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HOMOCLINIC ORBITS AT INFINITY FOR SECOND-ORDER HAMILTONIAN SYSTEMS WITH FIXED ENERGY

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Abstract. We obtain the existence of homoclinic orbits at infinity for a class of second-order Hamiltonian systems with fixed energy. We use the limit for a sequence of approximate solutions which are obtained by variational methods.

1. Introduction and main results

In this article, we consider the second-order Hamiltonian system

$$\ddot{u}(t) + \nabla V(u(t)) = 0 \tag{1.1}$$

with

$$\frac{1}{2}|\dot{u}(t)|^2 + V(u(t)) = H. \tag{1.2}$$

where $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $V \in C^1(\mathbb{R}^N, \mathbb{R})$. Subsequently, $\nabla V(x)$ denotes the gradient with respect to the x variable, $(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ denotes the standard Euclidean inner product in $\mathbb{R}^N$ and $|\cdot|$ is the induced norm. In this article, we say a solution $u(t)$ of problem (1.1)-(1.2) is homoclinic at infinity (following the terminology of Serra [19]) if $|u(t)| \to +\infty$ and $|\dot{u}(t)| \to H$ as $t \to \pm\infty$.

In previous two decades, many mathematicians have considered the existence of homoclinic and periodic orbits for problem (1.1); see [1-4,6-10,12-18,20-23] and the reference therein. Equation (1.1) can be used to describe the motion of heaven bodies under the law of universal gravitation. But in celestial mechanic, the potential V possesses singularities at any collision points. In 2000, Felmer and Tanaka [8] considered the existence of hyperbolic orbits for problem (1.1)-(1.2) with singular potential. Recently, Wu and Zhang [24] obtained the similar conclusion under some weaker conditions. As to the smooth potential, it can be referred to the restricted three-body problems which is a reduced model of N-body problems. The restricted three-body problem consists in determining u such that

$$\ddot{u}(t) + \frac{\alpha u(t)}{(|u(t)|^2 + |r(t)|^2)^{\frac{\alpha+2}{2}}} = 0, \tag{1.3}$$

where $r(t) = r(t + 2\pi) > 0$ for any $t \in \mathbb{R}$. Obviously, the potential in (1.3) has no singularity. In 1990, Rabinowitz [15] used variational methods to study the

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existence of orbits for (1.1) which are homoclinic to zero with the so called (AR) condition. Since the pioneering work of Rabinowitz, there are many works on the existence of homoclinic solutions to zero for problem (1.1). But as to the homoclinic orbits for non-singular Hamiltonian systems with a fixed energy, there are only few paper involving this topic. In 1994, Serra [19] obtained the existence of a class of homoclinic orbits at infinity for a class of second order conservative systems. In his paper, He treated the systems with zero energy and the approximated homoclinic orbits with a sequence of brake orbits which are obtained by variational methods. He obtain the following theorem.

Theorem 1.1 ([19]). *Suppose that the potential $V \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies*

- (A1) $V(x) < 0$ *for all* $x \in \mathbb{R}^N$,
- (A2) *there exist* $R_0 > 0$, $\gamma > 2$ *such that*
$$V(x) = -\frac{1}{|x|^\gamma} + W(x), \quad \forall |x| \geq R_0,$$
- (A3) $\lim_{|x|\to+\infty} W(x)|x|^\gamma = 0$,
- (A4) $(x, \nabla W(x)) > 0$, *for all* $|x| \geq R_0$.

Then there exists at least one homoclinic solution at infinity for (1.1)-(1.2) *with H=0.*

Motivated by above papers, we shall obtain the homoclinic orbits at infinity for problem (1.1)-(1.2) with the symmetrical potential V, but we do not (A2). Through out this article, we assume $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and the following conditions:

- (A5) $(x, \nabla V(x)) \to 0$ as$|x| \to +\infty$.
- (A6) there exist constants $\beta > 2$, $M_0 > 0$ and $r_0 > 0$ such that $|x|^\beta |V(x)| \leq M_0$ for all $|x| \geq r_0$.

Remark 1.2. It follows from (A6) that $V(x) \to 0$ as $|x| \to +\infty$.

We set
$$A = \inf\{V(x)|x \in \mathbb{R}^N\}, \quad B = \sup\{V(x)|x \in \mathbb{R}^N\}. \tag{1.4}$$
Since V is of C^1 class in $\mathbb{R}^N$ and satisfies (A6), we can conclude that $-\infty < A \leq B < +\infty$. Under above conditions, we have the following theorem.

Theorem 1.3. *Suppose* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ $(N \geq 2)$ *satisfies* (A5)-(A6). *If* $V(-x) = V(x)$ *for all* $x \in \mathbb{R}^N$, *then* (1.1)-(1.2) *possesses at least one homoclinic orbit to infinity for any given* $H > B$.

Remark 1.4. It follows from Remark 1.2 that $B \geq 0$. So the total energy H must be positive.

Remark 1.5. In Theorem 1.3, V can change sign. The potential in (1.3) satisfies the conditions of Theorem 1.3 for $\alpha > 2$. There are functions satisfying Theorem 1.3 but not Theorem 1.1. For example,
$$V(x) = \begin{cases} -\frac{1}{4}(|x|+1)^2 + 1 & \text{for } 0 \leq |x| \leq 1, \\ -\frac{1}{|x|^3} + \frac{1}{|x|^4} & \text{for } |x| \geq 1. \end{cases}$$

2. Variational settings

We obtain the homoclinic orbits at infinity as the limits of solutions for the following equations

$$\ddot{q}(t) + \nabla V(q(t)) = 0 \quad \forall t \in (-T_R, T_R) \tag{2.1}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = H \quad \forall t \in (-T_R, T_R) \tag{2.2}$$

Where T_R is a suitable number defined in the proof of the following lemma. We consider equations (2.1)-(2.2) on the set

$$G_R = \{q \in E_R : q(t + \frac{1}{2}) = -q(t)\},$$

where

$$E_R = \{q \in H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) : |q(0)| = |q(1)| = R\}.$$

Here R stands for the constraint on the Euclidean norm of the functions in E_R at the end of the time interval. If $q \in G_R$, it is easy to check that $\int_0^1 q(t)dt = 0$, then by Poincaré-Wirtinger's inequality, we have the equivalent norm

$$\|q\|_{H^1} = \Big(\int_0^1 |\dot{q}(t)|^2 dt\Big)^{1/2}.$$

Let $L^\infty([0,1], \mathbb{R}^N)$ be a space of measurable functions from $[0,1]$ into $\mathbb{R}^N$ and essentially bounded under the norm

$$\|q\|_{L^\infty([0,1],\mathbb{R}^N)} = \operatorname{ess\,sup}\{|q(t)| : t \in [0,1]\}.$$

Then functional $f : G_R \to \mathbb{R}$ can be defined as

$$f(q) = \frac{1}{2}\|q\|^2 \int_0^1 (H - V(q(t)))dt. \tag{2.3}$$

Then

$$\langle f'(q), q(t)\rangle = \|q\|^2 \int_0^1 \Big(H - V(q(t)) - \frac{1}{2}(\nabla V(q(t)), q(t))\Big)dt. \tag{2.4}$$

To prove Theorem 1.3, we approach the homoclinic orbits with a sequence of approximate solutions obtained using minimizing theory. The following lemma shows that the critical points of f are the solutions of (1.1)-(1.2) after some kind of time scaling.

Lemma 2.1 ([3]). *Let*

$$f(q) = \frac{1}{2}\int_0^1 |\dot{q}(t)|^2 dt \int_0^1 (H - V(q(t)))dt$$

and $\tilde{q} \in H^1$ be such that $f'(\tilde{q}) = 0$, $f(\tilde{q}) > 0$. Set

$$T^2 = \frac{\frac{1}{2}\int_0^1 |\dot{\tilde{q}}(t)|^2 dt}{\int_0^1 (H - V(\tilde{q}(t))dt}.$$

Then $\tilde{u}(t) = \tilde{q}(t/T)$ is a non-constant T-periodic solution for (1.1) *and* (1.2).

Lemma 2.2 ([21]). *Let σ be an orthogonal representation of a finite or compact group Π in the real Hilbert space H such that for any $\sigma \in \Pi$,*

$$f(\sigma \cdot x) = f(x),$$

where $f \in C^1(H, R^1)$. Let $S = \{x \in H | \sigma x = x, \forall \sigma \in \Pi\}$, then the critical point of f in S is also a critical point of f in H.

Remark 2.3. Since $V(x)$ is even in x, by the principle of symmetric criticality, we can see that all the critical points of f on G_R are the critical points of f on H^1 if we set the group $\Pi = \{-e, e\}$, $P : H^1 \to H^1$ such that $Pq(t) = -q(t + \frac{1}{2})$ and $\sigma(-e) = P$, $\sigma(e) = P^2 = id$, where id is the identity operator.

3. Existence of approximate solutions

Firstly, we prove the existence of the approximate solutions, then we study the limit process.

Lemma 3.1. *Suppose the conditions of Theorem 1.3 hold, then for any $R > 0$, there exists at least one approximate solution on G_R for systems* (2.1)-(2.2) *with some suitable T_R.*

Proof. We notice that H^1 is a reflexive Banach space and G_R is a weakly closed subset of H^1. By the definition of f and $H > B$, we obtain that f is a functional bounded from below and

$$\begin{aligned} f(q) &= \frac{1}{2}\|q\|^2 \int_0^1 (H - V(q(t)))dt \\ &\geq \frac{H - B}{2}\|q\|^2 \to +\infty \quad \text{as } \|q\| \to +\infty. \end{aligned}$$

Furthermore, it is easy to check that f is weakly lower semi-continuous. Then, we can see that for every $R > 0$ there exists a minimizer $q_R \in G_R$ such that

$$f'(q_R) = 0, \quad f(q_R) = \inf_{q \in G_R} f(q) \geq 0. \tag{3.1}$$

It is easy to see that $\|q_R\|^2 = \int_0^1 |\dot{q}_R(t)|^2 dt > 0$, otherwise we deduce that $q_R(t) \equiv Re_0$ for some $e_0 \in S^{N-1}$, which is a contradiction, since the anti-symmetry of q_R. Let

$$T_R^2 = \frac{\frac{1}{2}\int_0^1 |\dot{q}_R(t)|^2 dt}{\int_0^1 (H - V(q_R(t)))dt}, \tag{3.2}$$

Then by Lemma 2.1, $u_R(t) = q_R(\frac{t+T_R}{2T_R}) : (-T_R, T_R) \to H^1$ is a non-constant approximate solution satisfying (2.1) and (2.2). The proof is complete. □

Remark 3.2. In Lemma 3.1, we minimize the functional on the set G_R, but we can not show that $u_R(t)$ solves the equations at $\pm T_R$. But we do not need $u_R(t)$ to be a solution at these two moments, since we will let $R \to +\infty$ in the end.

4. Estimations on approximate solutions

Subsequently, we need to let $R \to +\infty$. But before doing this, we need to prove u_R can not approach infinity as $R \to +\infty$, which is the following lemma.

Lemma 4.1. *Suppose that $u_R(t) : (-T_R, T_R) \to H^1$ is the solution obtained in Lemma 3.1, then $\min_{t\in(-T_R,T_R)} |u_R(t)|$ is bounded uniformly. More precisely, there is a constant $M > 0$ independent of R such that*

$$\min_{t\in(-T_R,T_R)} |u_R(t)| \le M \quad \text{for all } R > 0.$$

Proof. Since $q_R \in G_R$ is a minimizer of f, we have $f'(q_R) = 0$ which implies that

$$\int_{-T_R}^{T_R} 2H - (2V(u_R(t)) + (\nabla V(u_R(t)), u_R(t)))dt = 0.$$

Then there exists $t_0 \in (-T_R, T_R)$ such that

$$2H - (2V(u_R(t_0)) + (\nabla V(u_R(t_0)), u_R(t_0))) \le 0,$$

which implies

$$2H \le 2V(u_R(t_0)) + (\nabla V(u_R(t_0)), u_R(t_0)).$$

It follows from Remark 1.4 that $H > 0$. Then by hypotheses (A5) and Remark 1.2 that there exists a constant $M_1 > 0$ independent of R such that

$$\min_{t\in(-T_R,T_R)} |u_R(t)| \le M_1.$$

Then the proof is complete. □

Lemma 4.2. *Suppose that $R > \max\{M, r_0\}$ and $u_R(t)$ is the solution for* (2.1)-(2.2) *obtained in Lemma 3.1, where M is from Lemma 4.1 and r_0 is defined in* (A6)*. Set*

$$t_+ = \sup\{t \in (-T_R, T_R) : |u_R(t)| \le L\}, \tag{4.1}$$

$$t_- = \inf\{t \in (-T_R, T_R) : |u_R(t)| \le L\} \tag{4.2}$$

where L is a constant independent of R such that $\max\{M, r_0\} < L < R$. Then we obtain

$$T_R - t_+ \to +\infty, \quad t_- + T_R \to +\infty \quad \text{as } R \to +\infty.$$

Proof. By the definition of B, we have

$$\begin{aligned} \int_{t_+}^{T_R} \sqrt{H - V(u_R(t))}|\dot u_R(t)|dt &\ge \sqrt{H - B}\int_{t_+}^{T_R} |\dot u_R(t)|dt \\ &\ge \sqrt{H - B}\Big|\int_{t_+}^{T_R} \dot u_R(t)dt\Big| \\ &\ge \sqrt{H - B}(R - L). \end{aligned} \tag{4.3}$$

Similarly, we can get

$$\int_{-T_R}^{t_-} \sqrt{H - V(u_R(t))}|\dot u_R(t)|dt \ge \sqrt{H - B}(R - L). \tag{4.4}$$

It follows from (1.4) and (2.2) that

$$\begin{aligned} \int_{t_+}^{T_R} \sqrt{H - V(u_R(t))}|\dot u_R(t)|dt &= \sqrt{2}\int_{t_+}^{T_R} (H - V(u_R(t)))dt \\ &\le \sqrt{2}(H - A)\,(T_R - t_+) \end{aligned}$$

From this inequality and (4.3), we obtain

$$\sqrt{H - B}(R - L) \le \sqrt{2}(H - A)\,(T_R - t_+)\,.$$

Then we have $T_R - t_+ \to +\infty$, as $R \to +\infty$. The limit for $t_- + T_R$ is obtained in the similar way. The proof is complete. $\square$

Lemma 4.3. *Suppose that $u_R(t)$ is the solution for (2.1)−(2.2) obtained in Lemma 3.1. Then there exists a constant $M_2 > 0$ independent of $R > r_0$ such that*

$$\int_{-T_R}^{T_R} \sqrt{H - V(u_R(t))}|\dot{u}_R(t)|dt \le 2\sqrt{H}R + M_2,$$

where r_0 comes from (A6).

Proof. Define the function $\xi(t)$ on $[1, +\infty)$ as a solution of the differential equation

$$\begin{gathered}\dot{\xi}(t) = \sqrt{2(H - V(\xi(t)e))}\\ \xi(1) = r_0,\end{gathered}$$

where $e \in S^{N-1}$. Let $\tau_R > 1$ be a real number such that $\xi(\tau_R) = R$. Furthermore, $\xi(t)$ can be odd extended to $(-\infty, -1]$ and define $\tau_{-R} = -\tau_R$ such that $\xi(\tau_{-R}) = -R$. Then we can fix $\varphi(t) \in H^1([-1, 1], \mathbb{R}^N)$ such that $\tilde{\gamma}_R(t) \in G_R$ where

$$\tilde{\gamma}_R(t) = \gamma_R(t(\tau_R - \tau_{-R}) + \tau_{-R}), \quad \gamma_R(t) = \begin{cases} \xi(t)e & \text{for } t \in [\tau_{-R}, -1] \bigcup [1, \tau_R], \\ \varphi(t) & \text{for } t \in [-1, 1]. \end{cases}$$

Subsequently, we set $u_r(t) = \tilde{\gamma}_R(\frac{t+r}{2r})$. And it is easy to see that $u_r(t) = \gamma_R(t)$ if $\tau_{\pm R} = \pm r$. Similar to [8], we can deduce that for $r > 0$

$$\begin{aligned}(2f(\tilde{\gamma}_R))^{1/2} &= \inf_{r>0} \frac{1}{\sqrt{2}} \int_{-r}^{r} \frac{1}{2}|\dot{u}_r(t)|^2 + H - V(u_r(t))dt \\ &\le \frac{1}{\sqrt{2}} \int_{-\tau_R}^{\tau_R} \frac{1}{2}|\dot{\gamma}_R(t)|^2 + H - V(\gamma_R(t))dt.\end{aligned} \tag{4.5}$$

Since $[-\tau_R, \tau_R] = [-\tau_R, -1] \bigcup [-1, 1] \bigcup [1, \tau_R]$, by (A6), we can estimate (4.5) by three integrals. Firstly, we estimate the integral on $[1, \tau_R]$, which is

$$\begin{aligned}I_{[1,\tau_R]} &= \frac{1}{\sqrt{2}} \int_1^{\tau_R} \frac{1}{2}|\dot{\gamma}_R(t)|^2 + H - V(\gamma_R(t))dt \\ &= \frac{1}{\sqrt{2}} \int_1^{\tau_R} H - V(\xi(t)e)dt \\ &= \int_1^{\tau_R} \sqrt{H - V(\xi(t)e)}\dot{\xi}(t)dt = \int_{r_0}^{R} \sqrt{H - V(se)}ds \\ &\le \int_{r_0}^{R} \sqrt{H} + \sqrt{|V(se)|}ds = \sqrt{H}(R - r_0) + \int_{r_0}^{R} \sqrt{|V(se)|}ds \\ &\le \sqrt{H}R + \sqrt{M_0} \int_{r_0}^{R} s^{-\frac{\beta}{2}} ds \le \sqrt{H}R + \sqrt{M_0} \int_{r_0}^{+\infty} s^{-\frac{\beta}{2}} ds \\ &\le \sqrt{H}R + M_3\end{aligned}$$

where

$$M_3 = \frac{\beta\sqrt{M_0}}{2} r_0^{\frac{2-\beta}{2}}.$$

Similarly, we have

$$I_{[-\tau_R,-1]} \le \sqrt{H}R + M_3.$$

Since $I_{[-1,1]}$ is independent of R, we obtain that

$$\frac{1}{\sqrt{2}}\int_{-\tau_R}^{\tau_R}\frac{1}{2}|\dot{\gamma}_R(t)|^2+H-V(\gamma_R(t))dt\le 2\sqrt{H}R+M_4$$

for some $M_4>0$ independent of R. Then by (4.5) and $q_R(t)$ is the minimizer of f on G_R, we have

$$\begin{aligned}
\int_{-T_R}^{T_R}\sqrt{H-V(u_R(t))}|\dot{u}_R(t)|dt &\le \Big(\int_{-T_R}^{T_R}H-V(u_R(t))dt\Big)^{1/2}\Big(\int_{-T_R}^{T_R}|\dot{u}_R(t)|^2dt\Big)^{1/2}\\
&=(2f(q_R))^{1/2}\le(2f(\tilde{\gamma}_R))^{1/2}\\
&\le\frac{1}{\sqrt{2}}\int_{-\tau_R}^{\tau_R}\frac{1}{2}|\dot{\gamma}_R(t)|^2+H-V(\gamma_R(t))dt\\
&\le 2\sqrt{H}R+M_2.
\end{aligned}$$

This completes the proof of this lemma. □

5. Proof of Theorem 1.3

Subsequently, we set

$$t^*=\inf\{t\in(-T_R,T_R)||u_R(t)|=M\},$$
$$u_R^*(t)=u_R(t^*-t),$$

where M is defined in Lemma 4.1. Since all the functions in G_R are continuous, it follows from Lemma 4.1 that $\{t\in(-T_R,T_R)||u_R(t)|=M\}$ is not empty when R is large enough.

Lemma 5.1. *Let $u_R\in E_R$ be the solution of* (2.1)-(2.2) *and u_R^* be defined as above. Then there exists a subsequence $\{u_{R_j}^*\}$ of $\{u_R^*\}_{R>0}$ that convergences to u_∞ in $C_{\rm loc}(\mathbb{R},\mathbb{R}^N)$. Furthermore, u_∞ is a homoclinic solution at infinity of* (1.1)-(1.2).

Proof. Step 1: We show that $\{u_R^*\}_{R>0}$ possesses a subsequence in $C_{\rm loc}(\mathbb{R},\mathbb{R}^N)$. By the definition of L and t^*, we can deduce that $t_+\ge t^*\ge t_-$. Then it follows from Lemma 4.2 that

$$-T_R+t^*\to-\infty,\quad T_R+t^*\to+\infty\quad \text{as} R\to+\infty.$$

By the energy equation (2.2), we obtain that

$$|\dot{u}_R^*(t)|^2=2(H-V(u_R^*(t)))\le 2(H-A),\quad \forall t\in(-T_R+t^*,T_R+t^*),\tag{5.1}$$

which implies that

$$|u_R^*(t_1)-u_R^*(t_2)|\le\Big|\int_{t_2}^{t_1}\dot{u}_R^*(s)ds\Big|\le\int_{t_2}^{t_1}|\dot{u}_R^*(s)|ds\le\sqrt{2(H-A)}|t_1-t_2|\tag{5.2}$$

for each $R>0$ and $t_1,t_2\in[-T_R+t^*,T_R+t^*]$, which shows $\{u_R^*\}$ is equicontinuous.

Subsequently, we show that u_R^* is uniformly bounded on any compact set of $\mathbb{R}$. Take $a,b\in\mathbb{R}$ such that $a<b$. When R is large enough, by Lemma 4.2, we can see that $[a,b]\subseteq[-T_R+t^*,T_R+t^*]$. Then, for any $t\in[a,b]$, it follows from (5.1) and the definition of t^* that

$$|u_R^*(t)|=\Big|\int_0^t\dot{u}_R^*(t)dt+u_R^*(0)\Big|$$

$$\begin{aligned}
&\le \Big|\int_0^t \dot{u}_R^*(t)dt\Big| + |u_R^*(0)| \\
&\le \Big|\int_0^t |\dot{u}_R^*(t)|dt\Big| + |u_R(t^*)| \\
&\le \sqrt{2(H-A)}|t| + M \\
&\le \sqrt{2(H-A)}(|a|+|b|) + M,
\end{aligned}$$

which implies

$$\max_{t\in[a,b]} |u_R^*(t)| \le \sqrt{2(H-A)}(|a|+|b|) + M. \tag{5.3}$$

We have shown that u_R^* is uniformly bounded on any compact set of $\mathbb{R}$ and uniformly equi-continuous on $\mathbb{R}$. By Arzelá-Ascoli theorem, it follows from inequalities (5.2) and (5.3) that there is a subsequence $\{u_{R_j}^*\}_{j>0}$ converging to u_∞ uniformly in $C_{\rm loc}(\mathbb{R},\mathbb{R}^N)$.

Step 2: We show that u_∞ is a homoclinic solution at infinity of (1.1)-(1.2). By Lemma 3.1 and the definition of $u_{R_j}^*$, we have

$$\ddot{u}_{R_j}^*(t) + \nabla V(u_{R_j}^*(t)) = 0,$$

with

$$\frac{1}{2}|\dot{u}_{R_j}^*(t)|^2 + V(u_{R_j}^*(t)) = H,$$

for each $j>0$ and $t\in(-T_R+t^*, T_R+t^*)$. Take $a,b\in\mathbb{R}$ such that $a<b$. Since V is of C^1 class, $\ddot{u}_{R_j}(t)$ is continuous on $[a,b]$ and $\ddot{u}_{R_j}(t)\to -\nabla V(t,u_\infty(t))$ uniformly on $[a,b]$. It follows that $\ddot{u}_{R_j}$ is a classical derivative of $\dot{u}_{R_j}$ in (a,b) for each $j>0$. Moreover, since $\dot{u}_{R_j}\to\dot{u}_\infty$ uniformly on $[a,b]$, we get

$$\ddot{u}_\infty(t) + \nabla V(u_\infty(t)) = 0,$$

with

$$\frac{1}{2}|\dot{u}_\infty(t)|^2 + V(u_\infty(t)) = H,$$

for all $t\in[a,b]$. Since a and b are arbitrary, we conclude that u_∞ satisfies (1.1) $-$ (1.2).

Furthermore, we need to prove that $|u_\infty(t)|\to+\infty$ as $t\to\pm\infty$. First, we show that $|u_\infty(t)|\to+\infty$ as $t\to+\infty$. Otherwise, there exists a sequence, denoted by t_n such that $t_n\to+\infty$ as $n\to+\infty$ and

$$|u_\infty(t_n)| \le M_\infty \quad \text{for all } n\in\mathbb{N}^+ \tag{5.4}$$

for some $M_\infty>0$. On one hand, it follows from Lemma 4.3, (4.3) and (4.4) that

$$\begin{aligned}
2\sqrt{H}R_j + M_2 &\ge \int_{-T_{R_j}+t^*}^{T_{R_j}+t^*} \sqrt{H - V(u_{R_j}^*(t))}|\dot{u}_{R_j}^*(t)|dt \\
&\ge \Big(\int_{t^*+t_-}^{t^*+t_+} + \int_{t_++t^*}^{T_{R_j}+t^*} + \int_{-T_{R_j}+t^*}^{t_-+t^*}\Big)\sqrt{H - V(u_{R_j}^*(t))}|\dot{u}_{R_j}^*(t)|dt \\
&\ge \int_{t^*+t_-}^{t^*+t_+} \sqrt{H - V(u_{R_j}^*(t))}|\dot{u}_{R_j}^*(t)|dt + 2\sqrt{H}(R_j - L).
\end{aligned}$$

The above inequality and (2.2) imply

$$
\begin{aligned}
2\sqrt{H}L + M_2 &\geq \int_{t^*+t_-}^{t^*+t_+} \sqrt{H - V(u^*_{R_j}(t))}|\dot{u}^*_{R_j}(t)|dt \\
&= \sqrt{2}\int_{t^*+t_-}^{t^*+t_+} (H - V(u^*_{R_j}(t)))dt \\
&\geq \sqrt{2}(H - B)(t_+ - t_-).
\end{aligned}
\tag{5.5}
$$

On the other hand, in the proof of Lemma 5.1, we choose $L > \max\{M, M_\infty, r_0\}$. By (4.2) and the definition of G_R, it is easy to see that $t_- < 0$. From (5.5), we can deduce that there exists $M_5 > 0$ independent of j such that $t_+ \leq M_5$. By our assumption, we can choose t_{n_0} such that $t_{n_0} > M_5$ and $|u_\infty(t_{n_0})| \leq M_\infty$. By the uniformly convergence of $\{u_{R_j}\}$, there exists $j_0 > 0$ such that

$$
|u_{R_j}(t_{n_0}) - u_\infty(t_{n_0})| \leq \frac{L - M_\infty}{2}
$$

for any $j > j_0$, which implies that $|u_{R_j}(t_{n_0})| \leq \frac{L+M_\infty}{2} < L$ for any $j > j_0$, which contradicts (4.1). Then $|u_\infty(t)| \to +\infty$ as $t \to +\infty$. The proof for $t \to -\infty$ is similar. Then we complete the proof. $\square$

From the above lemmas, we have proved there is at least one homoclinic solution at infinity for (1.1)-(1.2) with $H > B$. We finish the proof of Theorem 1.3.

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