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OPTIMAL PARTIAL REGULARITY FOR QUASILINEAR ELLIPTIC SYSTEMS WITH VMO COEFFICIENTS BASED ON A-HARMONIC APPROXIMATIONS

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Abstract. In this article, we consider quasi-linear elliptic systems in divergence form with discontinuous coefficients under controllable growth. We establish an optimal partial regularity of the weak solutions by a modification of A-harmonic approximation argument introduced by Duzaar and Grotowski.

1. Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^n (n \geq 2)$ and $u : \Omega \to \mathbb{R}^N$ be a vectorial-valued function in Sobolev spaces $W^{1,2}(\Omega, \mathbb{R}^N)$. In this article, we obtain optimal partial regularity in Hölder spaces to the weak solution of quasi-linear elliptic systems in divergence form under the controllable growth as follows:

$$-D_\alpha(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j) = B_i(x,u,Du), \quad \text{a. e. } x \in \Omega,\ i = 1,2,\dots,N; \tag{1.1}$$

where $A(x,u) = (A_{ij}^{\alpha\beta}(x,u))$ is a VMO function in $x \in \Omega$ uniformly with respect to $u \in \mathbb{R}^N$ and continuous in u uniformly with respect to $x \in \Omega$, and $B_i(x,u,Du)$ satisfies the controllable growth. In the context, we adopt Einstein's convention by summing over repeated indices with $\alpha, \beta = 1,2,\dots,n$ and $i, j = 1,2,\dots,N$. Therefore, a vectorial-valued function $u \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^N)$ is understood as a weak solution of (1.1) in the following distributional sense:

$$\int_\Omega A(x,u)Du \cdot D\varphi \, dx = \int_\Omega B(x,u,Du)\varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^N). \tag{1.2}$$

Before stating our basic assumptions and main result, let us briefly review some recent studies on the topic. As we know, the discontinuity of the coefficients is not so crucial for Hölder continuity of the weak solutions of the scalar partial differential equations, which is due to the famous De Giorgi-Moser-Nash iterating technique, see [16]. However, for the vectorial-valued case (i.e. $N > 1$) some counterexamples showed that nonlinear elliptic systems, even in the Euclidian metric, do not possess everywhere regularity conclusion, see Giaquinta's monograph [15]. In addition, to get the regularity of weak solutions of elliptic systems, one needs to assume the continuity of coefficients in general. In fact, the system (1.1) arises naturally in many different contexts. Giaquinta and Modica [18, 15]

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first studied partial regularity of weak solutions of the system (1.1) in the Morrey space and in the Campanato space [15, 24] when each entry of the leading coefficients $A(x,u)$ is assumed to be continuous four order tensorial-valued function.

It is an important observation that many stochastic processes with discontinuous coefficients reappeared in connected with diffusion approximation [20]. However, according to the famous counterexample of Nadirashvili there could not exist theory of solvability of systems with general discontinuous coefficients even if they are uniformly bounded and elliptic, and solutions are understood in a very weak sense. This reminds us of the significance to treat particular cases of discontinuity. As an important turning point, Sarason [25] introduced the function classes of the so-called Vanishing Mean Oscillations (briefly called VMO), which is a class of functions that neither contains nor is contained within $C^0(\Omega)$ and contains discontinuous functions. Moreover, the VMO functions own a good property similar to the class of continuous functions, which is not shared by general bounded measurable functions and BMO functions. Since then, the Calderón-Zygmund's theory of linear and nonlinear PDEs with VMO coefficients were immensely developed which naturally originated from the singular integral operators and the estimates of commutators with a VMO function [3, 1]. In the meantime, the regularity in Morrey spaces of weak solutions to PDEs with the discontinuous leading coefficients was also investigated in a similar approach by Fazio [13] and Fan-Lu-Yang [14]. Very recently, it developed some new different arguments to deal with the divergence or non-divergence elliptic and parabolic PDEs with the VMO leading coefficients, for example a few celebrated approaches of Chiarenza-Frasca-Longo [3], Syun-Wang [2] and Krylov-Dong-Kim [21, 9]. Now we are in the position to recall some assumptions imposed on $A(x,u)$ and $B(x,u,Du)$.

(H1) (uniform ellipticity) There exist two constants $0 < \lambda \le \Lambda$ such that

$$\lambda|\xi|^2 \le A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i\xi_\beta^j \le \Lambda|\xi|^2, \quad \forall x \in \Omega,\ u \in \mathbb{R}^N,\ \xi \in \mathbb{R}^{nN}. \tag{1.3}$$

(H2) ($A(x,u)$ is VMO in x and continuous in u) $A(\cdot,u)$ is VMO in x uniformly with respect to $u \in \mathbb{R}^N$ and is continuous in u uniformly with respect to $x \in \Omega$; that is, $\lim_{s\to 0} M_s(A(\cdot,u_0)) = 0$, where $M_s(A(\cdot,u))$ referred to section 2, and there exist a constant and a continuous concave function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$, $0 \le \omega \le 1$ such that

$$|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)| \le C\omega(|u-v|^2), \quad \forall u,v \in \mathbb{R}^N,\ x \in \Omega. \tag{1.4}$$

The modulus of continuity may take a continuous concave function by $\omega(t) = inf\{\lambda(t) : \lambda(t)$ concave and continuous with $\lambda(t) \ge \alpha(t)$ for any modulus of continuity $\alpha(t)\}$.

(H3) (controllable growth) The lower order item $B(x,u,Du)$ satisfies the following controllable growth with a constant $L > 0$:

$$|B_i(x,u,Du)| \le L(|Du|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + g_i), \tag{1.5}$$

where

$$\gamma = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2, \\ \text{any } \gamma > 2, & \text{if } n = 2; \end{cases} \quad g_i \in L^q(\Omega),\ q > \frac{n}{2};$$

for $\alpha = 1,2,\dots,n$ and $i = 1,2,\dots,N$.

Let us review some studies on the analogous questions. Gironimo-Esposito-Sgambati in [17] obtained the partial regularity in Morrey spaces to quasi-linear quadratic functionals with leading coefficient $A(x,u)$ allowing VMO dependence on x and continuous dependence on u. Later, Zheng [28] and Zheng-Feng [29] derived the partial regularity in Morrey spaces for quasi-linear elliptic systems with VMO leading coefficients with the controllable growth and the natural growth by a reverse Hölder inequality and perturbation argument, respectively. Chen-Tan [4] also got an optimal interior partial regularity for nonlinear elliptic systems under the controllable growth condition by the A-harmonic approximation, but their principle coefficients $A(x,u)$ are essentially Hölder continuous in (x,u). Here, we would like to study the above topic by way of an approach called A-harmonic approximation. As we know, the argument of harmonic approximation can go back to De Giorgi's work [6] who started to use the idea of approximating almost minimizers and the equation of minimal surfaces by systems with constant coefficients. Afterwards, the harmonic approximation argument was efficiently employed to study ε-regularity of harmonic maps, see [26]. Recently, Duzaar-Mingione-Grotowski-Steffen in [11, 10, 23, 6] developed this approach to so-called A-harmonic approximation, p-harmonic approximation and A-caloric approximation in proving the regularity for nonlinear elliptic systems with continuous or Hölder continuous coefficients, p-harmonic maps and parabolic settings, respectively. In particular, Daněček-John-Stará [5] employed so-called modified A-harmonic approximation approach to prove the regularity in Morrey's space of weak solutions of Stokes systems with VMO coefficients. Inspired by his work, in this paper we should like to prove an optimal partial regularity for quasi-linear elliptic systems with VMO coefficients under the controllable growth by a modification of A-harmonic approximation argument, which avoids to use the reverse Hölder inequality. We state our main results as follows.

Theorem 1.1. *In the case of vectorial-valued functions with $N > 1$, suppose that $u \in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$ is a locally weak solution of the system* (1.1)*, and $A(x,u)$, $B(x,u,Du)$ satisfy the basic assumptions* (H1)–(H3)*. Then there exists an open subset $\Omega_0 \subset \Omega$ with $\dim_H(\Omega\setminus\Omega_0) \le n-2$ such that $u \in C^{0,\alpha}_{\rm loc}(\Omega_0,\mathbb{R}^N)$, $\alpha = 2-\frac{n}{q}$ if $\frac{n}{2} < q < n$ or $u \in C^{0,\alpha}_{\rm loc}(\Omega_0,\mathbb{R}^N)$ for all $\alpha \in (0,1)$ if $q \ge n$, which $\dim_H$ expresses the Hausdorff's dimension.*

This article is organized as follows. In section 2, we recall some notations and facts, and give the proof of modification of so-called A-harmonic approximation, Caccioppoli inequality. Section 3 is devoted to prove the main conclusions.

2. Preliminaries

We adopt the usual convention of denoting by C a general constant, which may vary from line to line in the same chain of inequalities. Let us first recall some notation and basic facts [25, 27].

Definition 2.1. A locally integrable function f is said to belong to $BMO(\Omega)$(the spaces of bounded mean oscillation), if $f \in L_{\rm loc}(\Omega)$ and for any $0 < s < \infty$, we have

$$M_s(f,\Omega) = \sup_{x\in\Omega, 0<\rho<s} |\Omega(x,\rho)|^{-1} \int_{\Omega(x,\rho)} |f(y) - f_{x,\rho}|\,dy < +\infty,$$

where $\Omega(x,\rho) = \Omega \cap B(x,\rho)$ with any open ball $B(x,\rho)$ in $\mathbb{R}^n$ centered at x of radius ρ, and $f_{x,\rho} := \fint_{\Omega(x,\rho)} f(y)\,dy = \frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} f(y)\,dy$.

Definition 2.2. A function $f \in L_{\rm loc}(\Omega)$ is said to be in $VMO(\Omega)$(vanishing mean oscillation in Ω), if

$$M_0(f) = \lim_{s\to 0} M_s(f,\Omega) = 0.$$

As we know, Caccioppoli's inequality is usually a very beginning of studying regularity to elliptic and parabolic PDEs, see [15]. Here, we provide the so-called second Caccioppoli's inequality.

Lemma 2.3. *Let $u \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^N)$ be a weak solution of* (1.1) *and $A(x,u)$, $B(x,u,Du)$ satisfy the assumption* (H1)–(H3). *Then for any $B_\rho(x_0) \subset \Omega$, we have*

$$\int_{B_{\frac{\rho}{2}}(x_0)} |Du|^2\,dx \le \frac{C_1}{\rho^2}\int_{B_\rho(x_0)} |u-m|^2\,dx + C_2\Big(\int_{B_\rho(x_0)} (|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}, \tag{2.1}$$

where m is a vectorial-valued constant in $\mathbb{R}^N$.

Proof. For any $x_0 \in \Omega, 0 < \rho < \operatorname{dist}(x_0, \partial\Omega)$, denoting $B_\rho := B_\rho(x_0)$, we take $\eta \in C_0^\infty(B_\rho(x_0))$ as a cut-off function with $0 \le \eta \le 1$, $|D\eta| \le \frac{4}{\rho}$ and $\eta \equiv 1$ on $B_{\frac{\rho}{2}}(x_0)$. As usual, we can take the function $\varphi = \eta^2(u-m)$ as a test function with any vectorial-valued constant $m \in \mathbb{R}^N$. By (1.2), we have

$$\int_{B_\rho} A(x,u)Du \cdot [2\eta D\eta(u-m) + \eta^2 Du]dx = \int_{B_\rho} B(x,u,Du)\eta^2(u-m)dx,$$

which implies

$$\int_{B_\rho} \eta^2 A(x,u)Du\cdot Du\,dx = -2\int_{B_\rho} A(x,u)Du\cdot(\eta(u-m)D\eta) + \int_{B_\rho} B(x,u,Du)\eta^2(u-m)dx.$$

By the ellipticity (H1) and the controllable growth (H3) we obtain

$$\begin{aligned}&\lambda\int_{B_\rho}|\eta Du|^2\,dx\\ &\le 2\Lambda\int_{B_\rho}|\eta Du|\cdot|(u-m)D\eta|\,dx + L\int_{B_\rho}\Big(|Du|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + |g|\Big)|\varphi|dx := I + II.\end{aligned} \tag{2.2}$$

For I, by Young's inequality we have

$$I \le \varepsilon\int_{B_\rho}|\eta Du|^2\,dx + \frac{C(\varepsilon)}{\rho^2}\int_{B_\rho}|u-m|^2\,dx. \tag{2.3}$$

For II, by Hölder's inequality and Sobolev's inequality and Young's inequality we have

$$\begin{aligned}II &\le CL\int_{B_\rho}(|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})^{1-\frac{1}{\gamma}}|\varphi|dx\\ &\le CL\Big(\int_{B_\rho}(|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{1-\frac{1}{\gamma}}\Big(\int_{B_\rho}|\varphi|^\gamma\,dx\Big)^{\frac{1}{\gamma}}\\ &\le CL\Big(\int_{B_\rho}(|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{1-\frac{1}{\gamma}}\Big(\int_{B_\rho}|D\varphi|^2\,dx\Big)^{1/2}\\ &\le \varepsilon\int_{B_\rho}|D\varphi|^2\,dx + C(\varepsilon)\Big(\int_{B_\rho}(|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}.\end{aligned}$$

Note that $D\varphi = 2\eta D\eta(u-m) + \eta^2 Du$. Then

$$II \le \varepsilon\int_{B_\rho}|\eta Du|^2\,dx + C(\varepsilon)\int_{B_\rho}|D\eta|^2|u-m|^2\,dx + C(\varepsilon)\Big(\int_{B_\rho}(|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}. \tag{2.4}$$

Now by combining (2.3) and (2.4) it yields

$$(\lambda - 2\varepsilon)\int_{B_\rho}|\eta Du|^2\,dx \le \frac{C(\varepsilon)}{\rho^2}\int_{B_\rho}|u-m|^2\,dx + C(\varepsilon)\Big(\int_{B_\rho}(|Du|^2 + |u|^\gamma + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}. \tag{2.5}$$

So, we only choose some $\varepsilon < \lambda/2$, it yields the desired result. □

We are in position to introduce a modification of so-called A-harmonic approximation lemma. Let us first recall the definition of locally A-harmonic.

Definition 2.4. Let $A \in Bil(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2\times N^2})$ be a bilinear form with constant coefficients, which satisfies the assumptions of (1.3). We call a map $h \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ A-harmonic in $B_R(x_0)$ if it satisfies

$$\int_{B_R(x_0)} A(Dh, D\varphi)dx = 0, \quad \forall \varphi \in C_0^1(B_R(x_0), \mathbb{R}^N).$$

Since $A \in Bil(B_R(x_0)\times\mathbb{R}^N, \mathbb{R}^{n^2\times N^2})$ is a bilinear form with constant coefficients, it's well known that for any A-harmonic function h we have the following inequality.

Lemma 2.5 ([15]). *Let $h(x) \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ be a weak solution of the following linear system with constant coefficients*

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta h_j) = 0, \quad i = 1, \dots, N.$$

Then there exists a constant $C = C(n, \lambda, \Lambda)$ such that for any $x_0 \in \Omega$, $0 < \rho < R \le \operatorname{dist}(x_0, \partial\Omega)$, it holds

$$\int_{B_\rho(x_0)} |Dh|^2\, dx \le C(\frac{\rho}{R})^n \int_{B_R(x_0)} |Dh|^2\, dx. \tag{2.6}$$

Now we give the modified A-harmonic approximation which is based on the usual A-harmonic lemma originated by Duzaar and Grotowski' works [10, 6]. In the sequel, suppose that there exist two constants $0 < \lambda \le \Lambda < \infty$ such that the bilinear form $A \in Bil(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2\times N^2})$ satisfies

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j \ge \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^{nN}, \tag{2.7}$$

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \bar{\xi}_\beta^j \le \Lambda|\xi||\bar{\xi}|, \quad \forall \xi, \bar{\xi} \in \mathbb{R}^{nN}; \tag{2.8}$$

Lemma 2.6. *Consider fixed positive constants λ, Λ and $n, N \in \mathbb{N}$ with $n \ge 2$ as above. Then for any given $\varepsilon > 0$, there exists $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon) \in (0, 1]$ with the following property: for any bilinear form $A \in Bil(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2\times N^2})$ with (2.7),(2.8), assume $g \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ satisfies*

$$R^{-n} \int_{B_R(x_0)} |Dg|^2\, dx \le 1, \tag{2.9}$$

$$\Big|R^{-n} \int_{B_R(x_0)} A(Dg, D\varphi)dx\Big| \le \delta \sup_{B_R(x_0)} |D\varphi|, \quad \forall \varphi \in C_0^\infty(B_R(x_0), \mathbb{R}^N); \tag{2.10}$$

there exists an A-harmonic function

$$\omega \in H = \{h \in W^{1,2}(B_R(x_0), \mathbb{R}^N) : R^{-n} \int_{B_R(x_0)} |Dh|^2\, dx \le 1\}$$

with

$$R^{-n-2} \int_{B_R(x_0)} |\omega - g|^2\, dx \le \varepsilon. \tag{2.11}$$

Thanks to the A-harmonic approximation above, we obtain its modified version by imitating an argument from Stoke system by Daněček-John-Stará [5].

Lemma 2.7 (Modification of A-harmonic approximation)**.** *Let $0 < \lambda \le \Lambda < \infty$ and $n \ge 2$ as the above lemma. Then, for any given $\varepsilon > 0$ there exists $k = k(n, N, \lambda, \Lambda, \varepsilon) > 0$ with the following property: for any $A \in Bil(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2 \times N^2})$ satisfying* (2.7),(2.8) *and any $u \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$, there exists an A-harmonic function $h \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ such that*

$$\int_{B_R(x_0)} |Dh|^2\,dx \le \int_{B_R(x_0)} |Du|^2\,dx; \tag{2.12}$$

moreover, there exists $\varphi \in C_0^\infty(B_R(x_0), \mathbb{R}^N)$ with

$$\|D\varphi\|_{L^\infty(B_R(x_0),\mathbb{R}^N)} \le \frac{1}{R}; \tag{2.13}$$

such that

$$\int_{B_R(x_0)} |u-h|^2\,dx \le \varepsilon R^2 \int_{B_R(x_0)} |Du|^2\,dx + k(\varepsilon)\Big[R^{4-n}\Big(\int_{B_R(x_0)} ADu\cdot D\varphi\,dx\Big)^2\Big]. \tag{2.14}$$

Proof. First, observe that it is sufficient to prove the lemma for $x_0 = 0$ and $R = 1$ by a standard scaling argument. In the context, we let $B = B_1(0)$. For any given $\varepsilon > 0$, we pick $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon)$ as the above Lemma 2.6. Consider $u \in W^{1,2}(B, \mathbb{R}^N)$, we take

$$g = u\Big(\int_B |Du|^2\,dx\Big)^{-1/2},$$

therefore, $\int_B |Dg|^2\,dx \le 1$ which implies (2.9). Next, we consider the estimates divided into two cases.

Case 1. If for g there holds the inequality (2.10). By Lemma 2.6 there exists an A-harmonic function ω satisfying $\int_{B_\rho(x_0)} |D\omega|^2\,dx \le 1$ and $\int_B |\omega - g|^2\,dx \le \varepsilon$.

Let $h = (\int_B |Du|^2\,dx)^{1/2}\omega$, which satisfies (2.12). In fact, we can easily know h is A-harmonic and

$$\int_B |Dh|^2\,dx = \int_B |Du|^2\,dx \int_B |D\omega|^2\,dx \le \int_B |Du|^2\,dx.$$

Moreover, we have

$$|u-h|^2 = \int_B |Du|^2\,dx \cdot |g-\omega|^2,$$

which implies

$$\int_B |u-h|^2\,dx \le \int_B |Du|^2\,dx \int_B |g-\omega|^2\,dx \le \varepsilon \int_B |Du|^2\,dx.$$

Hence, the inequality (2.14) is valid.

Case 2. If for g the inequality (2.10) is false. Then there exists a non-constant function $\psi \in C_0^\infty(B, \mathbb{R}^N)$ such that

$$\Big|\int_B A(Dg, D\psi)dx\Big| > \delta(\varepsilon)\sup_B |D\psi|.$$

By taking $\varphi = \psi/\sup_B |D\psi|$ it yields $\|D\varphi\|_{L^\infty} = 1$, which implies

$$\frac{1}{\delta(\varepsilon)}\Big|\int_B A(Dg, D\varphi)dx\Big| > 1.$$

Now we take $h = \bar{u}$. By Poincaré inequality and recalling $Dg = (\int_B |Du|^2)^{-1/2} \cdot Du$, it follows that

$$\int_B |u-h|^2\,dx = \int_B |u-\bar{u}|^2\,dx \le C\int_B |Du|^2\,dx$$

$$\le \frac{C}{\delta^2(\varepsilon)}\int_B |Du|^2\,dx\Big|\int_B A(Dg,D\varphi)dx\Big|^2$$
$$\le \frac{C}{\delta^2(\varepsilon)}\Big|\int_B A(Du,D\varphi)dx\Big|^2.$$

By combining Cases 1 and 2, and taking $k(\varepsilon)=\frac{C}{\delta^2(\varepsilon)}$, we obtain the inequality (2.14). The proof is complete. ◻

Lemma 2.8 ([12]). *Let Ω be an open subset of $\mathbb{R}^n$ and $u\in L_{\rm loc}(\Omega,\mathbb{R}^N)$. Then for $0\le s<n$ and set*

$$E_s := \{x\in\Omega : \liminf_{\rho\to 0}\rho^{-s}\int_{B_\rho(x)}|u|\,dy>0\}, \tag{2.15}$$

there holds the estimate $H^s(E_s)=0$.

3. Proof of main result

In the section, we prove our main result by way of the idea from modification of A-harmonic approximation argument and perturbation approach.

Proof of Theorem 1.1. For any $x_0\in\Omega$ and fixed $0<R\le\frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$. Without loss of generality, we let $x_0=0$ and for any $0<\rho<R$ write B_ρ in place of $B_\rho(0)$. Now letting $m=u_{0,\rho}=u_\rho$ in Lemma 2.3, it follows that

$$\int_{B_{\frac{\rho}{2}}}|Du|^2\,dx\le\frac{C_1}{\rho^2}\int_{B_\rho}|u-u_\rho|^2\,dx+C_2\Big(\int_{B_\rho}(|Du|^2+|u|^\gamma+|g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}, \tag{3.1}$$

Let $\bar A=A(\cdot,u_R)_R$ be defined by

$$\bar A := A(x,u_R)_R = \fint_{B_R}A(x,u_R)dx.$$

Thanks to the modification of A-harmonic Lemma 2.7, there exists an $\bar A$-harmonic function $h\in W^{1,2}(B_R,\mathbb{R}^N)$ such that the inequalities (2.12),(2.13) and (2.14) are valid. Therefore, from (3.1) we have

$$\begin{aligned}
&\int_{B_{\frac{\rho}{2}}}|Du|^2\,dx\\
&\le\frac{2C_1}{\rho^2}\Big(\int_{B_\rho}|u-u_\rho-(h-h_\rho)|^2\,dx+\int_{B_\rho}|h-h_\rho|^2\,dx\Big)\\
&\quad+C\Big(\int_{B_\rho}(|Du|^2+|u|^\gamma+|g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}\\
&:=\frac{C}{\rho^2}(I_1+I_2)+C\Big(\int_{B_\rho}(|Du|^2+|u|^\gamma+|g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}.
\end{aligned}\tag{3.2}$$

Next we estimate I_1 and I_2. For the estimation of I_1, by Poincaré inequality and Lemma 2.5 on the system with constant coefficients it follows that

$$I_1=\int_{B_\rho}|h-h_\rho|^2\,dx\le C\rho^2\int_{B_\rho}|Dh|^2\,dx\le C\rho^2(\frac{\rho}{R})^n\int_{B_R}|Dh|^2\,dx.$$

Hence, from (2.12) it yields

$$I_1\le C\rho^2(\frac{\rho}{R})^n\int_{B_R}|Du|^2\,dx. \tag{3.3}$$

For I_2, by employing Poincaré inequality again and (2.14) in Lemma 2.7, we have

$$\begin{aligned} I_2 &= \int_{B_\rho} |u - u_\rho - (h - h_\rho)|^2 \, dx \le 2 \int_{B_\rho} |u - h|^2 \, dx \\ &\le C\varepsilon\rho^2 \int_{B_\rho} |Du|^2 \, dx + Ck(\varepsilon)\rho^{4-n}\Big(\int_{B_\rho} \bar{A}Du \cdot D\varphi \, dx\Big)^2 \\ &\le C\varepsilon\rho^2 \int_{B_R} |Du|^2 \, dx + Ck(\varepsilon)\rho^{4-n}\Big(\int_{B_\rho} \bar{A}Du \cdot D\varphi \, dx\Big)^2, \end{aligned} \tag{3.4}$$

where φ satisfies $\|D\varphi\|_{L^\infty(B_R,\mathbb{R}^N)} \le \frac{1}{R}$. Next, we estimate the term $\int_{B_\rho} \bar{A}Du \cdot D\varphi \, dx$. Note that u is a weak solution of (1.1), then

$$\begin{aligned} \int_{B_\rho} \bar{A}Du \cdot D\varphi \, dx &= \int_{B_\rho} [\bar{A} - A(x, u_\rho)]Du \cdot D\varphi \, dx + \int_{B_\rho} [A(x, u_\rho) - A(x, u)]Du \cdot D\varphi \, dx \\ &\quad + \int_{B_\rho} B(x, u, Du)\varphi \, dx; \end{aligned}$$

that is,

$$\begin{aligned} \Big(\int_{B_\rho} \bar{A}Du \cdot D\varphi \, dx\Big)^2 &\le C\Big(\int_{B_\rho} [\bar{A} - A(x, u_\rho)]Du \cdot D\varphi \, dx\Big)^2 \\ &\quad + C\Big(\int_{B_\rho} [A(x, u_\rho) - A(x, u)]Du \cdot D\varphi \, dx\Big)^2 + C\Big(\int_{B_\rho} B(x, u, Du)\varphi \, dx\Big)^2. \end{aligned} \tag{3.5}$$

Since $\|D\varphi\|_{L^\infty(B_R,\mathbb{R}^N)} \le \frac{1}{R}$ in (2.13) and $A(\cdot, u) \in VMO \bigcap L^\infty(\Omega)$ of the assumptions (H1)–(H2), it follows

$$\begin{aligned} \Big(\int_{B_\rho} [\bar{A} - A(x, u_\rho)]Du \cdot D\varphi \, dx\Big)^2 &\le \frac{1}{\rho^2} \int_{B_\rho} |Du|^2 \, dx \int_{B_\rho} |A(x, u_\rho) - \bar{A}|^2 \, dx \\ &\le \frac{1}{\rho^2} \cdot 2\Lambda\alpha_n\rho^n \fint_{B_\rho} |A(x, u_\rho) - \bar{A}| dx \int_{B_\rho} |Du|^2 \, dx \\ &\le C(n, \Lambda)M_s(A(x, u_\rho))\alpha_n\rho^{n-2} \int_{B_\rho} |Du|^2 \, dx, \end{aligned} \tag{3.6}$$

where α_n is the volume of unit ball in $\mathbb{R}^n$. Similarly, in terms of the continuous assumptions of $A(x, \cdot)$ in u uniformly with respect to $x \in \Omega$ we have the following estimates

$$\begin{aligned} &\Big(\int_{B_\rho} [A(x, u_\rho) - A(x, u)]Du \cdot D\varphi \, dx\Big)^2 \\ &\le \frac{1}{\rho^2} \int_{B_\rho} |Du|^2 \, dx \int_{B_\rho} |A(x, u_\rho) - A(x, u)|^2 \, dx \\ &\le \frac{1}{\rho^2} \cdot 2\Lambda\alpha_n\rho^n \fint_{B_\rho} |A(x, u_\rho) - A(x, u)| dx \cdot \int_{B_\rho} |Du|^2 \, dx \\ &\le C\frac{1}{\rho^2} \cdot \Lambda\alpha_n\rho^n \fint_{B_\rho} \omega(|u - u_\rho|) dx \int_{B_\rho} |Du|^2 \, dx \\ &\le C\Lambda\alpha_n\rho^{n-2}\omega\Big(\fint_{B_\rho} |u - u_\rho|^2 \, dx\Big) \int_{B_\rho} |Du|^2 \, dx \\ &\le C(n, \Lambda)\rho^{n-2}\omega\Big(\rho^2 \fint_{B_\rho} |Du|^2 \, dx\Big) \int_{B_\rho} |Du|^2 \, dx, \end{aligned} \tag{3.7}$$

where we use the Jensen's inequality in the fourth step and the Poincaré's inequality in the last step. Finally, we consider the controllable growth condition (H3) it yields

$$
\begin{aligned}
\Big(\int_{B_\rho} B(x,u,Du)\varphi\,dx\Big)^2 &\le \Big(\int_{B_\rho} |B(x,u,Du)|\,dx\Big)^2 \\
&\le C\Big(\int_{B_\rho} (|Du|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + |g|)dx\Big)^2 \\
&\le C\Big(\int_{B_\rho} (|Du|^2 + |u|^{\gamma} + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}(\alpha_n\rho^n)^{\frac{2}{\gamma}} \\
&= C(n)\rho^{n-2}\Big(\int_{B_\rho} (|Du|^2 + |u|^{\gamma} + |g|^{\frac{\gamma}{\gamma-1}})dx\Big)^{2(1-\frac{1}{\gamma})}
\end{aligned}
\tag{3.8}
$$

Now, substitute estimates (3.6), (3.7) and (3.8) into (2.4), it yields

$$
\begin{aligned}
&\Big(\int_{B_\rho} \bar{A}Du\cdot D\varphi\,dx\Big)^2 \\
&\le C(n,\Lambda)\rho^{n-2}\Big(M_s(A(x,u_\rho)) + \omega(\rho^2 ⨍_{B_\rho}|Du|^2\,dx)\Big)\int_{B_\rho}|Du|^2\,dx \\
&\quad + C(n)\rho^{n-2}\Big(\int_{B_\rho}\Big(|Du|^2 + |u|^{\gamma} + |g|^{\frac{\gamma}{\gamma-1}}\Big)dx\Big)^{2\left(1-\frac{1}{\gamma}\right)}.
\end{aligned}
\tag{3.9}
$$

Denoting

$$
\sigma(\rho) = M_s(A(x,u_\rho)) + \omega\Big(\rho^2 ⨍_{B_\rho}|Du|^2\,dx\Big) \tag{3.10}
$$

and inserting (3.9) into the estimate of I_2, we obtain

$$
I_2 \le C\Big(\varepsilon + \sigma(\rho)\Big)\rho^2\int_{B_R}|Du|^2\,dx + C\rho^2\Big(\int_{B_\rho}\Big(|Du|^2 + |u|^{\gamma} + |g|^{\frac{\gamma}{\gamma-1}}\Big)dx\Big)^{2\left(1-\frac{1}{\gamma}\right)}.
$$

Substitute the estimates for I and II into (3.2), we obtain

$$
\begin{aligned}
\int_{B_{\frac{\rho}{2}}}|Du|^2\,dx &\le C\Big(\Big(\frac{\rho}{R}\Big)^n + \varepsilon + \sigma(\rho)\Big)\int_{B_R}|Du|^2\,dx \\
&\quad + C\Big(\int_{B_\rho}\Big(|Du|^2 + |u|^{\gamma} + |g|^{\frac{\gamma}{\gamma-1}}\Big)dx\Big)^{2\left(1-\frac{1}{\gamma}\right)}.
\end{aligned}
\tag{3.11}
$$

It remains to estimate the term of the controllable growth. Observe that $g_i \in L^q(\Omega)$ with $q > n/2$ and

$$
\gamma = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2, \\ \text{any } \gamma > 2, & \text{if } n = 2. \end{cases}
$$

As we know it is trivial if $n = 2$. So, we only consider the case of $n > 2$ so that $2(1-\frac{1}{\gamma}) = (n+2)/n$, by Hölder inequality it yields

$$
\begin{aligned}
\Big(\int_{B_\rho}\Big(|Du|^2 + |u|^{\gamma} + |g|^{\frac{\gamma}{\gamma-1}}\Big)dx\Big)^{2\left(1-\frac{1}{\gamma}\right)} &\le C\Big(\int_{B_\rho}|Du|^2 + |u|^{\gamma}dx\Big)^{1+\frac{2}{n}} + C\Big(\int_{B_\rho}|g|^{\frac{2n}{n+2}}dx\Big)^{\frac{n+2}{n}} \\
&\le C\Big(\int_{B_\rho}|Du|^2 + |u|^{\gamma}dx\Big)^{1+\frac{2}{n}} + C\alpha_n^{\frac{(n+2)q-2n}{nq}}R^{n+2-\frac{2n}{q}}\|g\|_{L^q}^2,
\end{aligned}
$$

putting it into (3.11), yields

$$\begin{aligned}\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx \le C\Big(\Big(\frac{\rho}{R}\Big)^n + \varepsilon + \sigma(\rho) + \Big(\int_{B_\rho} |Du|^2 + |u|^\gamma dx\Big)^{2/n}\Big)\int_{B_R}(|Du|^2 \\ + |u|^\gamma)dx + CR^{n+2-\frac{2n}{q}}\|g\|^2_{L^q}.\end{aligned} \tag{3.12}$$

On the other hand, by a direct calculation it follows that

$$\begin{aligned}\int_{B_{\frac{\rho}{2}}} |u|^\gamma\,dx &\le C\int_{B_{\frac{\rho}{2}}} |u_{x_0,\rho}|^\gamma\,dx + C\int_{B_{\frac{\rho}{2}}} |u - u_{x_0,\rho}|^\gamma\,dx \\ &\le C(n)\Big(\frac{\rho}{R}\Big)^n \int_{B_R} |u|^\gamma\,dx + C\Big(\int_{B_R} |Du|^2\,dx\Big)^{\frac{\gamma}{2}-1}\Big(\int_{B_R}(|Du|^2 + |u|^\gamma)dx\Big).\end{aligned}$$

Now add the item $\int_{B_{\frac{\rho}{2}}} |u|^\gamma\,dx$ to both sides of (3.12) to obtain

$$\int_{B_{\frac{\rho}{2}}} |Du|^2 + |u|^\gamma\,dx \le C\Big(\Big(\frac{\rho}{R}\Big)^n + \varepsilon + \sigma(\rho) + \delta(\rho)\Big)\int_{B_R}\Big(|Du|^2 + |u|^\gamma\Big)dx + CR^{n+2-\frac{2}{q}n}\|g\|^2_{L^q}, \tag{3.13}$$

where

$$\delta(\rho) = \Big(\int_{B_\rho}(|Du|^2 + |u|^\gamma)\,dx\Big)^{2/n} + \Big(\int_{B_\rho} |Du|^2\,dx\Big)^{2/(n-2)}. \tag{3.14}$$

Note that $\delta(\rho) \to 0$ as $\rho \to 0$ due to the absolute continuity of $\int_{B_\rho}(|Du|^2+|u|^\gamma)\,dx$ on domain of integration, and if we assume $\rho^2 ⨍_{B_{\rho(x)}} |Du|^2 dy \to 0$ on $x \in \Omega_0 \subset \Omega$ as $\rho \to 0$, then it yields $\sigma(\rho) = M_s(A(x, u_\rho)) + \omega(\rho^2 ⨍_{B_\rho} |Du|^2\,dx) < \varepsilon$ as $\rho \to 0$ due to the VMO property of $A(x, u)$ in $x \in \Omega$. Observe that $n - 2 < n + 2 - \frac{2}{q}n < n$ if $\frac{n}{2} < q < n$, by the iteration lemma it follows

$$\int_{B_{\frac{\rho}{2}}}\Big(|Du|^2 + |u|^\gamma\Big)\,dx \le C\Big(\frac{\rho}{R}\Big)^{n+2-\frac{2}{q}n}\int_{B_R}\Big(|Du|^2 + |u|^\gamma\Big)\,dx + C\rho^{n+2-\frac{2}{q}n}\|g\|^2_{L^q(B_R)}, \tag{3.15}$$

which implies $Du \in L^{2,\lambda}(\Omega_0)$ with $\lambda = n + 2 - \frac{2n}{q}$. If $q \ge n$, also by the iteration lemma for any $\epsilon > 0$ we have

$$\int_{B_{\frac{\rho}{2}}}\Big(|Du|^2 + |u|^\gamma\Big)\,dx \le C\Big(\frac{\rho}{R}\Big)^{n-\epsilon}\int_{B_R}\Big(|Du|^2 + |u|^\gamma\Big)\,dx + C\rho^{n-\epsilon}\|g\|^2_{L^q(B_R)}, \tag{3.16}$$

which implies $Du \in L^{2,\lambda}(\Omega_0)$ with $\lambda = n-\epsilon$. Summarizing, in terms of the famous Morrey's lemma one concludes that $u \in C^{0,\alpha}_{\rm loc}(B_\rho, \mathbb{R}^N)$, $\alpha = 2 - \frac{n}{q}$ if $n/2 < q < n$ or $u \in C^{0,\alpha}_{\rm loc}(B_\rho, \mathbb{R}^N)$ for all $\alpha \in (0, 1)$ if $q \ge n$.

Finally, let us recall a "small" hypothesis of the following so-called an excess quantity

$$E(\rho) = \rho^{2-n}\int_{B_\rho(x_0)} |Du|^2\,dx,$$

According to the definition of Ω_0, we attain

$$\Omega \setminus \Omega_0 = \{x \in \Omega : \liminf_{\rho\to 0} \rho^{2-n}\int_{B_\rho} |Du|^2\,dx > 0\}.$$

Therefore, by Lemma 2.8, $\mathcal{H}^{n-2}(\Omega \setminus \Omega_0) = 0$. This completes proof. □

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References

[1] Bramanti, M.; Cerutti, M. C.; $W_p^{1,2}$ *solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients*. Comm. Partial Differential Equations 18(9-10)(1993), 1735-1763.

[2] Byun, S.; Wang, L.; *Elliptic equations with BMO coefficients in Reifenberg domains*. Comm. Pure Appl. Math. 57 (10) (2004), 1283-1310.

[3] Chiarenza, F.; Frasca, M.; Longo, P.; $W^{2,p}$ *solvability of the Dirichlet Problem for nonlinear elliptic equations with VMO coefficients.* Trans. Amer. Math. Soc., 336(2) (1993), 841-853.

[4] Chen, S. H.; Tan, Z.; *The method of A-harmonic approximation and optimal interior partial regularity for nonlinear elliptic systems under the controllable growth condition.* J. Math. Anal. Appl., 335 (2007), 20-42.

[5] Daněček, J.; John, O.; Stará, J.; *Morrey space regularity for weak solutions of Stokes systems with VMO coefficients.* Annali di Matematica, 190 (2011), 681-701.

[6] Duzaar,F; Mingione, G.; *Harmonic type approximation lemmas.* J. Math. Anal. Appl., 352(1)(2009), 301-335.

[7] Di Fazio, G.; Ragusa, M. A.; *Interior estimate in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients.* J. Funct. Anal., 112(2) (1993), 241-256.

[8] Dong, Y.; Niu, P. C.; *Estimates in Morrey spaces and Hölder continuity for weak solutions to degenerate elliptic systems.* Manuscripta Math., 138 (2012), 419-437.

[9] Dong, H. J. ; Kim, D.; *Parabolic and elliptic systems with VMO coefficients*. Methods Appl. Anal. 16(3) (2010), 365-388.

[10] Duzaar, F.; Grotowsli, J.; *Optimal interior partial regularity for nonlinear elliptic systems: The method of A-harmonic approximation.* Manuscripta Math., 103 (2000), 267-298.

[11] Duzaar, F.; Steffen, K.; *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals.* J. Reine Angew. Math., 546 (2002), 73-138.

[12] Evans, L. C.; Gariepy, R. F.; *Measure theory and fine properties of functions*. Studies in Advanced Mathematics, CRC Press, 1992

[13] Di Fazio, G.; Ragusa, M. A.; *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coeffcients*. J. Funct. Anal. 112(2) (1993),241-256.

[14] Fan, D. S.; Lu, S. Z.; Yang, D. C.; *Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients*. Georgian Math. J., 5(5) (1998), 425-440.

[15] Giaquinta, M.; *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* in 'Ann. Math. Stud.', Princeton Univ. Press, Vol. 105, 1983.

[16] Gilbarg, D.; Trudinger, N. S.; *Elliptic partial differential equations of second order* (3th edition). Spinger-Verlag, Berlin; 2001.

[17] Di Gironimo, P.; Esposito, L.; Sgambati, L.; *A remark on* $L^{2,\lambda}$ *regularity for minimizers of quasilinear functionals.* Manuscripta Math., 113 (2004), 143-151.

[18] Giaquinta, M.; Modica, G.; *Regularity results for some classes of higher order nonlinear elliptic systems.* J. Reine Angew. Math., 311 (1979), 145-169.

[19] Giaquinta, M.; Modica, G.; *Non-linear systems of the type of the ststionary Navier-Stokes system.* J. Reine Angew. Math., 330 (1982), 173-214.

[20] Krylov, N. V.; *On weak uniqueness for some diffusions with discontinuous coefficients*. Stoch. Proc. and Appl. 113(1)(2004),37-64.

[21] Krylov, N. V.; *Parabolic and elliptic equations with VMO coefficients*. Comm. Partial Differential Equations 32 (1-3) (2007), 453-475.

[22] Maugeri, A.; Palagachev, D. K.; Softova, L. G.; *Elliptic and parabolic equations with discontinuous coefficients.* Berlin: Wiley-vch Verlag, 2000.

[23] Migione, G.; *Regularity if minima: an invitation to the dark side of the calculus of variations.* Appl. Math., 51(4)(2006),355-425.

[24] Nečas, J.; *Introduction to the theory of nonlinear elliptic equations.* Teubner Verlagsge-Sellschaft, Leipzig, 1983.

[25] Sarason, D.; *Functionsof vanishing mean oscillation.* Trans. Amer. Math. Soc., 207(1975), 391-405.

[26] Simon, L.; *Theorems on regularity and singularity of energy minimizing maps*. Basel, Birkhäuscr, 1996.

[27] Stein, E. M.; *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*. Princeton: Princeton Univ. Press, NJ, 1993.

[28] Zheng, S. Z.; *Partial regularity for quasi-linear elliptic systems with VMO coefficients under natural growth condition.* Chin. Ann. Math., 29(2008),49-58.

[29] Zheng, S. Z.; Feng, Z. S.; *Regularity for quasi-linear elliptic systems with discontinuous coefficients.* Dynamics of PDE, 5(1) (2008), 87-99.

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