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K-DIMENSIONAL NONLOCAL BOUNDARY-VALUE PROBLEMS AT RESONANCE

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Abstract. In this article we show the existence of at least one solution to the system of nonlocal resonant boundary-value problem

$$x'' = f(t,x), \quad x'(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,dg(s),$$

where $f : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$, $g : [0,1] \to \mathbb{R}^k$.

1. Introduction

In this article we study the system of ordinary differential equations

$$x'' = f(t,x), \quad x'(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,dg(s), \tag{1.1}$$

where $f = (f_1,\dots,f_k) : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous, and $g = (g_1,\dots,g_k) : [0,1] \to \mathbb{R}^k$ has bounded variation. Observe that (1.1) can be written down as the system of equations

$$\begin{gathered} x_i''(t) = f_i(t,x(t)), \\ x_i'(0) = 0, \\ x_i'(1) = \int_0^1 x_i'(s)dg_i(s), \end{gathered}$$

where $t \in [0,1]$, $i = 1,\dots,k$ and the integrals $\int_0^1 x_i'(s)dg_i(s)$ are meant in the sense of Riemann-Stieltjes.

Our main goal is to show that the problem (1.1) has at least one solution. We impose on the function f a sign condition, which we called: the asymptotic integral sign condition. The idea comes from [16], where the author shows that the first order equation $x' = f(t,x)$ has periodic solutions. The method can be successfully applied to other BVPs (not necessarily only for differential equations of the first or second order but, for instance, involving p-Laplacians), for which the function f does not depend on x'.

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As far as we are aware, (1.1) has not been studied in this generality so far. Note that a special case of (1.1) is the Neumann BVP

$$x'' = f(t,x), \quad x'(0) = 0, \quad x'(1) = 0.$$

Under suitable monotonicity conditions or nonresonance conditions, some existence or uniqueness theorems or methods for Neumann BVPs have been presented (see, for instance, [1, 4, 12, 18, 17, 19, 20, 21, 22] and the references therein).

In [8], the authors study the Neumann boundary value problem $x'' + \mu(t)x_+ - \nu(t)x_- = p(t,x)$, $x'(0) = 0 = x'(\pi)$, where μ, ν lie in $L^1(0,\pi)$, $p(t,x)$ is a Carathéodory function, $p \ge 0$, $x_+(t) = \max(x(t),0)$, and $x_-(t) = \max(-x(t),0)$. They obtain several necessary and sufficient conditions on p so that the Neumann problem has a positive solution or a solution with a simple zero in $(0,\pi)$.

In [9], the author uses phase plane and asymptotic techniques to discuss the number of solutions of the problems $-x'' = f(t,x)$, $x'(0) = \sigma_1$, $x'(\pi) = \sigma_1$. It is assumed that $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is a continuous jumping nonlinearity with nonnegative asymptotic limits: $x^{-1}f(t,x) \to \alpha$ as $x \to -\infty$ and $x^{-1}f(t,x) \to \beta$ as $x \to \infty$. The limit problem where $f(t,x) = \alpha x_- + \beta x_+$ plays a key role in his methods. The authors describe how the number of solutions of the problem depends on the four parameters: $\alpha, \beta, \sigma_1, \sigma_2$. The results differ from those obtained by various authors who were mainly concerned with forcing the equation with large positive functions and keeping the boundary conditions homogeneous.

The boundary-value problem

$$x'' = f(t,x,x'), \quad x'(0) = 0, \quad x'(1) = 0,$$

is considered in [6]. The authors obtain some results of existence of solutions assuming that there is a constant $M > 0$ such that $yf(t,x,y) > 0$ for $|y| > M$ and the function f satisfies the Bernstein growth condition (or the Bernstein-Nagumo growth condition).

In [14] the author shows the existence of a solution to the Neumann problem for the equation

$$(d/dt)[A(t)dx/dt] = f(t,x,x'),$$

where $A : [0,1] \to L(\mathbb{R}^k, \mathbb{R}^k)$ and $f : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$, applying the coincidence degree theory.

The generalization of the Neumann problem (1.1) is a nonlocal problem. BVPs with Riemann-Stieltjes integral boundary conditions include as special cases multi-point and integral BVPs.

The multi-point and integral BCs are widely studied objects. The study of multi-point BCs was initiated in 1908 by Picone [15]. Reviews on differential equations with BCs involving Stieltjes measures has been written in 1942 by Whyburn [24] and in 1967 by Conti [2].

Since then, the existence of solutions for nonlocal nonlinear BVPs has been studied by many authors by using, for instance, the Leray-Schauder degree theory, the coincidence degree theory of Mawhin, the fixed point theorems for cones. For such problems and comments on their importance, we refer the reader to [3, 5, 10, 23, 25, 26] and the references therein.

2. The perturbed problem

First, we shall introduce notation and terminology. Throughout the paper $|\cdot|$ will denote the Euclidean norm on $\mathbb{R}^k$, while the scalar product in $\mathbb{R}^k$ corresponding to the Euclidean norm will be denoted by $(\cdot|\cdot)$. Denote by $C^1([0,1],\mathbb{R}^k)$ the Banach space of all continuous functions $x:[0,1]\to\mathbb{R}^k$ which have continuous first derivatives x' with the norm

$$\|x\| = \max\big\{\sup_{t\in[0,1]}|x(t)|, \sup_{t\in[0,1]}|x'(t)|\big\}. \tag{2.1}$$

The Lemma below, which is a straightforward consequence of the classical Arzelà-Ascoli theorem, gives a compactness criterion in $C^1([0,1],\mathbb{R}^k)$.

Lemma 2.1. *For a set $Z\subset C^1([0,1],\mathbb{R}^k)$ to be relatively compact, it is necessary and sufficient that:*

(1) *there exists $M>0$ such that for any $x\in Z$ and $t\in[0,1]$ we have $|x(t)|\le M$ and $|x'(t)|\le M$;*
(2) *for every $t_0\in[0,1]$ the families $Z:=\{x:x\in Z\}$ and $Z':=\{x':x\in Z\}$ are equicontinuous at t_0.*

Now, let us consider problem (1.1) and observe that the homogeneous linear problem, i.e.,

$$x''=0,\quad x'(0)=0,\quad x'(1)=\int_0^1 x'(s)\,dg(s),$$

has always nontrivial solutions, hence we deal with a resonant situation.

The following assumptions will be needed throughout this article:

(i) $f=(f_1,\dots,f_k):[0,1]\times\mathbb{R}^k\to\mathbb{R}^k$ is a continuous function.
(ii) $g=(g_1,\dots,g_k):[0,1]\to\mathbb{R}^k$ has bounded variation on the interval $[0,1]$.
(iii) There exists a uniform finite limit

$$h(t,\xi):=\lim_{\lambda\to\infty} f(t,\lambda\,\xi)$$

with respect to t and $\xi\in\mathbb{R}^k$, $|\xi|=1$.
(iv) Set

$$h_0(\xi):=\int_0^1 h(u,\xi)du-\int_0^1\int_0^s h(u,\xi)du\,dg(s).$$

For every $\xi\in\mathbb{R}^k$, $|\xi|=1$, we have $(\xi:h_0(\xi))<0$.

Problem (1.1) is resonant. Hence, there is no equivalent integral equation. The existence of a solution will be obtained by considering the perturbed boundary-value problem

$$x''=f(t,x),\quad t\in[0,1], \tag{2.2}$$

$$x'(0)=0, \tag{2.3}$$

$$x'(1)=\int_0^1 x'(s)\,dg(s)+\alpha_n x(0),\quad \alpha_n\in(0,1),\quad \alpha_n\to 0. \tag{2.4}$$

Notice that problem (2.2), (2.3), (2.4) is always nonresonant.

Now, let us consider the equation (2.2) and integrate it from 0 to t. By (2.3), we obtain

$$x'(t)=\int_0^t f(u,x(u))du. \tag{2.5}$$

By (2.4) and (2.5), we obtain

$$\int_0^1 f(u,x(u))du = \int_0^1 \int_0^s f(u,x(u))du\,dg(s) + \alpha_n x(0),$$

so

$$x(0) = \frac{1}{\alpha_n}\Big[\int_0^1 f(u,x(u))du - \int_0^1 \int_0^s f(u,x(u))du\,dg(s)\Big],$$

Moreover, by (2.5), we have

$$x(t) = x(0) + \int_0^t \int_0^s f(u,x(u))\,du\,ds.$$

Now, it is easily seen that the following Lemma holds.

Lemma 2.2. *A function $x \in C^1([0,1],\mathbb{R}^k)$ is a solution of* (2.2), (2.3), (2.4) *if and only if x satisfies the integral equation*

$$x(t) = \int_0^t \int_0^s f(u,x(u))\,du\,ds + \frac{1}{\alpha_n}\Big[\int_0^1 f(u,x(u))du - \int_0^1 \int_0^s f(u,x(u))du\,dg(s)\Big].$$

To search for solutions of (2.2), (2.3), (2.4), we first reformulate the problem as an operator equation. Given $x \in C^1([0,1],\mathbb{R}^k)$ and fixed $n \in \mathbb{N}$ let

$$\begin{aligned}(A_n x)(t) = &\int_0^t \int_0^s f(u,x(u))\,du\,ds \\ &+ \frac{1}{\alpha_n}\Big[\int_0^1 f(u,x(u))du - \int_0^1 \int_0^s f(u,x(u))du\,dg(s)\Big].\end{aligned}$$

Then

$$(A_n x)'(t) = \int_0^t f(u,x(u))du. \tag{2.6}$$

It is clear that $A_n x, (A_n x)' : [0,1] \to \mathbb{R}^k$ are continuous. It follows that the operator

$$A_n : C^1([0,1],\mathbb{R}^k) \to C^1([0,1],\mathbb{R}^k)$$

is well defined.

By assumption (iii), function f is bounded and we put

$$M := \sup_{t\in[0,1],x\in\mathbb{R}^k} |f(t,x)|.$$

By (2.6), we have

$$\sup_{t\in[0,1]} |(A_n x)'(t)| \le M. \tag{2.7}$$

Moreover, we obtain

$$\sup_{t\in[0,1]} |(A_n x)(t)| \le M + \frac{1}{\alpha_n}\left(M + M\,\mathrm{Var}(g)\right), \tag{2.8}$$

where $\mathrm{Var}(g)$ means the variation of g on the interval $[0,1]$.

From (ii), $L := \mathrm{Var}(g) < \infty$. Put $M_n := M + \frac{1}{\alpha_n}(M + M\,L)$, then $\|A_n x\| \le M_n$ for every $n \in \mathbb{N}$. Moreover, $(A_n x)''(t)$ and $(A_n x)'(t)$, $t \in [0,1]$, are bounded, hence the families $(A_n x)'$ and $(A_n x)$ are equicontinuous. Now, by Lemma 2.1, the following Lemma holds.

Lemma 2.3. *The operator A_n is completely continuous.*

Let $B_n := \{x \in C^1([0,1],\mathbb{R}^k) : \|x\| \le M_n\}$. Now, considering operator

$$A_n : B_n \to B_n,$$

by Schauder's fixed point Theorem, we get that the operator A_n has a fixed point in B_n for every n. We have proved the following result.

Lemma 2.4. *Under assumptions* (i)–(iii)*, problem* (2.2)*,* (2.3)*,* (2.4) *has at least one solution for every $n \in \mathbb{N}$.*

3. Main results

Let φ_n be a solution of the problem (2.2), (2.3), eqrefnon3, where n is fixed.

Lemma 3.1. *The sequence (φ_n) is bounded in $C^1([0,1],\mathbb{R}^k)$.*

Proof. Assume that the sequence (φ_n) is unbounded. Then, passing to a subsequence if necessary, we have $\|\varphi_n\| \to \infty$. We can proceed analogously as in (2.7) to show that

$$\sup_{t\in[0,1]} |(\varphi_n)'(t)| \le M,$$

for every n. Hence, $\sup_{t\in[0,1]} |\varphi_n(t)| \to \infty$, when $n \to \infty$.

Let us consider the following sequence $(\frac{\varphi_n}{\|\varphi_n\|}) \subset C^1([0,1],\mathbb{R}^k)$ and notice that the norm of the sequence equals 1. Hence, the sequence is bounded. Moreover, the family $(\frac{\varphi_n}{\|\varphi_n\|})$ (and simultaneously $(\frac{\varphi_n'}{\|\varphi_n\|})$) is equicontinuous, since $\frac{\varphi_n'(t)}{\|\varphi_n\|}$ (or $\frac{\varphi_n''(t)}{\|\varphi_n\|}$) is bounded. By Lemma 2.1, there exists a convergent subsequence of $(\frac{\varphi_n}{\|\varphi_n\|})$. To simplify the notation, let us denote this subsequence as $(\frac{\varphi_n}{\|\varphi_n\|})$.

First, observe that $\frac{\varphi_n'(t)}{\|\varphi_n\|} \to 0 \in \mathbb{R}^k$. Now, we shall show that

$$\frac{\varphi_n(t)}{\|\varphi_n\|} \to \xi, \tag{3.1}$$

where $\xi = (\xi_1,\dots,\xi_k)$ does not depend on t and $|\xi| = 1$.

Indeed, notice that $\frac{\varphi_n(t)}{\|\varphi_n\|}$ is given by

$$\begin{aligned}\frac{\varphi_n(t)}{\|\varphi_n\|} &= \frac{\int_0^t\int_0^s f(u,\varphi_n(u))\,du\,ds}{\|\varphi_n\|}\\ &\quad + \frac{\int_0^1 f(u,\varphi_n(u))du - \int_0^1\int_0^s f(u,\varphi_n(u))du\,dg(s)}{\alpha_n\|\varphi_n\|}.\end{aligned} \tag{3.2}$$

Since f is bounded, we obtain

$$\lim_{n\to\infty} \frac{\int_0^t\int_0^s f(u,\varphi_n(u))\,du\,ds}{\|\varphi_n\|} = 0 \in \mathbb{R}^k. \tag{3.3}$$

Now, by (3.2) and (3.3), we can easily observe that the limit (3.1) does not depend on t. The norm of the sequence $(\frac{\varphi_n}{\|\varphi_n\|})$ equals 1. Hence $\frac{\varphi_n(t)}{\|\varphi_n\|} \to \xi$, where $|\xi| = 1$.

On the other hand,

$$\begin{aligned}
\xi &= \lim_{n\to\infty} \frac{\varphi_n(t)}{\|\varphi_n\|} \\
&= \frac{\int_0^t \int_0^s f(u,\varphi_n(u))\,du\,ds}{\|\varphi_n\|} \\
&\quad + \frac{\int_0^1 f(u,\varphi_n(u))du - \int_0^1 \int_0^s f(u,\varphi_n(u))du\,dg(s)}{\alpha_n\|\varphi_n\|} \\
&= \lim_{n\to\infty} \Big(\frac{\int_0^1 f(u,\|\varphi_n\| \frac{\varphi_n(u)}{\|\varphi_n\|})du}{\alpha_n\|\varphi_n\|} - \frac{\int_0^1 \int_0^s f(u,\|\varphi_n\| \frac{\varphi_n(u)}{\|\varphi_n\|})du\,dg(s)}{\alpha_n\|\varphi_n\|} \Big).
\end{aligned} \tag{3.4}$$

Now, observe, that there exist a uniform limits of

$$\int_0^1 f(u,\|\varphi_n\| \frac{\varphi_n(u)}{\|\varphi_n\|})du$$

and

$$\int_0^1 \int_0^s f(u,\|\varphi_n\| \frac{\varphi_n(u)}{\|\varphi_n\|})du\,dg(s)$$

Moreover, by (iv), the sum of the limits is different from zero. Hence, since (3.1) holds, there exists $\gamma \in (0,\infty)$ such that $\gamma := \lim_{n\to\infty} 1/(\alpha_n\|\varphi_n\|)$.

Now, by assumption (iii), we obtain

$$\xi = \lim_{n\to\infty} \frac{\varphi_n(t)}{\|\varphi_n\|} = \gamma\Big[\int_0^1 h(u,\xi)du - \int_0^1 \int_0^s h(u,\xi)du\,dg(s)\Big]. \tag{3.5}$$

Finally, by (3.5) and (iv), we obtain

$$\begin{aligned}
1 = (\xi \mid \xi) &= \gamma\Big(\xi \mid \int_0^1 h(u,\xi)du - \int_0^1 \int_0^s h(u,\xi)du\,dg(s)\Big) \\
&= \gamma(\xi \mid h_0(\xi)) < 0
\end{aligned}$$

a contradiction. Hence, the sequence (φ_n) is bounded. □

Now, it is easy to see that the following lemma holds.

Lemma 3.2. *The set $Z = \{\varphi_n : n \in \mathbb{N}\}$ is relatively compact in $C^1([0,1],\mathbb{R}^k)$.*

By the above Lemmas, we get the proof of the following result.

Theorem 3.3. *Under assumptions* (i)–(iv) *problem* (1.1) *has at least one solution.*

Proof. Lemma 3.2 implies that (φ_n) has a convergent subsequence (φ_{n_l}), $\varphi_{n_l} \to \varphi$. We know that φ_{n_l} (φ'_{n_l}) converges uniformly to φ (φ') on $[0,1]$. Since (φ_{n_l}) is equibounded and f is uniformly continuous on compact sets, one can see that $f(t,\varphi_{n_l})$ is uniformly convergent to $f(t,\varphi)$. Since

$$\varphi''_{n_l}(t) = f(t,\varphi_{n_l}(t)),$$

the sequence $\varphi''_{n_l}(t)$ is also uniformly convergent. Moreover, $\varphi''_{n_l}(t)$ converges uniformly to $\varphi''(t)$.

Note that we have actually proved that function $\varphi \in C^1([0,1],\mathbb{R}^k)$ is a solution of the equation of problem (1.1). By (2.3) and (2.4), it is easy to see that φ satisfies boundary conditions of problem (1.1). This completes the proof. □

4. Applications

To illustrate our results we shall present some examples.

Example 4.1. Let us consider the Neumann BVP

$$x'' = f(t, x), \quad x'(0) = 0, \quad x'(1) = 0.$$

In this case $g_i(t) =$ constant, $i = 1, \dots, k$, $t \in [0, 1]$ and condition (ii) always holds. Moreover, we have

$$h_0(\xi) = \int_0^1 h(s, \xi) ds.$$

Hence for any f which satisfies conditions (i), (iii) and (iv) the Neumann BVP has at least one solution.

Example 4.2. Let $k = 1$, $g(t) = t$ and $f(t, x) = \frac{t-|x|x}{x^2+1}$. We have

$$h(t, \xi) = \lim_{\lambda\to\infty} f(t, \lambda\, \xi) = \begin{cases} -1, & \xi = 1 \\ 1, & \xi = -1\,. \end{cases}$$

Then $h_0(1) = -1/2$ and $h_0(-1) = 1/2$ and we get $(\xi|h_0(\xi)) < 0$. Hence, problem (1.1) has at least one nontrivial solution.

Example 4.3. Let $k = 3$, $g(t) = (t, t, t)$ and

$$\begin{gathered} f_1(t, x_1, x_2, x_3) = \frac{-x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2 + \sin^2 t + 1}}, \\ f_2(t, x_1, x_2, x_3) = \frac{-x_2 - t}{\sqrt{x_1^2 + x_2^2 + x_3^2 + 1}}, \\ f_3(t, x_1, x_2, x_3) = \frac{-x_3 + \arctan(x_2 - t)}{\sqrt{x_1^2 + x_2^2 + x_3^2 + 1}}. \end{gathered}$$

For every $\xi = (\xi_1, \xi_2, \xi_3)$ with $|\xi| = 1$, we obtain

$$\begin{gathered} h(t, \xi) = \lim_{\lambda\to\infty} f(t, \lambda\xi) = \Big(-\frac{\xi_1}{|\xi|}, -\frac{\xi_2}{|\xi|}, -\frac{\xi_3}{|\xi|} \Big), \\ h_0(\xi) = \Big(-\frac{\xi_1}{2|\xi|}, -\frac{\xi_2}{2|\xi|}, -\frac{\xi_3}{2|\xi|} \Big). \end{gathered}$$

Then

$$(\xi|h_0(\xi)) = -\frac{1}{2}\Big(\frac{\xi_1^2}{|\xi|} + \frac{\xi_2^2}{|\xi|} + \frac{\xi_3^2}{|\xi|} \Big) = -\frac{1}{2}|\xi| < 0.$$

Hence, problem (1.1) has at least one nontrivial solution.

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