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PROPERTIES OF SOLUTION DIAGRAMS FOR BISTABLE EQUATIONS

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Abstract. Bistable equation serves as a simple model of phase transition at an appropriate critical temperature. The structure of its stationary solutions determines the dynamics of the evolutionary model. The norm of a stationary solution depending on the diffusion coefficient is usually depicted in a solution diagram. As far as we know, the qualitative properties of such diagram like continuity and differentiability have not been proved rigorously yet. The purpose of our paper is to fill in this gap.

1. Introduction

Mathematical models of phase transition at an appropriate critical temperature that enables coexistence of two phases of a given substance are usually expressed in terms of the (fourth-order) Cahn-Hilliard equation, see e.g. [1, 2, 3, 4]. The structure of stationary solutions which plays the key role in understanding of its dynamics is quite complicated. That is why alternatively a simplified model based on the (second order) bistable equation, see e.g. [5, 9, 10], is frequently used to explain the slow dynamics in the time-dependent model. For the generalization to quasilinear problems involving the p-Laplacian, see [6, 7, 8].

Semilinear bistable equation is the following second-order parabolic equation,

$$\begin{gathered} u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - F'(u(x,t)), \quad x \in [0,1],\ t > 0 \\ u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in [0,1], \end{gathered} \tag{1.1}$$

where function F represents free energy and typically takes the form of a coercive C^2 function with two minima. It is called a double-well potential. We can associate an energy functional to the problem (1.1),

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_0^1 \big(u'(x)\big)^2\,\mathrm{d}x + \int_0^1 F(u(x))\,\mathrm{d}x, \quad u \in W^{1,2}(0,1).$$

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Its stationary points are time independent solutions of (1.1), i.e. weak solutions of the boundary value problem

$$\begin{gathered} \varepsilon^2 u''(x) - F'(u(x)) = 0, \\ u'(0) = u'(1) = 0. \end{gathered} \tag{1.2}$$

Standard regularity argument for the second-order ordinary differential equations yields that every weak solution u of (1.2) satisfies $u \in C^2[0,1]$. It is useful to depict the solutions in a *solution diagram*, showing their dependence on parameters ε and $\theta = u(0)$.

For the standard choice of double-well potential $F(z) = (1-z^2)^2$ the solution diagram is known and obtained by numerical computations. The novelty of this paper is to provide rigorous proofs of some properties of the solution diagram, even for wider class of *double-well potentials* such as given below in Definitions 1.1 and 1.2.

1.1. **Smooth double-well potential.** We introduce the following definition.

Definition 1.1 (Smooth Double-Well Potential)**.** Let $F \in C^2(\mathbb{R})$ be even and coercive and have exactly two global minimizers at ± 1. Let $F''(z) = 0$ have only two roots $\pm\xi$, $\xi \in (0,1)$ and let F be strictly decreasing in $(0,1)$ and strictly increasing in $(1,+\infty)$. Then we call F a *smooth double-well potential*.

For the sake of simplicity we further assume that $F(z) = 0 \iff z = \pm 1$ and thus $F(z) \geq 0$ for all $z \in \mathbb{R}$.

Typical examples of a smooth double-well potential are the fourth-order polynomial $F_P(z) = (1-z^2)^2, z \in \mathbb{R}$, or the goniometric function $F_G(z) = 1+\cos(\pi z), z \in (-1,1)$, suitably extended to the whole $\mathbb{R}$. The logarithmic potential,

$$\widetilde{F_L}(z) = -A_0 z^2 + A_1((1+z)\ln(1+z) + (1-z)\ln(1-z)), \quad z \in (-1,1), \tag{1.3}$$

where $A_0 > A_1 > 0$, is even and has two minimizers–transcendental points $\pm\nu$ and inflection points $\pm\xi = \pm\sqrt{1 - A_1/A_0}$. The rescaled shifted potential $F_L(z) = \widetilde{F_L}(\nu z) - \widetilde{F_L}(\nu)$, suitably extended to the whole $\mathbb{R}$, fulfils the properties required in Definition 1.1. Comparison of potentials F_P, F_G, F_L is shown in Figure 1.

1.2. **Non-smooth double-well potential.** Further we consider potentials losing the C^2-smoothness in their minimizers.

Definition 1.2. Let $F \in C^1(\mathbb{R})$ be even and coercive, strictly decreasing in $(0,1)$ and strictly increasing in $(1,+\infty)$. Let $F''(z)$ be defined in $\mathbb{R} \setminus \{\pm 1\}$, continuous in its domain and let $F''(z) = 0$ have only two roots $\pm\xi$, $\xi \in (0,1)$, F being strictly convex in $(\xi,+\infty)$ and strictly concave in $(0,\xi)$. Let there exist constants $\sigma \in (0, 1-\xi), \alpha \in (1,2)$ and $0 < \beta_1 < \beta_2$ such that

$$\beta_1 |z-\nu|^\alpha \leq F(z) - F(\nu) \leq \beta_2 |z-\nu|^\alpha \quad \text{for } \nu = \pm 1 \text{ and } \forall z \in (\nu-\sigma, \nu+\sigma). \tag{1.4}$$

Then F is called a *non-smooth double-well potential*.

From the definition above it follows that F has exactly two global minimizers at ± 1. By the same token as in the previous section we assume that $F(\pm 1) = 0$ and thus $F \geq 0$.

A typical example of non-smooth double-well potential is the function $F_\alpha(z) = |1-z^2|^\alpha$, $\alpha \in (1,2), z \in \mathbb{R}$.

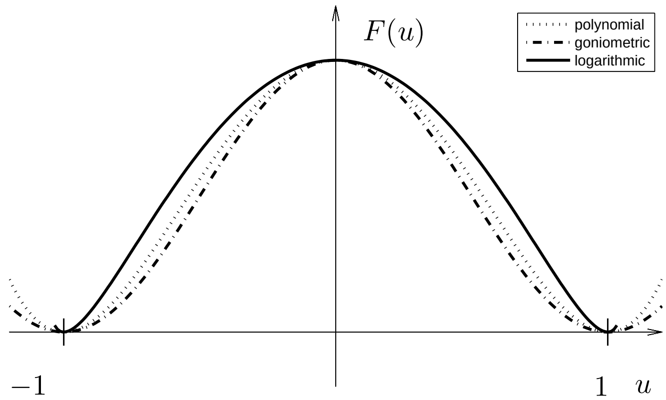

FIGURE 1. Comparison of smooth potentials: polynomial (F_P), goniometric (F_G) and logarithmic (F_L). Potentials were multiplied by a suitable constant to have equal value at 0.

1.3. **Definition of solution diagram.** Further we establish the definition of solution diagram, where $\mathbb{R}^+ := \{x \in \mathbb{R}, x > 0\}$.

Definition 1.3. The set

$$\begin{aligned} \mathcal{S} := \big\{(\theta,\varepsilon) \in \mathbb{R} \times \mathbb{R}^+ : &\text{(1.2) with parameter } \varepsilon \text{ has a non-constant solution} \\ &u \in C^2[0,1] \text{ satisfying } u(0) = \theta\big\}, \end{aligned}$$

is called *solution diagram* of (1.2) with potential F, or shortly, solution diagram for potential F.

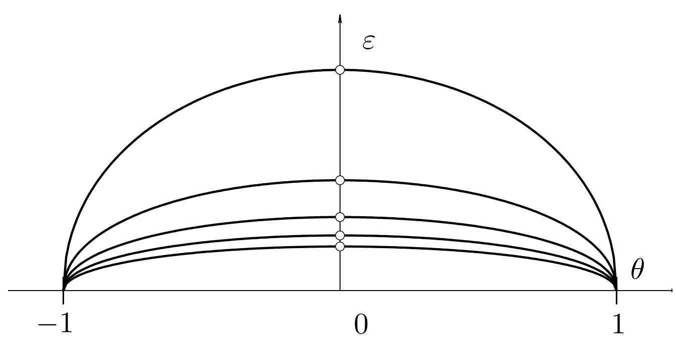

FIGURE 2. Sketch of the first five branches of the solution diagram for a smooth potential.

1.4. **Results.** The main novelty of this paper is to provide proofs of natural properties of the branches that build the solution diagram for both smooth and non-smooth double-well potentials, see the sketches in Figures 2 and 3, respectively. To keep the sketches clear we depict only the first five branches.

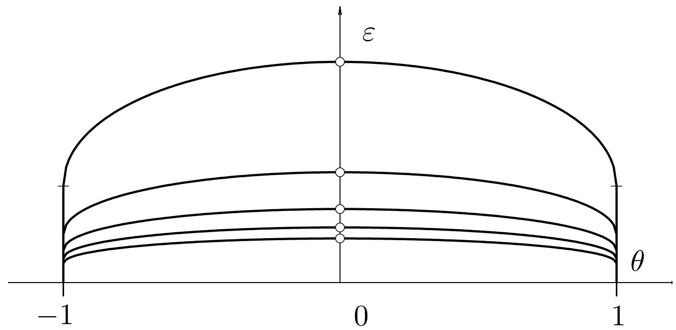

FIGURE 3. Sketch of the first five branches of the solution diagram for a non-smooth potential. The continua of solutions occur at ± 1 for small values of ε.

TABLE 1. Properties of solution diagrams

type of potential	**smooth**	**non-smooth**
continuity in $(0,1)$	yes (Prop. 4.1)	
limit at 0	$\in \mathbb{R}^+$, if it exists (Prop. 4.1)	
limit at 1^-	0 (Prop. 4.2)	$\in \mathbb{R}^+$ (Prop. 4.3)
continuous differentiability in $(0,1)$	yes (Prop. 4.4)	
limit of the derivative at 0	0, provided $F \in C^3(U(0,\delta))$ (Thm. 4.5)	
limit of the derivative at 1^-	$-\infty$ (Thm. 4.9)	$-\infty$ (Thm. 4.7)
monotonicity	criterion (Thm. 4.12)	
existence of limit at 0	two criteria (Thm. 4.5 & Cor. 4.14)	

Table 1 summarizes the properties of solution diagrams for both types of potentials we prove in this paper.

This article is organized as follows. In preliminary Section 2 we summarize some technical results concerning convex functions for later use. In Section 3 we introduce construction of solutions to boundary value problem (1.2), presented in [8]. Section 4 is devoted to continuity properties of the solution diagram.

2. Preliminaries - properties of convex functions

We will need several properties of convex (and concave) functions in our proofs. For the sake of clarity we introduce the following lemmas in a separate section.

Lemma 2.1. *Let $\mathcal{I} = (a,b)$ and $g \in C^2(\overline{\mathcal{I}})$ and let there exist $m > 0$ such that $g''(z) \geq 2m > 0$ (or $g''(z) \leq -2m < 0$) in $\overline{\mathcal{I}}$. Then there exists $K > 0$ such that for any $x, y, \xi_{xy} \in \mathcal{I}$ satisfying*

$$g(y) - g(x) = g'(\xi_{xy})(y - x),$$

it holds that

$$(\xi_{xy} - x) > K(y - x) \quad \textit{and} \quad (y - \xi_{xy}) > K(y - x).$$

In other words the Lagrangian point ξ_{xy} cannot be located *too close* to the end points of interval $\mathcal{I}$.

Proof. Let $g''(z) \geq 2m > 0, z \in \mathcal{I}$. Using the Taylor expansion with Lagrangian residue we can write

$$g(y) - g(x) \geq g'(x)(y - x) + m(y - x)^2, \tag{2.1}$$

for any $x, y \in \overline{\mathcal{I}}$. Using the Mean Value Theorem twice in (2.1) yields

$$g''(\tilde{\xi})(\xi_{xy} - x)(y - x) \geq m(y - x)^2,$$

for some $\tilde{\xi} \in \mathcal{I}$.

As $\sup_{\mathcal{I}} g''(z) < \infty$, we get the desired result with

$$K = \frac{m}{2 \sup_{\mathcal{I}} g''(z)}.$$

The other case is similar. □

The following lemma is a direct consequence of strict convexity (concavity) of g.

Lemma 2.2. *Let $g \in C^1(\overline{\mathcal{I}}) \cap C^2(\mathcal{I})$ be strictly convex (strictly concave) and $x_1 < x_2 < y$, ξ_i be the Lagrangian point of the interval $(x_i, y), i \in \{1, 2\}$, i.e.,*

$$g(y) - g(x_i) = g'(\xi_i)(y - x_i), \quad i = 1, 2.$$

Then $\xi_1 < \xi_2$.

An analogous lemma also holds when switching the roles of the border points, like in Figure 4.

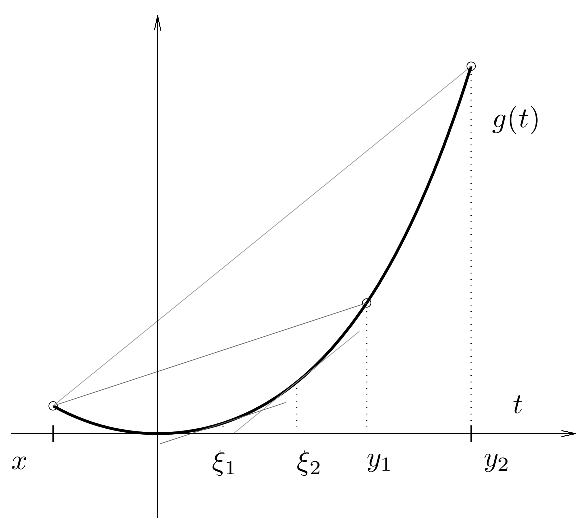

Figure 4. Illustration to the result in Lemma 2.2.

Lemma 2.3. *Let $\mathcal{I} = (a, b)$, $g \in C^1(\overline{\mathcal{I}}) \cap C^2(\mathcal{I})$ be strictly convex, $x, y \in \mathcal{I}, x < y$. Then for any $h \in (0, b - y)$ it holds that*

$$g(x) - g(y) > g(x + h) - g(y + h).$$

Proof. By the Mean Value Theorem there exist $\xi_0 \in (x, y), \xi_h \in (x + h, y + h)$ and $\xi_c \in (x, y + h)$ such that

$$g(x) - g(y) = -g'(\xi_0)(y - x), \tag{2.2}$$

$$g(x + h) - g(y + h) = -g'(\xi_h)(y - x), \tag{2.3}$$

$$g(x) - g(y + h) = -g'(\xi_c)(y + h - x). \tag{2.4}$$

Then applying the previous result we get $\xi_0 < \xi_c$ and $\xi_c < \xi_h$, hence $\xi_0 < \xi_h$. Since g' is strictly increasing, then

$$g'(\xi_0) < g'(\xi_h), \tag{2.5}$$

and substituting from (2.2) and (2.3) into (2.5) we get the desired inequality. □

3. Stationary solutions

In this section we briefly introduce stationary solutions of (1.1), i.e. solutions to the boundary value problem

$$\begin{gathered} \varepsilon^2 u''(x) = F'(u(x)), \\ u'(0) = u'(1) = 0, \end{gathered}$$

cf. [8]. First, we present the solutions that are common for both smooth and non-smooth double-well potentials F. At the end of this section we introduce briefly the continua of solutions that only occur in the non-smooth case, cf. Figure 3.

We get three constant solutions $\{\pm 1, 0\}$ to (1.2) and therefore three constant stationary points of J_ε, having the character of two global minima and saddle point, respectively. (The character of ± 1 is obvious, as $J_\varepsilon(u) \ge 0$ and $J_\varepsilon(\pm 1) = 0$. Local maximizer of the potential F gives $J_\varepsilon(0) = F(0)$. Then $J_\varepsilon(\delta) < F(0) < J_\varepsilon(\delta \sin j\pi x)$ for δ small enough and $j \in \mathbb{N}$ large enough. See [8] for details.) To get the non-constant solutions we employ the initial value problem

$$\begin{gathered} \varepsilon^2 u''(x) = F'(u(x)), \\ u(0) = \theta, \\ u'(0) = 0. \end{gathered} \tag{3.1}$$

The Existence and Uniqueness Theorem for (3.1) justifies the use of the shooting method. Then testing the condition $u'(1) = 0$ will give us couples of parameters (θ, ε) that provide a solution to the boundary value problem (1.2). We can restrict our attention just to $\theta \in (0, 1]$, as the monotonicity of F in $(1, +\infty)$ avoids the existence of solution for (1.2) for $\theta > 1$ and situation for $\theta < 0$ follows from the symmetry of F. The uniqueness also implies non-existence of non-constant solution for $\theta = 0$. The case $\theta = 1$ will be treated separately later in this section.

Let $\theta \in (0, 1)$. Using separation of variables, one comes to an implicit formula of solution to the initial value problem (3.1) in the form

$$x = \frac{\varepsilon}{\sqrt{2}} \int_{u(x)}^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}, \quad x \in [0, x_0], \tag{3.2}$$

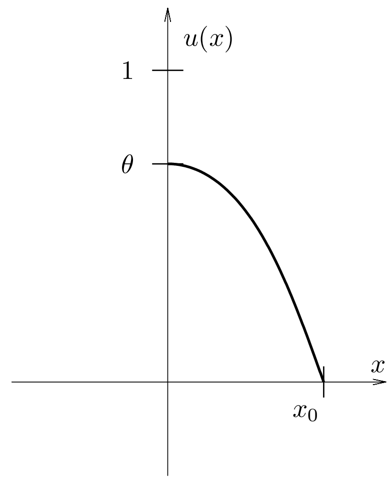

FIGURE 5. Sketch of construction of a solution to (1.2) via initial value problem (3.1).

where

$$x_0 = x_0(\theta, \varepsilon) = \frac{\varepsilon}{\sqrt{2}} \int_0^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}. \tag{3.3}$$

See the sketch in Figure 5.

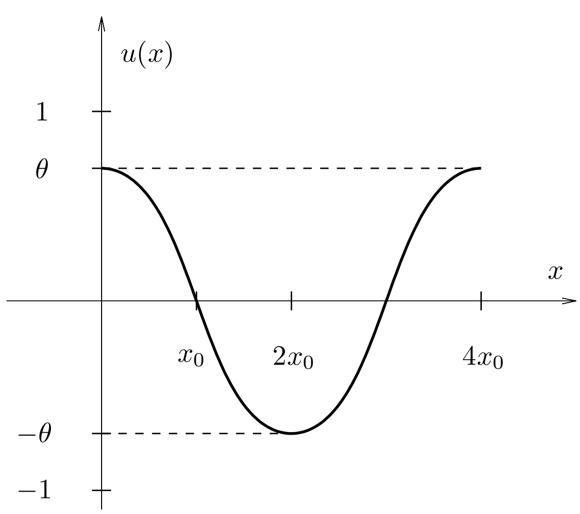

FIGURE 6. Solution of initial value problem (3.1), extended to interval $(0, 4x_0)$.

Note that $u(x_0) = 0$ and hence (3.2) only describes the non-negative part of solution u. The symmetry of F and the invariance of the equations in (3.1) with respect to the transformation $x \mapsto -x$ allows us to extend the solution as follows,

$$u(x) := \begin{cases} -u(2x_0 - x), & x \in (x_0, 2x_0], \\ u(4x_0 - x), & x \in (2x_0, 4x_0], \end{cases}$$

as it is sketched in Figure 6.

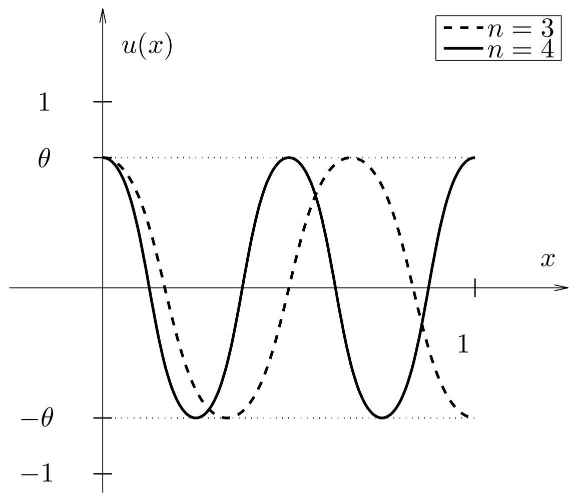

FIGURE 7. Solutions $u_{\theta,\varepsilon_3(\theta)}$ (dashed) and $u_{\theta,\varepsilon_4(\theta)}$ (solid).

As u given by the implicit formula (3.2) depends on both θ and ε, we will (for the purposes of the following paragraph) denote it by $u_{\theta,\varepsilon}$ and its first zero point by $x_0(\theta,\varepsilon)$. Further, we can extend $u_{\theta,\varepsilon}$ periodically on the entire $\mathbb{R}$. For any $\theta \in (0,1)$ and $n \in \mathbb{N}$ given we can construct an n-nodal solution $u_{\theta,1}$ to the initial value problem (3.1) with $\varepsilon = 1$ in $[0, L]$, where

$$L = L(n,\theta) := 2nx_0(\theta, 1),$$

and $u'(L) = 0$. Thanks to the linear dependence of x_0 on ε in (3.3), taking $\varepsilon_n(\theta) := 1/L(n,\theta)$ we get

$$2nx_0(\theta, \varepsilon_n(\theta)) = 1. \tag{3.4}$$

Hence $u_{\theta,\varepsilon_n(\theta)}$ has n nodal points in $(0,1)$ and is a solution to the boundary value problem (1.2) with $u(0) = \theta$ and $\varepsilon = \varepsilon_n(\theta)$, see Figure 7 for illustration.

We stress the above observations in the next lemma, which provides a complete description of the set of non-constant stationary points of J_ε satisfying $u(0) = \theta$.

Lemma 3.1. *For $n \in \mathbb{N}$ and $\theta \in (0,1)$ given we can find $\varepsilon = \varepsilon_n(\theta)$ such that the unique solution $u_{\theta,\varepsilon_n(\theta)}$ of the initial value problem* (3.1) *with n nodes satisfies $u'_{\theta,\varepsilon_n(\theta)}(1) = 0$ and therefore, it is a solution to the boundary value problem* (1.2).

Let $\theta = 1$. We treat this case separately for both types of double-well potential (smooth and non-smooth), as the result depends on the existence of the second derivative of F at 1.

For smooth double-well potential $F \in C^2(\mathbb{R})$ the Uniqueness Theorem avoids the existence of non-constant solutions of the initial value problem (3.1) with $\theta = 1$.

For a non-smooth potential F, one can show that the function given by implicit formula (3.2) with $\theta = 1$, i.e.

$$x = \frac{\varepsilon}{\sqrt{2}} \int_{u(x)}^{1} \frac{\mathrm{d}s}{\sqrt{F(s)}}, \quad x \in [0, 2x_0],$$

is indeed a solution to the initial value problem (3.1), see [7, Lemma 4.6.] for details. Like in (3.3) we set

$$x_0 = x_0(\varepsilon) = \frac{\varepsilon}{\sqrt{2}} \int_{0}^{1} \frac{\mathrm{d}s}{\sqrt{F(s)}}. \tag{3.5}$$

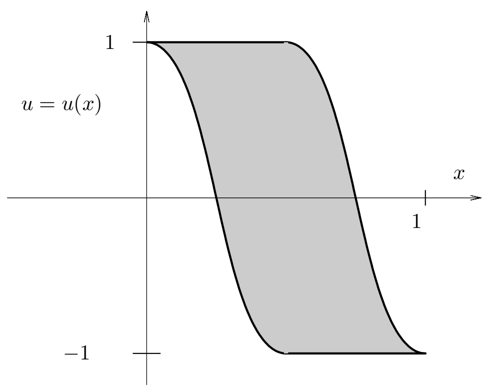

FIGURE 8. Sketch of solutions to (1.2) starting at $\theta = 1$ that build a one-dimensional continuum.

Then, for $\theta = 1$, the initial value problem (3.1) has a solution that is composed of constant solutions ± 1 and transitions between them, see Figure 8. For $\varepsilon > 0$ small, the transition can occur at an arbitrary point. Hence one gets a continua of solutions of the boundary value problem (1.2). More to these solutions can be found in [7] or [8]. Note that for $\varepsilon \to 0$, the number of continua increases as more solutions with multiple transitions between ± 1 occur. The dimension of every single continuum corresponds to the number of nodes. These continua are depicted in the solution diagram by vertical segments at ± 1, see Figure 3.

4. Solution diagram

For the whole article, we confine ourselves to $\theta \in (0, 1]$. Thanks to F being even it holds that

$$(\theta, \varepsilon) \in \mathcal{S} \iff (-\theta, \varepsilon) \in \mathcal{S}, \tag{4.1}$$

and there is no contribution to $\mathcal{S}$ by $|\theta| > 1$.

Our goal is to show the properties of *branches* $\varepsilon_n(\theta)$ that build the solution diagram.

4.1. **Continuity properties of the branches in** $(0,1)$**.** We introduce the following proposition about the continuity of $\varepsilon_n(\theta)$ in the open interval $(0,1)$ and the boundedness in neighbourhood of 0. These two results do not allow us to conclude the existence of limit at 0. In particular, we cannot exclude oscillatory behaviour of $\varepsilon_n(\theta)$ near 0 that may be affected by possible oscillatory blow-up of the third derivative of potential F.

Proposition 4.1. *Let F be a double-well potential (both smooth or non-smooth). Then,*

(a) *$\varepsilon_n(\theta)$ is continuous in $(0,1)$,*
(b) *there exist $\delta, C_1, C_2 > 0$ such that for all $\theta \in U^+(0,\delta)$ we have*

$$0 < \frac{C_1}{n} < \varepsilon_n(\theta) < \frac{C_2}{n}.$$

Proof. From (3.3) and (3.4) we obtain

$$\varepsilon_n(\theta) = \frac{\sqrt{2}}{2n\int_0^\theta \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}} = \frac{\sqrt{2}}{2nI(\theta)}, \tag{4.2}$$

where

$$I(\theta) = \int_0^\theta \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}. \tag{4.3}$$

The substitution $s = \theta a$ leads to the integral over $(0,1)$,

$$I(\theta) = \theta\int_0^1 \frac{\mathrm{d}a}{\sqrt{F(\theta a)-F(\theta)}}. \tag{4.4}$$

(a) The continuity of $\varepsilon_n(\theta)$ is proven by the Lebesgue Dominated Convergence Theorem. Take $\theta_0 \in (0,1)$ arbitrary but fixed. As $I(\theta) \neq 0$, thanks to (4.2) it suffices to show the continuity of $I(\theta)$ at θ_0. Focusing on (4.4) we need a dominating integrable function, independent of θ. Note that the pointwise convergence of the integrand (for $\theta \to \theta_0$) is obvious. We choose δ_0 small enough, such that $U(\theta_0,\delta_0) \subset (0,1)$. For $\theta \in U(\theta_0,\delta_0/2)$, there exists $\delta \in (0,1)$ such that $(1-\delta)\theta \in U(\theta_0,\delta_0)$. The assumptions on F allow to define the dominating function as follows:

$$M(a) := \begin{cases} \frac{2\sqrt{\theta_0+\delta_0/2}}{\sqrt{\underline{F_1}\delta}}, & \text{for } a \in (0,1-\delta], \\ \frac{2\sqrt{\theta_0+\delta_0/2}}{\sqrt{\underline{F_1}(1-a)}}, & \text{for } a \in (1-\delta,1), \end{cases}$$

where $\underline{F_1} := \inf_{z\in U(\theta_0,\delta_0)} |F'(z)| > 0$.

(b) By the Mean Value Theorem there exist c_1 and c_2 such that $\theta > c_1 > s > 0, c_1 > c_2 > 0$ and

$$|F(s)-F(\theta)| = |F'(c_1)(\theta-s)| = |(F'(c_1)-F'(0))(\theta-s)| = |F''(c_2)c_1(\theta-s)|.$$

From here we obtain

$$\underline{F_2}s(\theta-s) \leq |F(s)-F(\theta)| \leq \overline{F_2}\theta(\theta-s),$$

where $\underline{F_2} := \inf_{z\in U(0,\delta)} |F''(z)|$ and $\overline{F_2} = \sup_{z\in U(0,\delta)} |F''(z)|$. Substituting these two estimates into (4.3), we obtain

$$\frac{\pi}{\underline{F_2}} \geq I(\theta) \geq \frac{2}{\overline{F_2}}, \tag{4.5}$$

where we used the identity

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = \pi.$$

Substituting (4.5) to (4.2) completes the proof. $\square$

4.2. **Continuity of the branches at ± 1.** Smooth and non-smooth potentials exhibit different local behaviour at ± 1 which is also reflected in the appearance of its solution diagrams, compare Figures 2 and 3. The following two propositions show the difference in a rigorous way.

Proposition 4.2. *Let F be a smooth double-well potential. Then*

$$\lim_{\theta \to 1^-} \varepsilon_n(\theta) = 0.$$

Proof. Thanks to (4.2) we show equivalently $\lim_{\theta\to 1^-} I(\theta) = +\infty$. Using (4.4) and the Fatou's lemma, we obtain

$$\lim_{\theta \to 1^-} \theta \int_0^1 \frac{\mathrm{d}a}{\sqrt{F(\theta a) - F(\theta)}} \geq \int_0^1 \frac{\mathrm{d}a}{\sqrt{F(a)}} \geq \int_{1-\delta}^1 \frac{\mathrm{d}a}{\sqrt{F(a)}},$$

for any δ small. Using Taylor expansion of F at 1 we get that the last integral diverges as there exists $c \in (a,1)$ such that $F(a) = 1/2F''(c)(1-a)^2$ and $0 < F'' <$ *const*. near the minimizers of F. $\square$

The lack of C^2 continuity at ± 1 produces a shift of the limit of branches. Note that $\varepsilon_n(1) \in \mathbb{R}^+$ is well defined through (3.4) and (3.5).

Proposition 4.3. *Let F be a non-smooth double-well potential. Then*

$$\lim_{\theta \to 1^-} \varepsilon_n(\theta) = \varepsilon_n(1) \in \mathbb{R}^+.$$

Proof. Similarly to the proof of Proposition 4.2, thanks to (4.2) we equivalently show $I(\theta) \to I(1) := \int_0^1 \frac{\mathrm{d}a}{\sqrt{F(a)}}$ as $\theta \to 1^-$, using the Vitali Convergence Theorem. Note that for integrands in $I(\theta)$ and $I(1)$

$$\frac{\theta}{\sqrt{F(\theta a) - F(\theta)}} \to \frac{1}{\sqrt{F(a)}},$$

holds pointwise. The finite integrability of the limit function can be shown directly using the assumption (1.4). To finish the proof we show that there exists $c > 0$ such that $I(\theta) < c$ for any $\theta \in (1 - \sigma/2, 1)$, where σ is from Definition 1.2.

We split the integration to obtain

$$\begin{aligned} I(\theta) &= \int_0^\theta \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} \\ &= \int_0^{1-\sigma} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} + \int_{1-\sigma}^\theta \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} \\ &\leq \frac{1}{\sqrt{F(1-\sigma) - F(1-\sigma/2)}} + \int_{1-\sigma}^\theta \frac{\mathrm{d}s}{\sqrt{F(s + (1-\theta))}}, \end{aligned} \tag{4.6}$$

where in the last inequality we used monotonicity of F in $(0,1)$ and Lemma 2.3. Next, we can treat the latter term in (4.6) using (1.4) to obtain

$$\int_{1-\sigma}^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s+(1-\theta))}} \le \frac{1}{\beta_1^{1/2}(1-\alpha/2)}|1-\sigma-\theta|^{1-\alpha/2} \le \frac{\sigma^{1-\alpha/2}}{\beta_1^{1/2}(1-\alpha/2)}.$$

The conclusion that $I(\theta) \le c < \infty$ for any $\theta \in (1-\sigma/2, 1)$ is accomplished recalling that $\alpha \in (1,2)$. Therefore, the Vitali Theorem can be applied and hence $I(\theta) \to I(1)$ and also $\varepsilon_n(\theta) \to \varepsilon_n(1) \in \mathbb{R}^+$. □

4.3. **Continuous differentiability of the branches.**

Proposition 4.4. *Let F be a double-well potential (either smooth or non-smooth). Then the branches $\varepsilon_n(\theta)$ are continuously differentiable in $(0,1)$.*

Proof. From (4.2) we can write

$$\varepsilon_n'(\theta) = \frac{-\sqrt{2}}{2n} \frac{1}{[I(\theta)]^2} I'(\theta). \tag{4.7}$$

Thanks to the continuity of $I(\theta)$ and its positivity, it remains to show that $I'(\theta)$ is continuous at any $\theta_0 \in (0,1)$. From (4.4) we obtain

$$\begin{aligned} I'(\theta) &= \int_0^1 \frac{(F(\theta a) - \frac{1}{2}\theta a F'(\theta a)) - (F(\theta) - \frac{1}{2}\theta F'(\theta))}{(F(\theta a) - F(\theta))^{\frac{3}{2}}} \,\mathrm{d}a \\ &= \int_0^1 \frac{G(\theta a) - G(\theta)}{(F(\theta a) - F(\theta))^{\frac{3}{2}}} \,\mathrm{d}a, \end{aligned} \tag{4.8}$$

where we denoted

$$G(z) := F(z) - \frac{1}{2} z F'(z). \tag{4.9}$$

Let $\theta \to \theta_0$. We will use the Lebesgue Theorem to prove $I'(\theta) \to I'(\theta_0)$. The pointwise convergence of the integrand is obvious. We will construct an integrable dominating function $M = M(a)$ of the integrand in (4.8), independent of θ. We take δ_0 small enough such that $U(\theta_0, \delta_0) \subset (0,1)$. Then we define $\overline{G_1} := \sup_{z \in U(\theta_0,\delta_0)} |G'(z)|$ and $\underline{F_1} := \inf_{z \in U(\theta_0,\delta_0)} |F'(z)| > 0$. There exists $\delta \in (0,1)$ such that for any $\theta \in U(\theta_0, \delta_0/2)$ and $a \in (1-\delta, 1]$ we have $\theta a \in U(\theta_0, \delta_0)$ and the estimate

$$\frac{G(\theta a) - G(\theta)}{(F(\theta a) - F(\theta))^{\frac{3}{2}}} \le \frac{\overline{G_1}\theta(1-a)}{(\underline{F_1}\theta(1-a))^{\frac{3}{2}}} \le \frac{\overline{G_1}\,\underline{F_1}^{-\frac{3}{2}}}{\sqrt{(\theta_0 - \delta_0)(1-a)}},$$

holds.

For $\theta \in U(\theta_0, \delta_0/2)$ and $a \in (0, 1-\delta)$ the denominator of the integrand in (4.8) is bounded from below by $|F(\theta(1-\delta)) - F(\theta)|^{3/2} \ge |\underline{F_1}\theta\delta|^{3/2} > |\underline{F_1}(\theta_0 - \delta_0)\delta|^{3/2} > 0$, while the numerator is continuous in a compact interval and therefore bounded. Hence we constructed the dominating function

$$M(a) = \begin{cases} c(F, \theta_0, \delta_0, \delta) & \text{for } a \in (0, 1-\delta), \\ \frac{\overline{G_1}\,\underline{F_1}^{-3/2}}{\sqrt{\theta_0 - \delta_0}\sqrt{1-a}}, & \text{for } a \in (1-\delta, 1). \end{cases}$$

The proof is concluded observing that $M \in L^1(0,1)$. □

4.4. **Limit of derivative of branches at 0.**

Theorem 4.5. *Let $\delta \in (0,1)$ and $F \in C^3(U(0,\delta))$ be a (either smooth or non-smooth) potential. Then $\lim_{\theta\to 0} \varepsilon_n'(\theta) = 0$ and the following limit exists:*
$$\lim_{\theta\to 0} \varepsilon_n(\theta) \in \mathbb{R}^+. \tag{4.10}$$

Proof. We recall (4.7):
$$\varepsilon_n'(\theta) = -\frac{\sqrt{2}}{2n}\frac{I'(\theta)}{[I(\theta)]^2}.$$
From (4.5) we know that $I(\theta)$ is bounded from below by a positive constant for any $\theta \in U(0,\delta)$. Hence it suffices to show that
$$\lim_{\theta\to 0} I'(\theta) = 0, \tag{4.11}$$
to prove the first assertion. Moreover, from (4.11) we can also deduce the existence of $\lim_{\theta\to 0} \varepsilon_n(\theta)$. Indeed, (4.5) gives boundedness of $I(\theta)$ and (4.11) excludes its oscillations. Hence the existence of $\lim_{\theta\to 0} I(\theta) \in \mathbb{R}^+$ follows and (4.10) is proved. Therefore the proof reduces to proving (4.11).

We first show the existence of one-sided limit $\lim_{\theta\to 0^+} I'(\theta) = 0$. We recall that
$$I'(\theta) = \int_0^1 \frac{G(\theta a) - G(\theta)}{(F(\theta a) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}a, \tag{4.12}$$
where G is defined by (4.9). Since we are investigating the limit at 0, we can consider $\theta \in (0,\overline{\delta})$ only, where $\overline{\delta} = \min\{\delta, \xi/2\}$ (see Definitions 1.1 and 1.2 for ξ). Using the Mean Value Theorem one can express the numerator as
$$G(\theta a) - G(\theta) = -G'(x_G)\theta(1-a) = -G''(x_{GG})\theta(1-a)x_G,$$
for some Lagrangian points $x_{GG} < x_G$ in $(0,\theta)$. We have used that $G'(0) = 0$. Moreover, $G''(z) = -\frac{1}{2}zF'''(z)$ and therefore
$$G(\theta a) - G(\theta) \le |F'''(x_{GG})|\theta^3(1-a). \tag{4.13}$$

Next, let x_0 and x_F be Lagrangian points of F in intervals $(0,\theta)$ and $(\theta a,\theta)$, respectively. It follows from Lemma 2.2 applied to $g = F, x_1 = 0, x_2 = \theta a$ and $y = \theta$ that $x_0 < x_F$.

Hence
$$F(\theta a) - F(\theta) = -F'(x_F)\theta(1-a) > -F'(x_0)\theta(1-a), \tag{4.14}$$
due to the fact that F' is monotone decreasing in $(0,\overline{\delta})$. We use the Mean Value Theorem again for F' in the interval $(0,x_0)$ and get the Lagrangian point x_{00} such that
$$-F'(x_0)\theta(1-a) = -(F'(x_0) - F'(0))\theta(1-a) = -F''(x_{00})x_0\theta(1-a). \tag{4.15}$$
Notice that F satisfies the assumptions of Lemma 2.1 in $\mathcal{I} = (0,\overline{\delta})$. The choice $x = 0, y = \theta$ and $\xi_{xy} = x_0$ leads to a constant $K > 0$ independent of θ such that
$$x_0 > K\theta. \tag{4.16}$$
Denote $c(F) := (K \inf_{U^+(0,\overline{\delta})} |F''(z)|)^{3/2}$. Then it follows from (4.14)-(4.16) that denominator of (4.12) is estimated from below as follows
$$(F(\theta a) - F(\theta))^{\frac{3}{2}} \ge c(F)(1-a)^{3/2}\theta^3. \tag{4.17}$$

Substituting (4.13) and (4.17) into (4.12) we get a local estimate

$$|I'(\theta)| \le \int_0^1 \Big|\frac{F'''(x_{GG})\theta^3(1-a)}{c(F)\theta^3(1-a)^{3/2}}\,\mathrm{d}a\Big| \le 2\frac{\overline{F_3}(\theta)}{c(F)}, \quad \theta \in (0,\overline{\delta}), \tag{4.18}$$

where $\overline{F_3}(\theta) := \sup_{z\in(0,\theta)} |F'''(z)|$. Since F is even, then $F'''(0) = 0$ from which it follows that $\lim_{\theta\to 0^+} \overline{F_3}(\theta) = 0$ by the assumption $F \in C^3(U(0,\delta))$. Then from (4.18) we get $\lim_{\theta\to 0^+} I'(\theta) \to 0$ and by symmetry we get $\lim_{\theta\to 0^-} I'(\theta) \to 0$ which concludes the proof. □

Corollary 4.6. *Assume the same as in Theorem 4.5. Set $\varepsilon_n(0) := \lim_{\theta\to 0} \varepsilon_n(\theta)$. Then $\varepsilon_n \in C^1(-1,1)$.*

Proof. The assertion is a consequence of Propositions 4.1 and 4.4, symmetry property (4.1) and Theorem 4.5. □

For potential with oscillatory blow-up of F''' at zero we lose the upper bound (4.18) that prevents oscillations of $I(\theta)$. However, proving that the oscillation of branches do occur remains an open question.

4.5. **Limit of derivative of branches at ± 1.** Unlike the previous section, the assertion about limits at ± 1 does not require any additional assumptions on F. We must distinguish between smooth and non-smooth potentials; despite the assertions give the same result, the proofs are way too different in either case. Due to the symmetry we concentrate on limit at 1.

Theorem 4.7. *Let F be a non-smooth double-well potential. Then*

$$\lim_{\theta\to 1^-} \varepsilon_n'(\theta) = -\infty.$$

We start with the following technical estimate.

Lemma 4.8. *Let F be a non-smooth double-well potential. Then for any $s \in (1-\sigma, 1)$ we have*

$$-F'(s) > \beta_1(1-s)^{\alpha-1},$$

where the constants α, β_1, σ were introduced in Definition 1.2.

Proof. Combining the consequence of strict convexity of F in $(1-\sigma,1)$,

$$F(s) + F'(s)(1-s) < F(1),$$

with (1.4), we obtain

$$-F'(s)(1-s) > \beta_1(1-s)^{\alpha}, \quad s \in (1-\sigma, 1).$$

Dividing of both sides by $(1-s)$ completes the proof. □

Proof of Theorem 4.7. Using (4.7) and the fact that $\lim_{\theta\to 1^+} I(\theta) \in \mathbb{R}^+$ (from the proof of Proposition 4.3) we equivalently show that $I'(\theta) \to \infty$ as $\theta \to 1^-$.

Let $\theta \in (1-\sigma/2, 1)$ and $\delta \in (\sigma/2, \sigma)$. Then

$$\begin{aligned}\lim_{\theta\to 1^-} I'(\theta) &= \lim_{\theta\to 1^-} \int_0^1 \frac{G(\theta a) - G(\theta)}{(F(\theta a) - F(\theta))^{3/2}}\,\mathrm{d}a \\ &= \lim_{\theta\to 1^-} \Big(\int_0^{1-\delta} \dots + \int_{1-\delta}^1 \dots\Big) =: \lim_{\theta\to 1^-} \big(V_1(\theta) + V_2(\theta)\big).\end{aligned}$$

$V_1(\theta)$ is bounded, as the numerator is continuous and the denominator can be estimated from below by a positive constant. Fatou's lemma implies

$$\lim_{\theta\to 1^-} V_2(\theta) \geq \int_{1-\delta}^{1} \frac{G(a)}{(F(a))^{\frac{3}{2}}}\,\mathrm{d}a.$$

Thanks to (4.9) we can further estimate

$$\int_{1-\delta}^{1} \frac{G(a)}{F(a)^{\frac{3}{2}}}\,\mathrm{d}a = \int_{1-\delta}^{1} \Big(\frac{1}{\sqrt{F(a)}} + \frac{-1/2aF'(a)}{F(a)^{3/2}}\Big)\,\mathrm{d}a \geq \int_{1-\delta}^{1} \frac{-1/2aF'(a)}{F(a)^{\frac{3}{2}}}\,\mathrm{d}a.$$

From (1.4) we have

$$F(a)^{3/2} \leq \beta_2^{3/2}(1-a)^{3\alpha/2},$$

and from Lemma 4.8 we obtain

$$-1/2aF'(a) > \frac{1-\sigma}{2}\beta_1(1-a)^{\alpha-1},$$

for $a \in (1-\delta, 1)$. Hence there exists $c > 0$ such that

$$\int_{1-\delta}^{1} \frac{-1/2aF'(a)}{F(a)^{\frac{3}{2}}}\,\mathrm{d}a \geq c\int_{1-\delta}^{1} (1-a)^{\alpha-1-\frac{3\alpha}{2}}\,\mathrm{d}a = +\infty.$$

Therefore $\lim_{\theta\to 1^-} I'(\theta) = +\infty$ from which the desired result follows. $\square$

The claim for the smooth potential is the same, however the proof is more laborious.

Theorem 4.9. *Let F be a smooth double-well potential. Then*

$$\lim_{\theta\to 1^-} \varepsilon_n'(\theta) = -\infty.$$

Recalling the formula (4.7), the proof consists in estimates for limit behaviour of both $I(\theta)$ and $I'(\theta)$.

Lemma 4.10. *Let F be a smooth double-well potential. Then there exist $\theta_1 \in (0,1)$ and $C > 0$ such that for all $\theta \in (\theta_1, 1)$ we have*

$$I'(\theta) \geq \frac{C}{(1-\theta)}.$$

Proof. Using the substitution $\theta a = s$ we can rewrite $I'(\theta)$ from (4.12) as follows

$$I'(\theta) = \int_0^{\theta} \frac{G(s) - G(\theta)}{\theta(F(s) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}s. \tag{4.19}$$

Let θ_1 be chosen such that

$$(2\theta_1 - 1) \in (\xi, 1),$$

where ξ is the unique positive inflection point of potential F (see Definition 1.2). For $\theta \in (\theta_1, 1)$ we split the integral in (4.19) as

$$\int_0^{\theta} \frac{G(s) - G(\theta)}{\theta(F(s) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}s = \int_0^{2\theta-1} \dots\,\mathrm{d}s + \int_{2\theta-1}^{\theta} \dots\,\mathrm{d}s =: V_1(\theta) + V_2(\theta).$$

Let us estimate $V_2(\theta)$ first. Applying the Mean Value Theorem, we get $x_F, x_G \in (s,\theta)$ such that

$$V_2(\theta) \geq \frac{1}{\theta}\int_{2\theta-1}^{\theta} \frac{-G'(x_G)(\theta - s)}{(\overline{F_2}(\theta - s)(1 - x_F))^{\frac{3}{2}}}\,\mathrm{d}s,$$

where $\overline{F_2} := \sup_{z\in(2\theta_1-1,1)} F''(z)$. Clearly, $1-x_F \le 2(1-\theta)$ and hence

$$\begin{aligned}
&\frac{1}{\theta}\int_{2\theta-1}^{\theta} \frac{-G'(x_G)(\theta-s)}{(\overline{F_2}(\theta-s)(1-x_F))^{\frac{3}{2}}}\,\mathrm{d}s \\
&\ge \frac{1}{4\sqrt{2}\theta[\overline{F_2}(1-\theta)]^{\frac{3}{2}}}\int_{2\theta-1}^{\theta} \frac{x_G F''(x_G)-F'(x_G)}{\sqrt{\theta-s}}\,\mathrm{d}s \\
&\ge \frac{C_1}{(1-\theta)^{\frac{3}{2}}}\int_{2\theta-1}^{\theta} \frac{(2\theta_1-1)\inf_{z\in(2\theta_1-1,1)} F''(z)}{\sqrt{\theta-s}}\,\mathrm{d}s = C_2(1-\theta)^{-1},
\end{aligned}$$

where $C_1>0, C_2>0$ do not depend on $\theta\in(\theta_1,1)$. We used the fact, that as $2\theta_1-1>\xi$ we have $\inf_{z\in(2\theta_1-1,1)} F''(z)>0$.

Now it is sufficient to prove that there exists $C_3>0$ independent of $\theta\in(\theta_1,1)$ such that $V_1(\theta)\ge -C_3$. To this end, let us write, for $\theta\in(\theta_1,1)$,

$$V_1(\theta)=\frac{1}{\theta}\int_0^{\xi}\frac{G(s)-G(\theta)}{(F(s)-F(\theta))^{\frac{3}{2}}}\,\mathrm{d}s+\frac{1}{\theta}\int_{\xi}^{2\theta-1}\frac{G(s)-G(\theta)}{(F(s)-F(\theta))^{\frac{3}{2}}}\,\mathrm{d}s =: V_{11}(\theta)+V_{12}(\theta).$$

Then the denominator in $V_{11}(\theta)$ is bounded from below by a positive constant, while its numerator is continuous and thus bounded. Hence there exists $C_3>0$ such that $V_{11}(\theta)>-C_3$. Finally the fact that $G'(z)<0$ for $z\in(\xi,1)$, i.e. G is strictly decreasing in $(\xi,1)$, implies that $V_{22}(\theta)>0$, which concludes the proof. $\square$

Lemma 4.11. *Let F be a smooth double-well potential. Then there exist $\theta_2\in(0,1)$ and $C>0$ such that for all $\theta\in(\theta_2,1)$:*

$$I(\theta)\le\frac{C}{(1-\theta)^{1/4}}.$$

Proof. Let $\Gamma\in(\xi,1)$. There exists $\theta_2\in(\Gamma,1)$ such that for all $\theta\in(\theta_2,1)$ we have

$$\gamma(\theta):=\theta-(1-\theta)^{\frac{1}{2}}\in(\Gamma,\theta).$$

Hence for $\theta\in(\theta_2,1)$ we can split the integration as follows

$$\begin{aligned}
I(\theta)&=\int_0^{\theta}\frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}} \\
&=\int_0^{\Gamma}\dots+\int_{\Gamma}^{\gamma(\theta)}\dots+\int_{\gamma(\theta)}^{\theta}\dots =: W_1(\theta)+W_2(\theta)+W_3(\theta).
\end{aligned} \tag{4.20}$$

Then we estimate the integrals in (4.20) in the following way:

$$W_1(\theta)=\int_0^{\Gamma}\frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}\le\frac{\Gamma}{\sqrt{F(\Gamma)-F(\theta_2)}},$$

which is a constant independent of $\theta\in(\theta_2,1)$.

To estimate $W_2(\theta)$ we proceed as follows. The Mean Value Theorem implies that there exist $x_F\in(s,\theta)$ and $x_{FF}\in(x_F,1)$ such that $F(s)-F(\theta)=F'(x_F)(\theta-s)$ and $F'(x_F)=F'(x_F)-F'(1)=F''(x_{FF})(1-x_F)$. Hence,

$$W_2(\theta)=\int_{\Gamma}^{\gamma(\theta)}\frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}=\int_{\Gamma}^{\gamma(\theta)}\frac{\mathrm{d}s}{\sqrt{F''(x_{FF})(1-x_F)(\theta-s)}}. \tag{4.21}$$

Then we use Lemma 2.1 for $g = F$, $x = s$, $y = \theta, \xi_{xy} = x_F$ together with $s \in (\Gamma, \gamma(\theta))$ and $\theta < 1$ to get the estimate

$$(1 - x_F) \geq (\theta - x_F) \geq K(\theta - s) \geq K(\theta - \gamma(\theta)) = K(1-\theta)^{\frac{1}{2}}, \tag{4.22}$$

for any $\theta \in (\theta_2, 1)$. Further, notice that $\underline{F_2} := \inf_{z\in(\Gamma,1)} F''(z) > 0$. Now, combining (4.21) with (4.22) we deduce that there exists $C_1 > 0$ such that

$$W_2(\theta) \leq \frac{1}{\sqrt{\underline{F_2}K}(1-\theta)^{1/4}} \int_\Gamma^{\gamma(\theta)} \frac{\mathrm{d}s}{\sqrt{\theta - s}} \leq \frac{C_1\sqrt{1-\Gamma}}{(1-\theta)^{1/4}} \leq C_1(1-\theta)^{-1/4},$$

for all $\theta \in (\theta_2, 1)$.

Again, using the Mean Value Theorem, setting $\underline{F_2} := \inf_{z\in(\Gamma,1)} F''(z) > 0$ and using (4.22) there exists $C_2 > 0$ such that

$$\begin{aligned} W_3(\theta) &= \int_{\gamma(\theta)}^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} \\ &= \int_{\gamma(\theta)}^{\theta} \frac{\mathrm{d}s}{\sqrt{F''(x_{FF})(1-x_F)(\theta - s)}} \\ &\leq \frac{2(1-\theta)^{1/4}}{\sqrt{\underline{F_2}K}(1-\theta)^{1/4}} \leq C_2, \end{aligned}$$

for $\theta \in (\theta_2, 1)$. □

Proof of Theorem 4.9. Using the previous lemmas and taking $\theta_3 = \max\{\theta_1, \theta_2\}$, there is $C > 0$ such that for any $\theta \in (\theta_3, 1)$ we have

$$\varepsilon_n'(\theta) = -\frac{\sqrt{2}}{2n}\frac{I'(\theta)}{(I(\theta))^2} \leq -C\frac{(1-\theta)^{-1}}{(1-\theta)^{-1/2}} = -C(1-\theta)^{-1/2},$$

from which we conclude $\lim_{\theta\to 1^-} \varepsilon_n'(\theta) = -\infty$. □

4.6. **Monotonicity criterion.** In the last section we introduce a simple criterion, ensuring the strict monotonicity of the branches and therefore also the existence of the limit at zero.

Theorem 4.12 (Sufficient condition for monotonicity of the branches.)**.** *Let F be a (smooth or non-smooth) double-well potential and $\xi \in (0,1)$ its inflection point. Assume that F'' is monotone in $(0,\xi)$. Then $\varepsilon_n(\theta)$ is strictly decreasing in $(0,1)$.*

Note that all the basic prototypes of smooth and non-smooth potentials introduced in Section 1 satisfy the hypotheses of Theorem 4.12.

Proof. We show that $\varepsilon_n'(\theta) < 0$ in $(0,1)$. Thanks to (4.7) this is equivalent to $I'(\theta) > 0$ for all $\theta \in (0,1)$.

Recalling (4.12),

$$\begin{aligned} I'(\theta) &= \int_0^1 \frac{(F(\theta a) - \frac{1}{2}\theta a F'(\theta a)) - (F(\theta) - \frac{1}{2}\theta F'(\theta))}{(F(\theta a) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}a \\ &= \int_0^1 \frac{G(\theta a) - G(\theta)}{(F(\theta a) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}a, \end{aligned}$$

the sufficient condition for $I'(\theta)$ being positive is G being decreasing in $(0,1)$. Hence we concentrate on proving

$$0 > G'(z) = \frac{1}{2}\big(F'(z) - zF''(z)\big), \quad z \in (0,1).$$

Since $F' < 0$ in $(0,1)$ and $F'' > 0$ in $(\xi,1)$, we have $G'(z) < 0$ for all $z \in (\xi,1)$. Let $z \in (0,\xi]$. By the Mean Value Theorem, there exists $z_G \in (0,z)$ such that

$$F'(z) = F'(z) - F'(0) = F''(z_G)z.$$

Hence

$$G'(z) = \frac{1}{2}z[F''(z_G) - F''(z)],\ z \in (0,\xi].$$

Since F'' is increasing in $(0,\xi)$, we conclude that $G'(z) < 0$ in $(0,\xi]$. This completes the proof. □

The following remark illustrates that the above criterion for monotonicity of branches is not necessary. Even when the potential has non-convex derivative F' in $(0,\xi)$ but *not far* from being convex, the monotonicity is ensured.

Remark 4.13. *The condition for monotonicity in Theorem 4.12 is not necessary. Potential F defined through its derivative,*

$$F'(z) = \begin{cases} -z(z+1) + \frac{1}{50}(\cos(4\pi z) - 1) & \textit{for } z < 0, \\ z(z-1) - \frac{1}{50}(\cos(4\pi z) - 1) & \textit{for } z \geq 0 \end{cases}$$

violates the assumptions of Theorem 4.12 as $F'''(z) = 0$ has roots in $(0,\xi)$ but still $G'(z) < 0$ for all $z \in (0,1)$ and therefore $\varepsilon_n'(\theta) < 0$ in $(0,1)$.

Corollary 4.14. *Under the assumptions of Theorem 4.12, $\lim_{\theta\to 0}\varepsilon_n(\theta)$ exists.*

Proof. As $\varepsilon_n(\theta)$ is decreasing and bounded from above in $U^+(0,\delta)$, we easily conclude that $\lim_{\theta\to 0^+}\varepsilon_n(\theta)$ exists. Thanks to symmetry property (4.1) the limit $\lim_{\theta\to 0}\varepsilon_n(\theta) \in \mathbb{R}^+$ also exists. □

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