le \Big(\frac{1}{\Gamma(\alpha)(\alpha-n+1)} + \frac{2^{n-1}}{\Gamma(\alpha-n+2)(n-1)!}\Big) P\beta^\alpha \|y_2-y_1\|_\beta,
\end{aligned}$$

where $\frac{\partial^{n-1}((t-s)^{\alpha-1}t^{n-\alpha})}{\partial t^{n-1}}$ is calculated by Leibniz rule formula for derivative, and $\sum_{k=0}^{n-1} C_{n-1}^k = 2^{n-1}$ due to binomial formula $(a+b)^{n-1} = \sum_{k=0}^{n-1} C_{n-1}^k a^k b^{n-1-k}$ with $a = b = 1$. Thus, if

$$\Big(\frac{1}{\Gamma(\alpha)(\alpha-n+1)} + \frac{2^{n-1}}{\Gamma(\alpha-n+2)(n-1)!}\Big) P\beta^\alpha < 1,$$

then N_β is a contraction map.

Choose $\delta > 0$ such that $\Big(\frac{1}{\Gamma(\alpha)(\alpha-n+1)} + \frac{2^n}{\Gamma(\alpha-n+2)(n-1)!}\Big)P\delta^\alpha = 1$ and the proof is complete. □

Lemma 3.3. *For each $\beta > 0$, N_β is positive with respect to $\mathcal{P}$ and $\mathcal{P}_\beta$. In addition, $N_\beta : \mathcal{P}_\beta \setminus \{0\} \to \mathcal{P}_\beta^\circ$.*

Proof. The positivity of N_β with respect to $\mathcal{P}$ and $\mathcal{P}_\beta$ is an easy consequence of the sign properties of Green's function G and the kernel K. Now, we shall show that $N_\beta : \mathcal{P}_\beta \setminus \{0\} \to \mathcal{P}_\beta^\circ$. From Lemma 3.1, we have $\Omega_\beta \subset \mathcal{P}_\beta^\circ$. Next, we prove $N_\beta : \mathcal{P}_\beta \setminus \{0\} \to \Omega_\beta$.

Let $y \in \mathcal{P}_\beta \setminus \{0\}$, then there exists $[\gamma_1, \gamma_2] \subset [0, \beta]$ such that $p(t) > 0$ and $y(t) > 0$ for all $t \in [\gamma_1, \gamma_2]$. So

$$\begin{aligned} N_\beta y(t) &= \int_0^\beta G(\beta; t, s)p(s)y(s)ds \\ &\geq \int_{\gamma_1}^{\gamma_2} G(\beta; t, s)p(s)y(s)ds > 0, \quad \text{for all } t \in (0, \beta). \end{aligned}$$

Note $z(t) = \int_0^\beta K(\beta; t, s)p(s)y(s)ds$, we have

$$\begin{gathered} z(\beta) = \int_0^\beta K(\beta; \beta, s)p(s)y(s)ds \geq \int_{\gamma_1}^{\gamma_2} K(\beta; \beta, s)p(s)y(s)ds > 0, \\ z^{(n-1)}(0) = \int_0^\beta \frac{\partial^{n-1} K(\beta; 0, s)}{\partial t^{n-1}} p(s)y(s)ds > 0. \end{gathered}$$

Thus, $N_\beta y \in \Omega_\beta$ and $N_\beta : \mathcal{P}_\beta \setminus \{0\} \to \mathcal{P}_\beta^\circ$. □

Remark 3.4. According to Theorem 2.6, N_β is u_0-positive with respect to $\mathcal{P}_\beta$.

Lemma 3.5. *The mapping $\beta \mapsto r(N_\beta)$ with N_β defined on $\mathcal{B}$ for each $\beta \in (0, b]$ is continuous.*

Proof. We shall prove that the mapping $\beta \mapsto N_\beta$ is continuous in the uniform operator topology with N_β defined on $\mathcal{B}$ for each $\beta \in (0, b]$. Since $p(t)$ is continuous on $[0, \infty)$, the linear operator N_β defined on $\mathcal{B}$ can be proved to be compact as in [9]. Now, let $f : (0, b] \to \{N_\beta\}|_0^b$ be given by $f(\beta) = N_\beta$. Assume $y = t^{\alpha - n} z \in \mathcal{B}$ with $\|y\| = 1$. Note $P = \max_{0 \leq t \leq b} |p(t)|$. Let $0 < \gamma_1 < \gamma_2 \leq b$. Then

$$\begin{aligned} &\|f(\gamma_2) - f(\gamma_1)\| \\ &= \|N_{\gamma_2} - N_{\gamma_1}\| \\ &= \sup_{\|y\|=1} \|N_{\gamma_2} y - N_{\gamma_1} y\| \\ &= \sup_{\|y\|=1} \sup_{0 \leq t \leq b} \Big| \int_0^{\gamma_2} K(\gamma_2; t, s)p(s)y(s)ds - \int_0^{\gamma_1} K(\gamma_1; t, s)p(s)y(s)ds \Big|. \end{aligned}$$

Since $K(\beta; t, s)$ is continuous for each $\beta \in (0, b]$, for $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(\gamma_2; t, s) - K(\gamma_1; t, s)| < \frac{\varepsilon}{2bP}$ whenever $|\gamma_2 - \gamma_1| < \delta$.

Case (i) $t \leq \gamma_1$. Let $\sup_{0 \leq t \leq \gamma_1, \gamma_1 \leq s \leq \gamma_2} |K(\gamma_2; t, s)| \leq K_1$. Choose $\delta = \frac{\varepsilon}{2K_1 P}$. Then

$$\begin{aligned} &\Big| \int_0^{\gamma_2} K(\gamma_2; t, s)p(s)y(s)ds - \int_0^{\gamma_1} K(\gamma_1; t, s)p(s)y(s)ds \Big| \\ &\leq \int_0^{\gamma_1} |K(\gamma_2; t, s) - K(\gamma_1; t, s)| p(s)y(s)ds + \int_{\gamma_1}^{\gamma_2} K(\gamma_2; t, s)p(s)y(s)ds \\ &\leq \frac{\varepsilon}{2bP} \cdot \gamma_1 \cdot P \cdot 1 + K_1 \cdot P \cdot 1 \cdot |\gamma_2 - \gamma_1| < \varepsilon. \end{aligned}$$

Case (ii) $\gamma_1 \leq t \leq \gamma_2$. Let $\sup_{\gamma_1 \leq t \leq \gamma_2, 0 \leq s \leq \gamma_1} \big| \frac{\partial K(\gamma_2; t, s)}{\partial t} \big| \leq K_2$ and

$$\sup_{t, s \in [\gamma_1, \gamma_2]} |K(\gamma_2; t, s)| \leq K_3.$$

Choose $\delta = \frac{\varepsilon}{2(K_2 b + K_3)P}$. Then

$$
\begin{aligned}
&\Big|\int_0^{\gamma_2} K(\gamma_2; t, s)p(s)y(s)ds - \int_0^{\gamma_1} K(\gamma_1; \gamma_1, s)p(s)y(s)ds\Big| \\
&\le \int_0^{\gamma_1} |K(\gamma_2; t, s) - K(\gamma_1; \gamma_1, s)|p(s)y(s)ds + \int_{\gamma_1}^{\gamma_2} K(\gamma_2; t, s)p(s)y(s)ds \\
&\le \int_0^{\gamma_1} |K(\gamma_2; t, s) - K(\gamma_2; \gamma_1, s)|p(s)y(s)ds \\
&\quad + \int_0^{\gamma_1} |K(\gamma_2; \gamma_1, s) - K(\gamma_1; \gamma_1, s)|p(s)y(s)ds + \int_{\gamma_1}^{\gamma_2} K(\gamma_2; t, s)p(s)y(s)ds \\
&\le \int_0^{\gamma_1} \Big|\frac{\partial K(\gamma_2; \xi_{t,\gamma_1}, s)}{\partial t}\Big|(t - \gamma_1)p(s)y(s)ds + \frac{\varepsilon}{2bP} \cdot \gamma_1 \cdot P \cdot 1 + K_3 \cdot P \cdot 1 \cdot |\gamma_2 - \gamma_1| \\
&\le (K_2\gamma_1 + K_3)P|\gamma_2 - \gamma_1| + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}
$$

Case (iii) $t \ge \gamma_2$. The similar technique is used in Case (ii), so we omit it here. From above discussion we can see that $\beta \mapsto N_\beta$ is continuous in the uniform operator topology. Therefore, the mapping $\beta \mapsto r(N_\beta)$ is continuous due to Theorem 2.8. $\square$

Theorem 3.6. *For $0 < \beta \le b$, $r(N_\beta)$ is strictly increasing as a function of β.*

Proof. Let $\lambda > 0$ and $y \in \mathcal{P}_\beta \setminus \{0\}$. Theorem 2.7 implies that $N_\beta y(t) = \lambda y(t)$ for $t \in [0, \beta]$. Let $y(t) = y(\beta)$ for $t > \beta$. Then, for $t \in [0, b]$, $N_\beta y(t) = \lambda y(t)$, and $r(N_\beta) \ge \lambda > 0$, i.e., $r(N_\beta) > 0$.

Next, let $0 < \beta_1 < \beta_2 \le b$. Since $r(N_{\beta_1}) > 0$, by Theorem 2.9, there exists $y \in \mathcal{P}_{\beta_1}$ such that $N_{\beta_1} y = r(N_{\beta_1})y$. Let $u_1 = N_{\beta_1} y$ and $u_2 = N_{\beta_2} y$. Then for $t \in [0, \beta_1]$, we claim that $u_2 - u_1 \in \mathcal{P}^\circ_{\beta_1}$. In fact, by noting $(u_2 - u_1)(t) = t^{\alpha-2} z_{12}(t)$, we have

$$
\begin{aligned}
z_{12}(t) &= \int_0^{\beta_2} K(\beta_2; t, s)p(s)y(s)ds - \int_0^{\beta_1} K(\beta_1; t, s)p(s)y(s)ds \\
&= \int_0^{\beta_1} [K(\beta_2; t, s) - K(\beta_1; t, s)]p(s)y(s)ds + \int_{\beta_1}^{\beta_2} K(\beta_2; t, s)p(s)y(\beta_1)ds.
\end{aligned}
$$

Since $y \in \mathcal{P}_{\beta_1} \setminus \{0\}$ and $p(t)$ does not vanish identically on any compact subinterval $[0, \beta_1] \subset [0, b]$, it follows that $z_{12}(t) > 0$ as $K(\beta_2; t, s) > K(\beta_1; t, s)$. So, $u_2(t) > u_1(t)$ on $(0, \beta_1)$. In view of $\frac{\partial^i K(\beta; 0, s)}{\partial t^i} = 0$, for $\beta \in (0, b]$ and $s \in [0, b]$, $i = 0, 1, 2, \dots, n-2$, we have

$$
z_{12}^{(i)}(0) = \int_0^{\beta_2} \frac{\partial^i K(\beta_2; 0, s)}{\partial t^i} p(s)y(s)ds - \int_0^{\beta_1} \frac{\partial^i K(\beta_1; 0, s)}{\partial t^i} p(s)y(s)ds = 0,
$$

for $i = 0, 1, 2, \dots, n-2$. Since $\frac{\partial K(b;0,s)}{\partial t} > 0$ and $\frac{\partial}{\partial b}\Big(\frac{\partial K(b;0,s)}{\partial t}\Big) > 0$ for $s \in (0, b)$, we can get

$$
\begin{aligned}
z_{12}^{(n-1)}(0) &= \int_0^{\beta_2} \frac{\partial^{n-1} K(\beta_2; 0, s)}{\partial t^{n-1}} p(s)y(s)ds - \int_0^{\beta_1} \frac{\partial^{n-1} K(\beta_1; 0, s)}{\partial t^{n-1}} p(s)y(s)ds \\
&= \int_0^{\beta_1} \Big[\frac{\partial^{n-1} K(\beta_2; 0, s)}{\partial t^{n-1}} - \frac{\partial^{n-1} K(\beta_1; 0, s)}{\partial t^{n-1}}\Big] p(s)y(s)ds
\end{aligned}
$$

$$+\int_{\beta_1}^{\beta_2} \frac{\partial^{n-1}K(\beta_2;0,s)}{\partial t^{n-1}} p(s)y(\beta_1)ds > 0.$$

Also,

$$\begin{aligned} z_{12}(\beta_1) &= \int_0^{\beta_2} K(\beta_2;\beta_1,s)p(s)y(s)ds - \int_0^{\beta_1} K(\beta_1;\beta_1,s)p(s)y(s)ds \\ &= \int_0^{\beta_1} [K(\beta_2;\beta_1,s) - K(\beta_1;\beta_1,s)]p(s)y(s)ds \\ &\quad + \int_{\beta_1}^{\beta_2} K(\beta_2;\beta_1,s)p(s)y(\beta_1)ds > 0, \end{aligned}$$

due to $\frac{\partial K(\beta;t,s)}{\partial b} > 0$ for $\beta \in (0,b)$ and $K(\beta_2;\beta_1,s) > 0$ on (β_1,β_2).

Thus, the restriction of $u_2 - u_1$ to $[0,\beta_1]$ belongs to $\Omega_{\beta_1} \subset \mathcal{P}^\circ_{\beta_1}$. So there exists $\delta > 0$ such that $u_2 - u_1 \succeq \delta y$ with respect to $\mathcal{P}_{\beta_1}$. Let $u_1(t) = u_1(\beta_1)$ for $t > \beta_1$. In view of $u_2 \in \mathcal{P}_{\beta_2}$, it follows that $u_2 - u_1 \succeq \delta y$ with respect to $\mathcal{P}_{\beta_2}$. Thus,

$$u_2 \succeq u_1 + \delta y = r(N_{\beta_1})y + \delta y = (r(N_{\beta_1}) + \delta)y,$$

i.e., $N_{\beta_2}y \succeq (r(N_{\beta_1}) + \delta)y$. So by Theorem 2.10,

$$r(N_{\beta_2}) \geq r(N_{\beta_1}) + \delta > r(N_{\beta_1}).$$

Hence, $r(N_\beta)$ is strictly increasing for $0 < \beta \leq b$. □

Theorem 3.7. *The following three statements are equivalent:*

(i) *b_0 is the first extremal point of the BVP (b),* (1.1)-(1.2)*;*
(ii) *there exists a nontrivial solution y of the BVP(b_0),* (1.1)-(1.2) *such that $y \in \mathcal{P}_{b_0}$;*
(iii) *$r(N_{b_0}) = 1$.*

Proof. (iii) ⇒ (ii) is an immediate consequence of Theorem 2.9.

Next, we prove (ii) ⇒ (i). Let $y \in \mathcal{P}_{b_0} \setminus \{0\}$ satisfy BVP(b_0),(1.1)-(1.2) for $0 \leq t \leq b_0$. Extend $y(t) = y(b_0)$ for $t > b_0$. For $N_{b_0}y(t) = y(t)$, we have $r(N_{b_0}) \geq 1$.

If $r(N_{b_0}) = 1$, then by Theorem 3.6 that $r(N_\beta) < r(N_{b_0})$ for $0 < \beta < b_0$, i.e., $r(N_\beta) < 1$. So the BVP(β),(1.1)-(1.2) has the only trivial solution. Thus, b_0 is the first extremal point of BVP (b),(1.1)-(1.2).

If $r(N_{b_0}) > 1$. Let $v \in \mathcal{P}_{b_0} \setminus \{0\}$ such that $N_{b_0}v = r(N_{b_0})v$. From Lemma 3.3, we know that the restriction of v to $[0,b_0]$ belongs to $\mathcal{P}^\circ_{b_0}$. Thus, there exists $\delta > 0$ such that $y \succeq \delta v$ with respect to $\mathcal{P}_{b_0}$, $0 \leq t \leq b_0$. Extend $v(t) = v(b_0)$ for $t > b_0$. Then $y \succeq \delta v$ with respect to $\mathcal{P}$. Assume δ is maximal such that the inequality $y \succeq \delta v$ holds. Then,

$$y = N_{b_0}y \succeq N_{b_0}(\delta v) = \delta N_{b_0}v = \delta r(N_{b_0})v.$$

Since $r(N_{b_0}) > 1$, $\delta r(N_{b_0}) > \delta$. But this contradicts the assumption that δ is the maximal value satisfying the inequality $y \succeq \delta v$. So $r(N_{b_0}) = 1$.

Finally, to prove (i) ⇒ (iii) observe that $\lim_{b \to 0^+} r(N_b) = 0$. If b_0 is the first extremal point of BVP (b),(1.1)-(1.2), then $r(N_{b_0}) \geq 1$. If $r(N_{b_0}) > 1$, then by the continuity of r about b, there exists $\beta_0 \in (0,b_0)$ such that $r(N_{\beta_0}) = 1$, and for this β_0, the BVP (β_0),(1.1)-(1.2) has a nontrivial solution, which is a contradiction. □

4. A nonlinear problem

Consider a BVP for a nonlinear fractional differential equation of the form

$$D^{\alpha}_{0+}y + f(t,y) = 0, \quad 0 < t < b \tag{4.1}$$

with boundary conditions(1.2). Suppose that $f(t,y) : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is continuous, and $f(t,0) \equiv 0$, $f(t,y)$ is differentiable in y. Assume $\frac{\partial f(t,0)}{\partial y}$ is continuous and nonnegative on $[0,\infty)$ and does not vanish identically on each compact subinterval of $[0,\infty)$. Then the variational equation along the zero solution of (4.1) is

$$D^{\alpha}_{0+}y + \frac{\partial f(t,0)}{\partial y} y = 0, \quad 0 < t < b. \tag{4.2}$$

To obtain sufficient conditions for the existence of solutions of the BVP (4.1)-(1.2), we shall apply the following fixed point theorem, see [2, 11, 27].

Theorem 4.1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $M : \mathcal{B} \to \mathcal{B}$ be a completely continuous nonlinear operator such that $M : \mathcal{P} \to \mathcal{P}$ and $M(0) = 0$. Assume M is Fréchet differentiable at $u = 0$ whose Fréchet derivative $N = M'(0)$ has the property:*

- (A1) *There exist $w \in \mathcal{P}$ and $\mu > 1$ such that $Nw = \mu w$, and $Nu = u$ implies $u \notin \mathcal{P}$. Further, there exists $\rho > 0$ such that, if $u = \frac{1}{\lambda}Mu$, $u \in \mathcal{P}$ and $\|u\| = \rho$, then $\lambda \le 1$.*

Then the equation $u = Mu$ has a solution $u \in \mathcal{P} \setminus \{0\}$.

Now, we shall use this theorem and the main conclusions of Section 3 to prove the following result.

Theorem 4.2. *Suppose that b_0 is the first extremal point of BVP (4.2)-(1.2). For each $\beta > b_0$ assume the property:*

- (H1) *There exists $\rho(\beta) > 0$ such that if $y(t)$ is a nontrivial solution of the BVP*

$$D^{\alpha}_{0+}y + \frac{1}{\lambda} f(t,y) = 0, \quad 0 < t < b, \tag{4.3}$$

with boundary conditions (1.2), and if $y \in \mathcal{P}$ with $\|y\| = \rho(\beta)$, then $\lambda \le 1$.

Then the BVP(β),(4.1)-(1.2) has a nontrivial solution $y \in \mathcal{P}$ for all $\beta \ge b_0$.

Proof. For each $\beta > b_0$, let $N_\beta : \mathcal{B} \to \mathcal{B}$ be defined by (3.1), where $p(t) \equiv \frac{\partial f(t,0)}{\partial y}$. Define the nonlinear operator $M_\beta : \mathcal{B} \to \mathcal{B}$ by

$$M_\beta y(t) = \begin{cases} \int_0^\beta G(\beta;t,s) f(s,y(s))ds, & 0 \le t \le \beta, \\ \int_0^\beta G(\beta;\beta,s) f(s,y(s))ds, & \beta \le t \le b. \end{cases}$$

The differentiability of f with respect to y is sufficient to argue that M_β is Fréchet differentiable at $y = 0$ since

$$\begin{aligned} &\Big| \int_0^\beta G(\beta;t,s)[f(s,y(s)) - p(s)y(s)]ds \Big| \\ &= \Big| \int_0^\beta G(\beta;t,s)[f_y(s,\tilde{y}(s)) - p(s)]y(s)ds \Big| \\ &\le Q\beta\|y\| \int_0^\beta |f_y(s,\tilde{y}(s)) - p(s)|ds, \end{aligned}$$

where $0 \le \tilde{y}(t) \le y(t)$ for $t \in [0, \beta]$ and $Q = \sup_{t,s\in[0,b]} |G(\beta; t, s)|$. Moreover, $M'_\beta(0) = N_\beta$.

By Theorems 3.6 and 3.7, it follows that $r(N_{b_0}) = 1$ and $r(N_\beta) > 1$ if $\beta > b_0$. Moreover, since b_0 is the first extremal point of the BVP (b),(4.2)-(1.2), it also follows from Theorem 3.7 that if $N_\beta y = y$ and y is nontrivial for $\beta > b_0$, then $y \notin \mathcal{P}$. So, for $\beta > b_0$, we can apply property (H1) to check the condition (A1) in Theorem 4.1. Then we obtain the existence of a $y \in \mathcal{P} \setminus \{0\}$ such that $y = N_\beta y$ and the proof is complete. □

Remark 4.3. Condition (4.3) may always be satisfied when $f(t, y)$ is sublinear for large $|y|$, in the case when $\alpha = 2$ we can refer the readers to see [27].

Acknowledgments. A. Yang was supported by Project 201408330015 from the China Scholarship Council, and by the NNSF of China (61273016). This research was carried out while A. Yang was a Visiting Research Professor at Baylor University.

References

[1] W. Coppel; *Disconjugacy*, Lecture Notes in Mathematics, No. 220, Springer-Verlag, New York/Berlin (1971).
[2] K. Deimling; *Nonlinear Functional Analysis*, pringer-Verlag, NewYork (1985).
[3] P. W. Eloe, D. Hankerson, J. Henderson; *Positive solutions and conjugate points for multipoint boundary value problems*, J. Diff. Eqns., **95** (1992), 20-32.
[4] P. W. Eloe, D. Hankerson, J. Henderson; *Positive solutions and j-focal points for two point boundary value problems*, Rocky Mtn. J. Math., **22** (1992), 1283-1293.
[5] P. W. Eloe, J. Henderson; *Comparison of eigenvalues for a class of two-point boundary value problems*, Appl. Anal., **34** (1989), 25-34.
[6] P. W. Eloe, J. Henderson; *Focal points and comparison theorems for a class of two-point boundary value problems*, J. Diff. Eqns., **103** (1993), 375-386.
[7] P. W. Eloe, J. Henderson; *Focal point characterizations and comparisons for right focal differential operators*, J. Math. Appl. Anal., **181** (1994), 22-34.
[8] P. W. Eloe, J. Henderson, H. B. Thompson; *Extrimal points for impulsive Lidstone boundary value problems*, Math. Comp. Mod., **32** (2000), 687-698.
[9] P. W. Eloe, J. T. Neugebauer; *Existence and comparison of smallest eigenvalues for a fractional boundary-value problem*, Electron. J. Diff. Eqns., **2014** (2014), No. 43, 1-10.
[10] P. W. Eloe, J. T. Neugebauer; *Conjugate point for fractional differential equations*, Fract. Calc. Appl. Anal., **17** (2014), No. 3, 855-871.
[11] J. A. Gatica, H. L. Smith; *Fixed point techniques in a cone with application*, J. Math. Anal. Appl., **61** (1977), 58-71.
[12] L. Gaul, P. Klein, S. Kempfle; *Damping description involving fractional operators*, Mech. Systems Signal Processing, **5** (1991), 81-88.
[13] D. Hankerson, J. Henderson; *Comparison of eigenvalues for n-point boundary value problems for difference equations*, Differential Eqns. Stab. Control., **127** (1990), 203-208.
[14] D. Hankerson, J. Henderson; *Positive solutions and extremal points for differential equations*, Appl. Anal., **39** (1990), 193-207.
[15] P. Hartman; *Disconjugate nth order differential equations and principal solutions*, Bull. Amer. Math. Soc., **68** (1968), 125-129.
[16] J. Henderson, R. Luca; *Positive solutions for a system of nonlocal fractional boundary value problems*, Fract. Calc. Appl. Anal., **16** (2013), No. 4, 985-1008.
[17] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006.
[18] M. Krasnosel'skii; *Positive Solutions of Operator Equations.* Fizmatgiz, Moscow, 1962; English Translation P. Noordhoff Ltd. Gronigen, The Netherlands, 1964.

[19] M. G. Krein, M. A. Rutman; *Linear operators leaving a cone invariant in a Banach space*, Translations Amer. Math. Soc., Series 1, Volume 10, 199-325, American Mathematical Society, Providence, RI, 1962.

[20] A. Ju. Levin; *Non-oscillation of solutions of the equation* $x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p_n(t)x=0$, Russian Math. Surveys, **24** (1969), 43-99.

[21] F. Mainardi; *Fractional calculus: Some basic problems in continuum and statistical mechanics, in Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., Springer-Verlag, Wien, 1997.

[22] F. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmacher; *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys., **103** (1995), 7180-7186.

[23] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[24] R. D. Nussbaum; *Periodic sollutions of some nonlinear integral equations*, Proc. Internatl. Conf. on Differential Equations, Gainesville, Florida, (1976).

[25] I. Podlubny; *Fractional Differential Equations, Mathematics in Sciences and Engineering*, **198**, Academic Press, San Diego, 1999.

[26] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[27] K. Schmitt, H. L. Smith; *Positive solutions and conjugate points for systems of differential equations*, Nonlinear Anal., **2** (1978), 93-105.

Aijun Yang
Zhejiang University of Technology, College of Science, Hangzhou 310023, China.
Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
E-mail address: yangaij2004@163.com, Aijun_Yang@baylor.edu

Johnny Henderson
Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
E-mail address: Johnny_Henderson@baylor.edu

Charles Nelms Jr.
Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
E-mail address: Charles_Nelms@baylor.edu