Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 147, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INELASTIC COLLISION OF TWO SOLITONS FOR GENERALIZED BBM EQUATION WITH CUBIC NONLINEARITY

JINGDONG WEI, LIXIN TIAN, ZAILI ZHEN, WEIWEI GAO

Abstract. We study the inelastic collision of two solitary waves of different velocities for the generalized Benjamin-Bona-Mahony (BBM) equation with cubic nonlinearity. It shows that one solitary wave is smaller than the other one in the $H^1(\mathbb{R})$ energy space. We explore the sharp estimates of the nonzero residue due to the collision, and prove the inelastic collision of two solitary waves and nonexistence of a pure 2-soliton solution.

1. Introduction

In this article, we study the generalized Benjamin-Bona-Mahony (BBM) equation with cubic nonlinearity

$$(1-\partial_x^2)\partial_t u + \partial_x(u+u^3) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1}$$

where $u(t,x)$ is a function of time t and a single spatial variable x. If the nonlinearity term u^3 changes to u^2, then the above equation becomes the BBM equation. Equation (1.1) was introduced by Peregrine [30] and Benjamin, Bona and Mahony [2]. In particular, it is not completely integrable. No inverse-scattering theory can be developed for this equation [23, 29]. This situation is in contrast with the generalized Korteweg-de Vries equation (gKdV) equations

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0, \tag{1.2}$$

which is completely integrable for $f(u) = u^2$ (KdV equation), $f(u) = u^3$ (mKdV equation) and $f(u) = u^2 - \mu u^3$ (Gardner equation).

Let us review some classical works related to collision problems of solitons for the generalized KdV and BBM equations. The two equations have been studied since the 1960s from both experimental and numerical points of view; see examples [1, 3, 4, 5, 8, 10, 31, 33, 34, 35]. Many elegant results have been found on the existence of explicit solution and stability, local and global well-posedness, long time dynamical behavior, etc.

It is well-known that the KdV equation has explicit pure N-soliton solutions by using the inverse scattering transform [9, 24, 32]. The stability and asymptotic

2010 *Mathematics Subject Classification.* 35Q53, 35C10, 35B35, 35B40.
Key words and phrases. Generalized Benjamin-Bona-Mahony equation; cubic nonlinearity; solitary waves; pure 2-soliton; collision.

Submitted September 26, 2014. Published June 6, 2015.

stability of N-solitons of KdV equation were studied by Maddocks and Sachs [12] in $H^N(R)$ using variational techniques and in $H^1(\mathbb{R})$ by Martel, Merle and Tsai [21]. LeVeque [11] further investigated the behavior of the explicit 2-soliton solution of KdV equation for nearly equal size. Mizumachi [26] considered the large time behavior of two decoupled solitary wave of the generalized KdV equation. Martel and Merle [16] investigated the inelastic collision of two solitons with nearly equal size for gKdV equation (1.2) with $f(u) = u^4$. It shows that the 2-soliton structure is globally stable in $H^1(\mathbb{R})$ and the nonexistence of a pure 2-soliton in the regime. They also considered the so-called BBM equation [15]:

$$(1-\lambda\partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + u^2) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R},\ \lambda \in (0,1).$$

For the pure 2-soliton, we mean that the solution $u(t,x)$ of KdV equation satisfies

$$\|u(t,x) - \sum_{j=1}^{2} Q_{c_j}(x - c_j t - x_j)\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } t \to -\infty,$$
$$\|u(t,x) - \sum_{j=1}^{2} Q_{c_j}(x - c_j t - x'_j)\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } t \to +\infty,$$

for some x_j such that the shifts $\Delta_j = x'_j - x_j$ depend on c_1, c_2. This solution which is called the 2-soliton represents the pure collision of two solitons, with no residue terms before and after the collision. In other words, the collision is elastic.

Except for the collision of two solitons with nearly equal size of two equations, the collision of two solitons with different velocities has also been studied in [14, 18, 20, 28]. In [14], it was shown that the collision of two stable solitary waves is inelastic but almost elastic. As a consequence, the monotonicity properties are strict: the size of the large soliton increases and the size of the small soliton decreases through the collision with explicit lower and upper bounds. In [18], they considered the generalized KdV equation with a general nonlinearity $f(u)$:

$$u_t + (u_{xx} + f(u))_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad u(0) = u_0 \in H^1(\mathbb{R}),$$

assuming that for $p = 2,3,4$, $f(u) = u^p + f_1(u)$ where $\lim_{u\to 0} |\frac{f_1(u)}{u^p}| = 0$, and f_1 belongs C^{p+4}.

In [28], it is classified on the nonlinearities for which collision are elastic and inelastic. For integrable case, the collision of two solitons is elastic, such as KdV, mKdV and Gardner nonlinearities. For non-integrable case, the collision of two solitons is inelastic, such as gKdV, a general case f and BBM etc.

There are many unsolved questions related to the collision problems for partial differential equations. In this study we focus on the existence of a pure 2-soliton for the gBBM equations in the collision regime. To prove the nonexistence of a pure 2-soliton for non-integrable partial differential equations, we should consider four conditions [4, 18, 20]. The first one is that the related Cauchy problem should be globally well-posed. The second one is that the solutions of the equations should satisfy the mass and energy conservation laws. The third one is that the equations should have solitary wave solutions with certain properties. The last one is that the equations should have asymptotic stability of multi-solitons.

In this article, we consider the collision of two solitary waves with different velocities for the gBBM equation with cubic nonlinearity. Our purposes is to study the dynamical behavior of two solitons during the collision and to prove the nonexistence of a pure 2-soliton after the collision.

As we know, the Cauchy problem related to (1.1) is globally well-posed in $H^1(\mathbb{R})$ [2] and the solutions of (1.1) satisfy the following conservation laws:

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)dx + \frac{1}{4}\int_{\mathbb{R}} u^4(t,x)dx = E(u_0), \tag{1.3}$$

$$m(u(t)) = \frac{1}{2}\int_{\mathbb{R}} (u^2(t,x) + u_x^2(t,x))dx = m(u_0). \tag{1.4}$$

The quantity $\int_{\mathbb{R}} u(t,x)dx$ is also formally conserved. However, (1.1) admits no more conserved quantities.

Recall that (1.1) has a two-parameter family of solitary wave solutions $\{\phi_c(x - ct - x_0) : c > 1,\ x_0 \in \mathbb{R}\}$, where ϕ_c satisfies

$$c\phi_c'' - (c-1)\phi_c + \phi_c^3 = 0, \quad \text{in } \mathbb{R}. \tag{1.5}$$

The unique even solution of (1.5) is given by

$$\phi_c(x) = (c-1)^{1/2} Q\Big(\sqrt{\frac{c-1}{c}}x\Big),$$

where $Q(x) = \sqrt{2}\cosh^{-1}(x)$ solves

$$Q'' + Q^3 = Q. \tag{1.6}$$

Let $N \le 1$, $1 < c_N < \dots < c_1$ and $x_1, \dots, x_N \in \mathbb{R}$. There exists a unique solution u of (1.1) such that

$$\lim_{t\to-\infty} \|u(t,x) - \sum_{j=1}^{N} \phi_{c_j}(x - x_j - c_j t)\|_{H^1(\mathbb{R})} = 0, \tag{1.7}$$

which means that the behavior of the sum of N-soliton solutions with different velocities is asymptotical stable as $t \to -\infty$. Following [14, 18, 20], we assume that $1 < c_2 < c_1$ and $u(t)$ is the $H^1(\mathbb{R})$ solution of (1.1) such that

$$\lim_{t\to-\infty} \|u(t) - \sum_{j=1,2} \phi_{c_j}(x - c_j t)\|_{H^1(\mathbb{R})} = 0. \tag{1.8}$$

By the symmetry $x \to -x, t \to -t$ of (1.1), there exist solutions with similar behavior as $t \to +\infty$. However, what will occur after the collision of the two solitons and the global behavior of such solutions is still unknown. We will explore the collision of two solitons and the nonexistence of a pure 2-soliton solution.

Theorem 1.1. *Let $c_1 > c_2 > 1$ and u be the unique solution of* (1.1) *such that*

$$\lim_{t\to-\infty} \|u(t) - \sum_{j=1,2} \phi_{c_j}(x - c_j t)\|_{H^1(\mathbb{R})} = 0. \tag{1.9}$$

There exists $\varepsilon_0 = \varepsilon_0(c_1) > 0$ such that if $0 < c_2 - 1 < \varepsilon_0$ then there exists $c_1^+ > c_2^+ > 1, \rho_1(t), \rho_2(t)$ and $T_0, K > 0$ such that

$$w^+(t,x) = u(t,x) - \sum_{j=1,2} \phi_{c_j^+}(x - \rho_j(t))$$

satisfies

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(x > \frac{1}{2}(1+c_2)t)} = 0,, \tag{1.10}$$

$$\frac{1}{K}(c_2-1)^{\frac{9}{2}} \le c_1^+ - c_1 \le K(c_2-1)^{\frac{7}{2}}, \quad \frac{1}{K}(c_2-1)^5 \le c_2 - c_2^+ \le K(c_2-1)^4, , \tag{1.11}$$

$$\frac{1}{K}(c_2-1)^{9/4} \le \|\partial_x w^+(t)\|_{L^2(\mathbb{R})} + \sqrt{c_2-1}\|w^+(t)\|_{L^2(\mathbb{R})} \le K(c_2-1)^{7/4}, \tag{1.12}$$

for $t \ge T_0$.

Remark 1.2. Because the mass and energy conservation, the Sobolev space $H^1(\mathbb{R})$ appears to be an ideal space to study long time dynamical properties of (1.1).

Remark 1.3. The results of the Theorem 1.1 mean nonexistence of a pure 2-soliton in the regime. By (1.9) and (1.10), we see that an asymptotic 2-soliton at $-\infty$ cannot be an asymptotic 2-soliton at $+\infty$. We also see from (1.11) that the size of the large soliton increases and the size of the small soliton decreases through the collision, with explicit lower and upper bounds. The bound in (1.12) is thus a qualitative version of nonexistence of a pure 2-soliton.

This article is organized as follows. In Section 2, we construct an approximate solution to the problem in the collision regime. Section 3 is devoted to preliminary stability results. Section 4 is concerned with the proof of Theorem 1.1.

2. Construction of an approximate 2-soliton solution

The objective of this section is to construct an approximate solution for the gBBM equation with cubic nonlinearity, which describe the inelastic collision of two solitons ϕ_{c_1}, ϕ_{c_2} in the case where $0 < c_2 - 1 < \varepsilon_0$ is small enough. And the approximate solution $z(t,x)$ does only exist in the collision region. Also, the structure of $z(t,x)$ and $S(t)$ are more complicated than that of the BBM equation.

2.1. **Reduction of the problem.** Let

$$c_1 > 1, \quad \lambda = \frac{c_1 - 1}{c_1} \in (0,1). \tag{2.1}$$

We make the change of variables:

$$\hat{x} = \lambda^{1/2}(x - \frac{t}{1-\lambda}), \quad \hat{t} = \frac{\lambda^{3/2}}{1-\lambda}t, \quad z(\hat{t},\hat{x}) = \sqrt{\frac{1-\lambda}{\lambda}}\, u(t,x). \tag{2.2}$$

If $u(t,x)$ is a solution to (1.1), then $z(\hat{t},\hat{x})$ satisfies

$$(1-\lambda\partial_{\hat{x}}^2)\partial_{\hat{t}} z + \partial_{\hat{x}}(\partial_{\hat{x}}^2 z - z + z^3) = 0. \tag{2.3}$$

Lemma 2.1. *(i) Let $c > 1$. By (2.2), a solitary wave solution $\phi_c(x-ct)$ to (1.1) is transformed into $\tilde{Q}_\sigma(y_\sigma)$ which is a solution of (2.3) where*

$$\tilde{Q}_\sigma(x) = \sqrt{\sigma\theta_\sigma}Q(\sqrt{\sigma}x), \quad Q(x) = \sqrt{2}\cosh^{-1}(x), \tag{2.4}$$

$$\sigma = \frac{c-1}{c\lambda}, \quad \theta_\sigma = \frac{1-\lambda}{1-\lambda\sigma} = (1-\lambda)\sum_{j=0}^{\infty}(\lambda\sigma)^j, \quad \frac{1}{\theta_\sigma} = \frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}\sigma,$$

$$\mu_\sigma = \frac{1-\sigma}{1-\lambda\sigma} = 1 + (\lambda-1)\sigma\sum_{j=0}^{\infty}(\lambda\sigma)^j, \quad y_\sigma = \hat{x} + \mu_\sigma\hat{t}.$$

Especially, if $c = c_1$, then $\mu_\sigma = 0$, $y_\sigma = \hat{x}$, $\tilde{Q}_\sigma(y_\sigma) = Q(\hat{x})$ and

$$Q'' + Q^3 = Q, \quad (Q')^2 + \frac{1}{2}Q^4 = Q^2 \quad \text{in } \mathbb{R}. \tag{2.5}$$

(ii) Moreover, $\tilde{Q}_\sigma$ satisfies the equations

$$\tilde{Q}_\sigma'' + \frac{1}{\theta_\sigma}\tilde{Q}_\sigma^3 = \sigma\tilde{Q}_\sigma, \quad (\tilde{Q}_\sigma')^2 + \frac{1}{2\theta_\sigma}\tilde{Q}_\sigma^4 = \sigma\tilde{Q}_\sigma^2. \tag{2.6}$$

For $\sigma > 0$ small, we have

$$\|\tilde{Q}_\sigma\|_{L^\infty(\mathbb{R})} \sim \sqrt{(1-\lambda)\sigma}\|Q\|_{L^\infty(\mathbb{R})}, \quad \|\tilde{Q}_\sigma\|_{L^2(\mathbb{R})} \sim \sqrt{(1-\lambda)}\sigma^{1/4}\|Q\|_{L^2(\mathbb{R})}, \tag{2.7}$$

$$(\tilde{Q}_\sigma^3)'(y_\sigma + \delta) = \frac{(1-\lambda)^{3/2}}{\lambda^2}(\phi_c^3)'(x - ct + \frac{\delta}{\sqrt{\lambda}}). \tag{2.8}$$

The proof of the above lemma is similar to the proof of [20, Claim 2.1], so we omit it.

2.2. **Decomposition of approximate solution.** Firstly, we construct an approximate solution $z(t,x)$ of

$$(1-\lambda\partial_x^2)\partial_t z + \partial_x(\partial_x^2 z - z + z^3) = 0, \tag{2.9}$$

which is the sum of the function $Q(y)$, a small soliton $\tilde{Q}_\sigma(y_\sigma)$ and an error term $w(t,x)$. As in [20], we introduce the new coordinates and the approximate solution of the form

$$y_\sigma = x + \mu_\sigma t, \quad y = x - \alpha(y_\sigma),$$

$$\alpha(s) = \int_0^s \beta(r)dr, \quad \beta(y_\sigma) = \sum_{(k,l)\in\Sigma_0} a_{k,l}\sigma^l\tilde{Q}_\sigma^k(y_\sigma), \tag{2.10}$$

$$z(t,x) = Q(y) + \tilde{Q}_\sigma(y_\sigma) + w(t,x), \tag{2.11}$$

where

$$w(t,x) = \sum_{(k,l)\in\Sigma_0} \sigma^l(A_{k,l}(y)\tilde{Q}_\sigma^k(y_\sigma) + B_{k,l}(y)(\tilde{Q}_\sigma^k)'(y_\sigma)), \tag{2.12}$$

$$\Sigma_0 = \{(k,l) = (1,0),(1,1),(2,0),(2,1),(3,0),(4,0)\}, \tag{2.13}$$

Define the operator L by

$$Lf = -f'' + f - 3Q^2 f. \tag{2.14}$$

Definition 2.2. Let $\mathcal{M}$ be the set of C^∞ functions f such that

$$\forall j \in \mathbb{N},\ \exists K_j,\ r_j > 0,\ \forall x \in \mathbb{R},\ |f^{(j)}(x)| \le K_j(1+|x|)^{r_j}e^{-|x|}. \tag{2.15}$$

Following formulas (2.10)-(2.13), we set

$$S(z) = (1-\lambda\partial_x^2)\partial_t z + \partial_x(\partial_x^2 z - z + z^3) = S_{\rm mKdV}(z) + S_{\rm gBBM}(z),$$
$$S_{\rm mKdV}(z) = \partial_t z + \partial_x(\partial_x^2 z - z + z^3), \quad S_{\rm gBBM}(z) = -\lambda\partial_t\partial_x^2 z.$$

Then, it gives

$$S(z(t,x)) = S(Q(y)) + S(\tilde{Q}_\sigma(y_\sigma)) + \delta S(w(t,x)) + S_{int}(t,x), \tag{2.16}$$

where

$$\delta S(w) = \delta S_{\rm mKdV}(w) + S_{\rm gBBM}(w),\ \delta S_{\rm mKdV}(w) = \partial_t w - \partial_x L w, \tag{2.17}$$

$$\begin{aligned} S_{int}(t,x) = \partial_x(&w^3(t,x) + 3Q^2(y)\tilde{Q}_\sigma(y_\sigma) + 3Q(y)\tilde{Q}_\sigma^2(y_\sigma) + 3\tilde{Q}_\sigma^2(y_\sigma)w(t,x) \\ &+ 6Q(y)\tilde{Q}_\sigma(y_\sigma)w(t,x) + 3Q(y)w^2(t,x) + 3\tilde{Q}_\sigma(y_\sigma)w^2(t,x)). \end{aligned}$$

Since $\tilde{Q}_\sigma(y_\sigma)$ is a solution to (2.9), we get $S(\tilde{Q}_\sigma(y_\sigma)) = 0$.

Proposition 2.3. *There holds*

$$\begin{aligned} S(z) = &\sum_{(k,l)\in\Sigma_0} \sigma^l \tilde{Q}^k_\sigma(y_\sigma)\left(a_{k,l}((\lambda-3)Q''-3Q^3)'-(LA_{k,l})'+F_{k,l}\right)(y)\\ &+\sum_{(k,l)\in\Sigma_0} \sigma^l (\tilde{Q}^k_\sigma)'(y_\sigma)\Big((3-\lambda)A''_{k,l}+3Q^2A_{k,l}+a_{k,l}(2\lambda-3)Q''\\ &-(LB_{k,l})'+G_{k,l}\Big)(y)+\varepsilon(t,x), \end{aligned}$$

where

$$\begin{gathered} F_{1,0}=(3Q^2)',\quad G_{1,0}=3Q^2,\\ F_{1,1}=(3-2\lambda)A'_{1,0}+(3-\lambda)B''_{1,0}+3Q^2B_{1,0}+\lambda(\lambda-1)a_{1,0}Q''',\\ G_{1,1}=2a_{1,0}\lambda(\lambda-1)Q'',\\ F_{2,0}=a_{1,0}\big\{(\lambda-3)A''_{1,0}-3A_{1,0}Q^2-3Q^2\big\}'+(3Q+9A_{1,0}Q)'+(3-2\lambda)a^2_{2,0}Q''',\\ \begin{aligned} G_{2,0}=&\frac{a_{1,0}}{2}\left\{(6\lambda-9)A'_{1,0}+(\lambda-3)B''_{1,0}-3Q^2B_{1,0}\right\}'\\ &+(3B_{1,0}Q+3A_{1,0}B_{1,0}Q)'+3Q+9A_{1,0}Q+\frac{3}{2}(1-\lambda)a^2_{1,0}Q''. \end{aligned} \end{gathered}$$

In addition, the following statements hold:

(i) For all $(k,l)\in\Sigma_0$ such that $3\le k+l\le 4$, then $F_{k,l}$ and $G_{k,l}$ depend on $A_{k',l'}$ and $B_{k',l'}$ for $1\le k+l\le 2$. Moreover, if $A_{k',l'}$ is even and $B_{k',l'}$ is odd, then $F_{k,l}$ is odd and $G_{k,l}$ is even.

(ii) If the functions $A_{k',l'}$ and $B_{k',l'}$ are bounded, then $\varepsilon(t,x)$ satisfies

$$|\varepsilon(t,x)|\le K\sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)). \tag{2.18}$$

Before proving Proposition 2.3, we give some preliminary lemmas on ϕ_c.

Lemma 2.4 (Identities of ϕ_c)**.** *For all $c>1$,*

$$\begin{gathered} \int_{\mathbb{R}}\phi_c^2=c^{1/2}(c-1)^{1/2}\int_{\mathbb{R}}Q^2,\\ \int_{\mathbb{R}}\phi_c^4=\frac{4}{3}(c-1)\int_{\mathbb{R}}\phi_c^2,\quad \int_{\mathbb{R}}(\phi_c')^2=\frac{1}{3}\big(\frac{c-1}{c}\big)\int_{\mathbb{R}}\phi_c^2,\\ E(\phi_c)=\frac{1}{2}\int_{\mathbb{R}}\phi_c^2+\frac{1}{4}\int_{\mathbb{R}}\phi_c^4=\big(\frac{2c+1}{6}\big)c^{1/2}(c-1)^{1/2}\int_{\mathbb{R}}Q^2,\\ m(\phi_c)=\frac{1}{2}\Big(\frac{1}{3}\big(\frac{c-1}{c}\big)+1\Big)\int_{\mathbb{R}}\phi_c^2=c^{-1/2}(c-1)^{1/2}\big(\frac{4c-1}{6}\big)\int_{\mathbb{R}}Q^2,\\ E(\phi_c)-cm(\phi_c)=-\frac{1}{3}(c-1)^{3/2}c^{1/2}\int_{\mathbb{R}}Q^2,\\ \frac{d}{dc}E(\phi_c)=c\frac{d}{dc}m(\phi_c)>0. \end{gathered} \tag{2.19}$$

The proof of the above lemma can be completed by a straightforward calculations.

Lemma 2.5 (Properties of L, [14, Lemma 2.2])**.** *The operator L defined in $L^2(\mathbb{R})$ by (2.14) is self-adjoint and satisfies the following properties:*

(i) *by the first eigenfunction, then $LQ^2=-3Q^2$;*
(ii) *by the second eigenfunction, then $LQ'=0$; the kernel of L is $\{c_1Q',c_1\in\mathbb{R}\}$;*

(iii) *if a function $h \in L^2(\mathbb{R})$ is orthogonal to Q' by the L^2 scalar product, there exists a unique function $f \in H^2(\mathbb{R})$ orthogonal to Q' such that $Lf = h$. Moreover, if h is even (odd), then f is even (odd).*

Lemma 2.6 ([20, Claim B.1]). *Let $h(t,x) = g(y) = g(x - \alpha(y_\sigma))$, where g is a C^3 function. Then we have*

$$
\begin{gathered}
\partial_t h = -\mu_\sigma \beta(y_\sigma) g'(y), \quad \partial_x h = (1-\beta(y_\sigma))g'(y), \\
\partial_x^2 h = (1-\beta(y_\sigma))^2 g''(y) - \beta(y_\sigma) g'(y), \\
\partial_x \partial_t h = -\mu_\sigma (1-\beta(y_\sigma))\beta(y_\sigma) g''(y) - \mu_\sigma \beta'(y_\sigma) g'(y), \\
\partial_x^3 h = (1-\beta(y_\sigma))^3 g'''(y) - 3(1-\beta(y_\sigma))\beta'(y_\sigma) g''(y) - \beta''(y_\sigma) g'(y), \\
\begin{aligned}
\partial_x^2 \partial_t h = \mu_\sigma \Big\{ &-(1-\beta(y_\sigma))^2 \beta(y_\sigma) g'''(y) + 3\beta(y_\sigma)\beta'(y_\sigma) g''(y) \\
&- 2\beta'(y_\sigma) g''(y) - \beta''(y_\sigma) g'(y) \Big\}.
\end{aligned}
\end{gathered}
$$

Lemma 2.7. *Let A and q be C^3-functions. Then we have*

$$
\begin{aligned}
&\delta S_{\mathrm{mKdV}}(A(y)q(y_\sigma)) \\
&= q(y_\sigma)\Big\{ -(LA)'(y) + \beta(y_\sigma)(-3A'' - 3AQ^2 + (1-\mu_\sigma)A)'(y) - \beta'(y_\sigma)(3A'')(y) \\
&\quad + \beta^2(y_\sigma)(3A''')(y) + (\beta^2)'(3A''/2)(y) - \beta''(y_\sigma)A'(y) - \beta^3(y_\sigma)A'''(y) \Big\} \\
&\quad + q'(y_\sigma)\Big\{ 3A''(y) + 3A(y)Q^2(y) + (\mu_\sigma - 1)A(y) - \beta(y_\sigma)(6A'')(y) \\
&\quad - \beta'(y_\sigma)(3A')(y) + \beta^2(y_\sigma)(3A'')(y) \Big\} + q''(y_\sigma)\left\{ 3(1-\beta(y_\sigma))A'(y) \right\} \\
&\quad + q'''(y_\sigma)A(y).
\end{aligned}
$$

Proof. Using Lemma 2.6, we have

$$
\partial_t (A(y)q(y_\sigma)) = -\mu_\sigma \beta(y_\sigma) A' q(y_\sigma) + \mu_\sigma A q'(y_\sigma),
$$

and

$$
\begin{aligned}
&-\partial_x L(A(y)q(y_\sigma)) \\
&= \partial_x \big\{ (\partial_x^2 A - A + 3AQ^2) q(y_\sigma) + 2(\partial_x A) q'(y_\sigma) + A q''(y_\sigma) \big\} \\
&= \{ \partial_x (\partial_x^2 A - A + 3AQ^2) \} q(y_\sigma) + (\partial_x^2 A - A + 3AQ^2) q'(y_\sigma) \\
&\quad + 2(\partial_x^2 A) q'(y_\sigma) + 3(\partial_x A) q''(y_\sigma) + A q'''(y_\sigma) \\
&= q(y_\sigma)\Big\{ (1-\beta(y_\sigma))^3 A''' - 3(1-\beta(y_\sigma))\beta'(y_\sigma) A'' - \beta''(y_\sigma) A' - (1-\beta(y_\sigma)) A' \\
&\quad + 3(1-\beta(y_\sigma))(AQ^2)' \Big\} + q'(y_\sigma)\big\{ 3(1-\beta(y_\sigma))^2 A'' - 3\beta'(y_\sigma) A' - A + 3AQ^2 \big\} \\
&\quad + q''(y_\sigma)\left\{ 3(1-\beta(y_\sigma))A' \right\} + q'''(y_\sigma) A.
\end{aligned}
$$

Combining the above equalities, we complete the proof. □

Lemma 2.8 ([20, Claim B.3]). *Let A and q be C^3-functions. Then we have*

$$
\begin{aligned}
&S_{\mathrm{gBBM}}(A(y)q(y_\sigma)) \\
&= \lambda \mu_\sigma q(y_\sigma)\big\{ \beta(y_\sigma) A'''(y) + \beta'(y_\sigma)(2A''(y)) \big\} + \lambda \mu_\sigma q(y_\sigma)\Big\{ \beta^2(y_\sigma)(-2A''')(y) \\
&\quad + (\beta^2)'(y_\sigma)(-3A''/2)(y) + \beta''(y_\sigma) A'(y) + \beta^3(y_\sigma) A'''(y) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\mu_\sigma q'(y_\sigma)\Big\{-A''(y)+\beta(y_\sigma)(4A'')(y)+\beta'(y_\sigma)(3A')(y)+\beta^2(y_\sigma)(-3A'')(y)\Big\}\\
&+\lambda\mu_\sigma q''(y_\sigma)\big\{-2A'(y)+\beta(y_\sigma)(3A')(y)\big\}+\lambda\mu_\sigma q'''(y_\sigma)(-A)(y).
\end{aligned}
$$

Lemma 2.9. *Let*

$$\beta = a_{1,0}\tilde{Q}_\sigma + a_{1,1}\sigma\tilde{Q}_\sigma + a_{2,0}\tilde{Q}_\sigma^2 + a_{2,1}\sigma\tilde{Q}_\sigma^2 + a_{3,0}\tilde{Q}_\sigma^3 + a_{4,0}\tilde{Q}_\sigma^4.$$

Then

$$
\begin{aligned}
\beta' &= a_{1,0}(\tilde{Q}_\sigma)' + a_{1,1}\sigma(\tilde{Q}_\sigma)' + a_{2,0}(\tilde{Q}_\sigma^2)' + a_{2,1}\sigma(\tilde{Q}_\sigma^2)' + a_{3,0}(\tilde{Q}_\sigma^3)' + a_{4,0}(\tilde{Q}_\sigma^4)',\\
\beta'' &= \sigma\tilde{Q}_\sigma a_{1,0} + \tilde{Q}_\sigma^3\big(-\frac{a_{1,0}}{1-\lambda}\big) + \sigma\tilde{Q}_\sigma^2(4a_{2,0}) + \tilde{Q}_\sigma^4\big(-\frac{3a_{2,0}}{1-\lambda}\big)\\
&\quad + \sigma\tilde{Q}_\sigma^3\big(\frac{\lambda a_{1,0}}{1-\lambda} - \frac{a_{1,1}}{1-\lambda} + 9a_{3,0}\big) + \sigma^2\tilde{Q}_\sigma a_{1,1} + \tilde{Q}_\sigma^5\big(-\frac{6a_{3,0}}{1-\lambda}\big) + \sigma^{5/2}O(\tilde{Q}_\sigma),\\
\beta^2 &= a_{1,0}^2\tilde{Q}_\sigma^2 + 2a_{1,0}a_{2,0}\tilde{Q}_\sigma^3 + 2a_{1,0}a_{1,1}\sigma\tilde{Q}_\sigma^2 + (2a_{1,0}a_{3,0} + a_{2,0}^2)\tilde{Q}_\sigma^4\\
&\quad + (2a_{2,0}a_{3,0} + 2a_{1,0}a_{4,0})\tilde{Q}_\sigma^5 + (2a_{1,0}a_{2,1} + 2a_{1,1}a_{2,0})\sigma\tilde{Q}_\sigma^3 + \sigma^{5/2}O(\tilde{Q}_\sigma),\\
(\beta^2)' &= a_{1,0}^2(\tilde{Q}_\sigma^2)' + 2a_{1,0}a_{2,0}(\tilde{Q}_\sigma^3)' + 2a_{1,0}a_{1,1}\sigma(\tilde{Q}_\sigma^2)' + (2a_{1,0}a_{3,0} + a_{2,0}^2)(\tilde{Q}_\sigma^4)'\\
&\quad + \sigma^{5/2}O(\tilde{Q}_\sigma).
\end{aligned}
$$

In the next four lemmas, we expand the various terms in (2.16).

Lemma 2.10. *There holds*

$$
\begin{aligned}
S(Q) = &\sum_{(k,l)\in\Sigma_0}\sigma^l\Big(\tilde{Q}_\sigma^k(y_\sigma)a_{k,l}\big\{(\lambda-3)Q''-3Q^3\big\}'(y)\\
&+(\tilde{Q}_\sigma^k)'(y_\sigma)a_{k,l}(2\lambda-3)Q''(y)\Big)\\
&+\sum_{(k,l)\in\Sigma_0}\sigma^l\Big(\tilde{Q}_\sigma^k(y_\sigma)F_{k,l}^I(y)+(\tilde{Q}_\sigma^k)'(y_\sigma)G_{k,l}^I(y)\Big)\\
&+\sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)),
\end{aligned}
\tag{2.20}
$$

where

$$
\begin{gathered}
F_{1,0}^I = 0,\quad G_{1,0}^I = 0,\quad F_{1,1}^I = \lambda(\lambda-1)a_{1,0}Q''',\quad G_{1,1}^I = 2a_{1,0}\lambda(\lambda-1)Q''\\
F_{2,0}^I = (3-2\lambda)a_{2,0}^2Q''',\quad G_{2,0}^I = \frac{3}{2}(1-\lambda)a_{1,0}^2Q'',
\end{gathered}
$$

and for all $(k,l)\in\Sigma_0$ such that $3\le k+l\le 4$, $F_{k,l}^I\in\mathcal{M}$ is odd, $G_{k,l}^I\in\mathcal{M}$ is even and both depend only on $a_{k',l'}$ for $1\le k'+l'\le 2$.

Proof. By Lemma 2.7, let $A(y)=Q(y)$ and $q=1$, then

$$
\begin{aligned}
S_{\mathrm{mKdV}}(Q(y)) &= \delta S_{\mathrm{mKdV}}(Q) - \partial_x(2Q^3)\\
&= (Q''-Q+Q^3)' + \beta(y_\sigma)(-3Q''-3Q^3\\
&\quad + (1-\mu_\sigma)Q)' - \beta'(y_\sigma)(3Q'') + \beta^2(y_\sigma)(3Q''') + (\beta^2)'(3Q''/2)\\
&\quad - \beta''(y_\sigma)Q' - \beta^3(y_\sigma)Q'''.
\end{aligned}
$$

Using $Q''=Q-Q^3$ and (2.4), we find

$$
\begin{aligned}
&S_{\mathrm{mKdV}}(Q(y))\\
&= \beta(y_\sigma)(-3Q''-3Q^3)' - \beta'(y_\sigma)(3Q'') + \beta^2(y_\sigma)(3Q''') + (\beta^2)'(y_\sigma)(3Q''/2)
\end{aligned}
$$

$$
\begin{aligned}
&- \sigma\beta(y_\sigma)(\lambda-1)Q' - \beta''(y_\sigma)Q' - \beta^3(y_\sigma)Q''' - (\lambda-1)\lambda\sigma^2\beta(y_\sigma)Q' \\
&+ \sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}
$$

By Lemma 2.8 and (2.4), we have

$$
\begin{aligned}
&S_{gBBM}(Q(y)) \\
&= \lambda\mu_\sigma\Big\{\beta(y_\sigma)Q''' + \beta'(y_\sigma)(2Q'') + \beta^2(y_\sigma)(-2Q''') + (\beta^2)'(y_\sigma)(-3Q''/2) \\
&\quad + \beta''(y_\sigma)Q' + \beta^3(y_\sigma)(Q''')\Big\} \\
&= \beta(y_\sigma)(\lambda Q''') + \beta'(y_\sigma)(2\lambda Q'') + \beta^2(y_\sigma)(-2\lambda Q''') + (\beta^2)'(y_\sigma)(-3\lambda Q''/2) \\
&\quad + \sigma\beta(y_\sigma)\lambda(\lambda-1)Q''' + \sigma\beta'(y_\sigma)\left\{2\lambda(\lambda-1)Q''\right\} \\
&\quad + \beta''(y_\sigma)(\lambda Q') + \beta^3(y_\sigma)(\lambda Q''') + \sigma\beta^2(y_\sigma)\lambda(\lambda-1)(-2Q''') \\
&\quad + \sigma(\beta^2)'(y_\sigma)\lambda(\lambda-1)(-3Q''/2) + \sigma\beta''(y_\sigma)\lambda(\lambda-1)Q' \\
&\quad + \sigma^2\beta(y_\sigma)\lambda^2(\lambda-1)Q''' + \sigma^2\beta'(y_\sigma)\lambda^2(\lambda-1)(2Q'') + \sigma^{5/2}Q_\sigma(y_\sigma)).
\end{aligned}
$$

Combining the above discussions, we deduce that

$$
\begin{aligned}
&S(Q) \\
&= \beta(y_\sigma)\left\{(\lambda-3)Q'' - 3Q^3\right\}' + \beta'(y_\sigma)(2\lambda-3)Q'' + \beta^2(y_\sigma)(3-2\lambda)Q''' \\
&\quad + (\beta^2)'(y_\sigma)(1-\lambda)(3Q''/2) + \beta''(y_\sigma)(\lambda-1)Q' + \sigma\beta(y_\sigma)(\lambda-1)\left\{\lambda Q'' - Q\right\}' \\
&\quad + \sigma\beta'(y_\sigma)\left\{2\lambda(\lambda-1)Q''\right\} + \beta^3(y_\sigma)(\lambda-1)Q''' + \sigma\beta^2(y_\sigma)\lambda(\lambda-1)(-2Q''') \\
&\quad + \sigma(\beta^2)'(y_\sigma)\lambda(\lambda-1)(-3Q''/2) + \sigma\beta''(y_\sigma)\lambda(\lambda-1)Q' \\
&\quad + \sigma^2\beta(y_\sigma)\lambda(\lambda-1)(\lambda Q'' - Q)' + \sigma^2\beta'(y_\sigma)\lambda^2(\lambda-1)(2Q'') + \sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}
$$

By Lemma 2.9, we derive

$$
\begin{aligned}
&S(Q) \\
&= \tilde{Q}_\sigma(y_\sigma)a_{1,0}\left\{(\lambda-3)Q'' - 3Q^3\right\}' + \tilde{Q}_\sigma'(y_\sigma)a_{1,0}(2\lambda-3)Q'' \\
&\quad + \tilde{Q}_\sigma^2(y_\sigma)\Big(a_{2,0}\left\{(\lambda-3)Q'' - 3Q^3\right\}' + (3-2\lambda)a_{2,0}^2Q'''\Big) \\
&\quad + (\tilde{Q}_\sigma^2)'(y_\sigma)\Big(a_{2,0}(2\lambda-3)Q'' + \frac{3}{2}(1-\lambda)a_{1,0}^2Q''\Big) \\
&\quad + \sigma\tilde{Q}_\sigma(y_\sigma)\Big(a_{1,1}\left\{(\lambda-3)Q'' - 3Q^3\right\}' + \lambda(\lambda-1)a_{1,0}Q'''\Big) \\
&\quad + \sigma\tilde{Q}_\sigma'(y_\sigma)\left(a_{1,1}(2\lambda-3)Q'' + 2\lambda(\lambda-1)a_{1,0}Q''\right) \\
&\quad + \sum_{3\le k+l\le 4}\sigma^l\Big(\tilde{Q}_\sigma^k(y_\sigma)a_{k,l}\left\{(\lambda-3)Q'' - 3Q^3\right\}'(y) + (\tilde{Q}_\sigma^k)'(y_\sigma)a_{k,l}(2\lambda-3)Q''(y)\Big) \\
&\quad + \sum_{3\le k+l\le 4}\sigma^l\Big(\tilde{Q}_\sigma^k(y_\sigma)F_{k,l}^I + (\tilde{Q}_\sigma^k)'(y_\sigma)G_{k,l}^I\Big) + \sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)),
\end{aligned}
$$

for $3 \le k+l \le 4$, $F_{k,l}^I \in \mathcal{M}$ and $G_{k,l}^I \in \mathcal{M}$ are as in the statement of Lemma 2.10. □

Lemma 2.11. *There holds*

$$\delta S_{\mathrm{mKdV}}(w)$$

$$
\begin{aligned}
&= \sum_{(k,l)\in\Sigma_0} \sigma^l \Big(\tilde{Q}_\sigma^k(y_\sigma)(-LA_{k,l})'(y) + (\tilde{Q}_\sigma^k)'(y_\sigma)((-LB_{k,l})' + 3A''_{k,l} + 3Q^2A_{k,l})(y)\Big)\\
&\quad + \sum_{(k,l)\in\Sigma_0} \sigma^l \Big(\tilde{Q}_\sigma^k(y_\sigma)F_{k,l}^{II}(y) + (\tilde{Q}_\sigma^k)'(y_\sigma)G_{k,l}^{II}(y)\Big) + \sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)),
\end{aligned}
$$

where

$$
\begin{gathered}
F_{1,0}^{II} = 0, \quad G_{1,0}^{II} = 0, \quad F_{1,1}^{II} = 3A'_{1,0} + 3B''_{1,0} + 3Q^2B_{1,0}, \quad G_{1,1}^{II} = 0,\\
F_{2,0}^{II} = a_{1,0}(-3A''_{1,0} - 3A_{1,0}Q^2)', \quad G_{2,0}^{II} = -\frac{3a_{1,0}}{2}\left(3A''_{1,0} + (B''_{1,0} + Q^2B_{1,0})'\right).
\end{gathered}
$$

For all $(k,l)\in\Sigma_0$ such that $1 \le k'+l' \le 2$, $F_{k,l}^{II}$ and $G_{k,l}^{II}$ depend on $A_{k',l'}$ and $B_{k',l'}$ for $1 \le k'+l' \le 2$. Moreover, if $A_{k',l'}$ is even and $B_{k',l'}$ is odd, then $F_{k,l}^{II}$ is odd and $G_{k,l}^{II}$ is even.

Proof. Note that

$$
\delta S_{\mathrm{mKdV}}(w) = \sum_{(k,l)\in\Sigma_0} \sigma^l\Big(\delta S_{\mathrm{mKdV}}(A_{k,l}(y)\tilde{Q}_\sigma(y_\sigma)) + \delta S_{\mathrm{mKdV}}(B_{k,l}(y)(\tilde{Q}_\sigma)'(y_\sigma))\Big).
$$

By Lemmas 2.7 and 2.9, we have

$$
\begin{aligned}
&\delta S_{\mathrm{mKdV}}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma))\\
&= \tilde{Q}_\sigma(y_\sigma)\Big\{-(LA_{1,0})' + a_{1,0}\tilde{Q}_\sigma(y_\sigma)(-3A''_{1,0} - 3A_{1,0}Q^2)' - a_{1,0}\tilde{Q}'_\sigma(y_\sigma)(3A''_{1,0})\\
&\quad + a_{1,0}^2\tilde{Q}_\sigma^2(y_\sigma)(3A'''_{1,0})\Big\} + \tilde{Q}'_\sigma(y_\sigma)\Big\{3A''_{1,0} + 3A_{1,0}Q^2 - a_{1,0}\tilde{Q}_\sigma(y_\sigma)(6A''_{1,0})\Big\}\\
&\quad + \tilde{Q}''_\sigma(y_\sigma)(3A'_{1,0}) + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}
$$

Using $(\tilde{Q}'_\sigma)^2(y_\sigma) = \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma))$, by (2.4) and (2.6), we have

$$
\tilde{Q}''_\sigma(y_\sigma)(3A'_{1,0}) = \Big(\sigma\tilde{Q}_\sigma(y_\sigma) - \frac{1}{1-\lambda}\tilde{Q}_\sigma^3(y_\sigma)\Big)(3A'_{1,0}) + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).
$$

Thus, we get

$$
\begin{aligned}
&\delta S_{\mathrm{mKdV}}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma))\\
&= \tilde{Q}_\sigma(y_\sigma)(-LA_{1,0})' + \tilde{Q}'_\sigma(y_\sigma)\left(3A''_{1,0} + 3A_{1,0}Q^2\right)\\
&\quad + \tilde{Q}_\sigma^2(y_\sigma)\left(a_{1,0}(-3A''_{1,0} - 3A_{1,0}Q^2)'\right) + (\tilde{Q}_\sigma^2)'(y_\sigma)\Big(-\frac{9}{2}a_{1,0}A''_{1,0}\Big)\\
&\quad + \tilde{Q}_\sigma^3(y_\sigma)(3a_{1,0}^2A'''_{1,0} - \frac{3A'_{1,0}}{1-\lambda}) + \sigma\tilde{Q}_\sigma(y_\sigma)(3A'_{1,0})\\
&\quad + \sigma^{5/2}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}
$$

We study $\delta S_{\mathrm{mKdV}}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma))$ in a similar way:

$$
\begin{aligned}
&\delta S_{\mathrm{mKdV}}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma))\\
&= \tilde{Q}'_\sigma(y_\sigma)\left\{-(LB_{1,0})' + a_{1,0}\tilde{Q}_\sigma(y_\sigma)(-3B''_{1,0} - 3B_{1,0}Q^2)'\right\}\\
&\quad + \tilde{Q}''_\sigma(y_\sigma)(3B''_{1,0} + 3B_{1,0}Q^2) + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma))\\
&= \tilde{Q}'_\sigma(y_\sigma)(-LB_{1,0})' + (\tilde{Q}_\sigma^2)'(y_\sigma)\Big(a_{1,0}\frac{(-3B''_{1,0} - 3B_{1,0}Q^2)'}{2}\Big)
\end{aligned}
$$

$$+\sigma\tilde{Q}_\sigma(y_\sigma)(3B''_{1,0}+3B_{1,0}Q^2)+\tilde{Q}^3_\sigma(y_\sigma)\Big(-\frac{3B''_{1,0}+3B_{1,0}Q^2}{1-\lambda}\Big)+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).$$

The proof is complete. □

Lemma 2.12. *There holds*

$$\begin{aligned}S_{gBBM}(w)=&\sum_{(k,l)\in\Sigma_0}\sigma^l(\tilde{Q}^k_\sigma)'(y_\sigma)(-\lambda A''_{k,l})(y)\\&+\sum_{(k,l)\in\Sigma_0}\sigma^l\left(\tilde{Q}^k_\sigma(y_\sigma)F^{III}_{k,l}(y))+(\tilde{Q}^k_\sigma)(y_\sigma)G^{III}_{k,l}(y))\right)\\&+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).\end{aligned}$$

where

$$F^{III}_{1,0}=0,\quad G^{III}_{1,0}=0,\quad F^{III}_{1,1}=-2\lambda A'_{1,0}-\lambda B''_{1,0},\quad G^{III}_{1,1}=0$$
$$F^{III}_{2,0}=\lambda a_{1,0}A'''_{1,0},\quad G^{III}_{2,0}=3\lambda a_{1,0}A''_{1,0}+\frac{\lambda a_{1,0}}{2}B'''_{1,0}.$$

For all $(k,l)\in\Sigma_0$ such that $1\le k'+l'\le 2$, $F^{III}_{k,l}$ and $G^{III}_{k,l}$ depend on $A_{k',l'}$ and $B_{k',l'}$ for $1\le k'+l'\le 2$. Moreover, if $A_{k',l'}$ is even and $B_{k',l'}$ is odd, then $F^{III}_{k,l}$ is odd and $G^{III}_{k,l}$ is even.

Proof. By definition, we see that

$$S_{gBBM}(w)=\sum_{(k,l)\in\Sigma_0}\sigma^l\left(S_{gBBM}(A_{k,l}(y)\tilde{Q}^k_\sigma(y_\sigma))+S_{gBBM}(B_{k,l}(y)(\tilde{Q}^k_\sigma)'(y_\sigma))\right).$$

It follows from Lemma 2.8 and (2.4)–(2.6) that

$$\begin{aligned}&S_{gBBM}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma))\\&=\lambda\mu_\sigma\tilde{Q}_\sigma(y_\sigma)\left\{\beta(y_\sigma)A'''_{1,0}+\beta'(y_\sigma)(2A''_{1,0})+\beta^2(y_\sigma)(-2A'''_{1,0})\right\}\\&\quad+\lambda\mu_\sigma\tilde{Q}'_\sigma(y_\sigma)\left\{-A''_{1,0}+\beta(y_\sigma)(4A''_{1,0})\right\}+\lambda\mu_\sigma\tilde{Q}''_\sigma(y_\sigma)(-2A'_{1,0})+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma))\\&=\lambda\tilde{Q}_\sigma(y_\sigma)\Big\{a_{1,0}\tilde{Q}_\sigma(y_\sigma)A'''_{1,0}+a_{1,0}\tilde{Q}'_\sigma(y_\sigma)(2A''_{1,0})+a^2_{1,0}\tilde{Q}^2_\sigma(y_\sigma)(-2A'''_{1,0})\Big\}\\&\quad+\lambda\tilde{Q}'_\sigma(y_\sigma)\Big\{-A''_{1,0}+a_{1,0}\tilde{Q}_\sigma(y_\sigma)(4A''_{1,0}\Big\}+\lambda\tilde{Q}''_\sigma(y_\sigma)(-2A'_{1,0})+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma))\\&=\tilde{Q}'_\sigma(y_\sigma)(-\lambda A''_{1,0})+\sigma\tilde{Q}_\sigma(y_\sigma)(-2\lambda A'_{1,0})+\tilde{Q}^2_\sigma(y_\sigma)(\lambda a_{1,0}A'''_{1,0})\\&\quad+(\tilde{Q}^2_\sigma)'(y_\sigma)(3\lambda a_{1,0}A''_{1,0})+\tilde{Q}^3_\sigma(y_\sigma)(-2\lambda a^2_{1,0}A'''_{1,0}+\frac{2\lambda}{1-\lambda}A'_{1,0})+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).\end{aligned}$$

Similarly, we can derive that

$$\begin{aligned}S_{gBBM}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma))&=\lambda\tilde{Q}'_\sigma(y_\sigma)a_{1,0}\tilde{Q}_\sigma(y_\sigma)B'''_{1,0}+\lambda\tilde{Q}''_\sigma(y_\sigma)(-B''_{1,0})\\&=\sigma\tilde{Q}_\sigma(y_\sigma)(-\lambda B''_{1,0})+(\tilde{Q}^2_\sigma)'(y_\sigma)\Big(\frac{\lambda a_{1,0}}{2}B'''_{1,0}\Big)\\&\quad+\tilde{Q}^3_\sigma(y_\sigma)\Big(\frac{\lambda}{1-\lambda}B''_{1,0}\Big)+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).\end{aligned}$$

In view of $2\le k+l\le 4$, we have

$$S_{gBBM}(\sigma^l\tilde{Q}^k_\sigma(y_\sigma)A_{k,l}(y))=\sigma(\tilde{Q}^k_\sigma)'(y_\sigma)(-\lambda A''_{k,l})+\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)),$$
$$S_{gBBM}(\sigma^l(\tilde{Q}^k_\sigma)'(y_\sigma)B_{k,l}(y))=\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).$$

The proof is complete □

Lemma 2.13. *There holds*

$$S_{int}(w) = \sum_{(k,l)\in\Sigma_0} \sigma^l \Big(\tilde{Q}^k_\sigma(y_\sigma) F^{\rm int}_{k,l}(y)) + (\tilde{Q}^k_\sigma)(y_\sigma) G^{\rm int}_{k,l}(y))\Big) + \sigma^{3/2} O(\tilde{Q}_\sigma(y_\sigma)),$$

where

$$F^{\rm int}_{1,0} = (3Q^2)', \quad G^{\rm int}_{1,0} = 3Q^2, \quad F^{\rm int}_{1,1} = G^{\rm int}_{1,1} = 0,$$
$$F^{\rm int}_{2,0} = 3Q' - 3a_{1,0}(Q^2)' + (9A_{1,0}Q)',$$
$$G^{\rm int}_{2,0} = 3Q + 9A_{1,0}Q + (3B_{1,0}Q)' + (3A_{1,0}B_{1,0}Q)'.$$

For all $(k,l)\in\Sigma_0$ such that $3 \le k'+l' \le 4$, $F^{\rm int}_{k,l}$ and $G^{\rm int}_{k,l}$, depend on $A_{k',l'}, B_{k',l'}$ for $1 \le k'+l' \le 2$. Moreover, if $A_{k',l'}$, are even and $B_{k',l'}$ are odd then $F^{\rm int}_{k,l}$ are odd and $G^{\rm int}_{k,l}$ are even.

Proof. By a direct calculation, we have

$$\begin{aligned}
\partial_x(w^3) &= \partial_x(A^3_{1,0}(y)\tilde{Q}^3_\sigma(y_\sigma)) + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)) \\
&= (1-\beta(y_\sigma))\Big\{(A^3_{1,0})'\tilde{Q}^3_\sigma(y_\sigma)\Big\} + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)) \\
&= \tilde{Q}^3_\sigma(y_\sigma)(A^3_{1,0})' + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)),
\end{aligned}$$

$$\begin{aligned}
&\partial_x(3Q^2\tilde{Q}_\sigma(y_\sigma) + 3Q\tilde{Q}^2_\sigma(y_\sigma)) \\
&= 3(1-\beta(y_\sigma))\Big\{\tilde{Q}_\sigma(y_\sigma)(Q^2)' + \tilde{Q}^2_\sigma(y_\sigma)Q'\Big\} + 3Q^2(\tilde{Q}_\sigma)'(y_\sigma) + 3Q(\tilde{Q}^2_\sigma)'(y_\sigma) \\
&= \tilde{Q}_\sigma(y_\sigma)(3Q^2)' + (\tilde{Q}_\sigma)'(y_\sigma)(3Q^2) + \tilde{Q}^2_\sigma(y_\sigma)(3Q' - 3a_{1,0}(Q^2)') \\
&\quad + (\tilde{Q}^2_\sigma)'(y_\sigma)(3Q) + \tilde{Q}^3_\sigma(y_\sigma)(3a_{1,0}Q'),
\end{aligned}$$

$$\partial_x(3\tilde{Q}^2_\sigma(y_\sigma)w) = \tilde{Q}^3_\sigma(y_\sigma)(3A'_{1,0}) + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)),$$

and

$$\begin{aligned}
&\partial_x(6Q\tilde{Q}_\sigma(y_\sigma)w) \\
&= \partial_x(6A_{1,0}Q\tilde{Q}^2_\sigma(y_\sigma) + 3B_{1,0}Q(\tilde{Q}^2_\sigma)'(y_\sigma)) \\
&= \tilde{Q}^2_\sigma(y_\sigma)(6A_{1,0}Q)' + (\tilde{Q}^2_\sigma)'(y_\sigma)\left(6A_{1,0}Q + (3B_{1,0}Q)'\right) - \tilde{Q}^3_\sigma(y_\sigma)(6a_{1,0}A_{1,0}Q)' \\
&\quad + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}$$

Thus, we further get

$$\begin{aligned}
\partial_x(3Qw^2) &= \partial_x\Big(3A_{1,0}Q\tilde{Q}^2_\sigma(y_\sigma) + 3A_{1,0}B_{1,0}Q(\tilde{Q}^2_\sigma)'(y_\sigma)\Big) \\
&= \tilde{Q}^2_\sigma(y_\sigma)(3A_{1,0}Q)' + (\tilde{Q}^2_\sigma)'(y_\sigma)\left((3QA_{1,0}B_{1,0})' + 3A_{1,0}Q\right) \\
&\quad + \tilde{Q}^3_\sigma(y_\sigma)(-3a_{1,0}(A_{1,0}Q)') + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma),
\end{aligned}$$

$$\partial_x(3\tilde{Q}_\sigma(y_\sigma)w^2) = \tilde{Q}^3_\sigma(y_\sigma)(3A'_{1,0}) + \sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma)).$$

By Lemmas 2.10–2.13 and Proposition 2.3, we obtain explicit expressions of $F_{k,l}$ and $G_{k,l}$ for $1 \le k+l \le 2$. immediately.

By Proposition 2.3, if the system, (2.21),

$$\begin{gathered}
(LA_{k,l})' = a_{k,l}((\lambda-3)Q'' - 3Q^3)' + F_{k,l} \\
(LB_{k,l})' = (3-\lambda)A''_{k,l} + 3Q^2A_{k,l} + a_{k,l}(2\lambda-3)Q'' + G_{k,l}
\end{gathered} \tag{2.21}$$

is solved for every $(k,l)\in\Sigma_0$, then $S(z) = \varepsilon(t,x)$ is small. $\square$

2.3. **Explicit resolution of system** (2.21)**.** In this part, we consider (2.21) with $k=1$ and $l=0$. We look for explicit solutions of the form

$$A_{k,l}=\tilde{A}_{k,l}+\gamma_{k,l},\quad B_{k,l}=\tilde{B}_{k,l}+b_{k,l}\phi, \tag{2.22}$$

where $\tilde{A}_{k,l}\in\mathcal{M}$ is even and $\tilde{B}_{k,l}\in\mathcal{M}$ is odd. Let

$$P_\lambda=\frac{3}{2}Q+\frac{3-\lambda}{2}xQ',\quad P=P_1=\frac{3}{2}Q+xQ'.$$

So we see that

$$LP=-2Q-Q^3,\quad LP_\lambda=((\lambda-3)Q''-3Q^3). \tag{2.23}$$

Lemma 2.14. *Assume that* (2.22) *and* $(a_{k,l},A_{k,l},B_{k,l})$ *satisfy* (2.21)*. Then*

$$a_{k,l}=\frac{12}{(\lambda-3)(\lambda-7)}\frac{1}{\int_{\mathbb{R}}Q^2}\Big\{-\gamma_{k,l}\int_{\mathbb{R}}P_\lambda+\int_{\mathbb{R}}G_{k,l}Q+\int_{\mathbb{R}}F_{k,l}\int_0^x P_\lambda\Big\}.$$

Proof. Multiplying the equation of $B_{k,l}$ by Q and using $LQ'=0$ gives

$$\begin{aligned}a_{k,l}(2\lambda-3)\int_{\mathbb{R}}(Q')^2&=\int_{\mathbb{R}}((3-\lambda)Q''+3Q^3)A_{k,l}+\int_{\mathbb{R}}G_{k,l}Q\\&=-\int_{\mathbb{R}}(LA_{k,l})P_\lambda+\int_{\mathbb{R}}G_{k,l}Q.\end{aligned}$$

Then, multiplying the equation of $A_{k,l}$ by $\int_0^x P_\lambda(y)dy$ yields

$$\begin{aligned}\int_{\mathbb{R}}(LA_{k,l})'\int_0^x P_\lambda&=-\int_{\mathbb{R}}(LA_{k,l})P_\lambda+\gamma_{k,l}\int_{\mathbb{R}}P_\lambda\\&=-a_{k,l}\int_{\mathbb{R}}(LP_\lambda)P_\lambda+\int_{\mathbb{R}}F_{k,l}\int_0^x P_\lambda.\end{aligned}$$

By combining the above two identities, we obtain

$$a_{k,l}\Big\{(2\lambda-3)\int_{\mathbb{R}}(Q')^2+\int_{\mathbb{R}}(LP_\lambda)P_\lambda\Big\}=-\gamma_{k,l}\int_{\mathbb{R}}P_\lambda+\int_{\mathbb{R}}G_{k,l}Q+\int_{\mathbb{R}}F_{k,l}\int_0^x P_\lambda,$$

which leads to the formula of $a_{k,l}$. According to Lemma 2.4,

$$a_{k,l}\Big\{(2\lambda-3)\int_{\mathbb{R}}(Q')^2+\int_{\mathbb{R}}(LP_\lambda)P_\lambda\Big\}=a_{k,l}\frac{(\lambda-3)(\lambda-7)}{12}\int_{\mathbb{R}}Q^2. \tag{2.24}$$

□

Lemma 2.15. *The following is a solution of* (2.21) *with* $k=1$ *and* $l=0$*:*

$$a_{1,0}=\frac{-6\lambda}{(\lambda-3)(\lambda-7)}\frac{\int_{\mathbb{R}}Q^3}{\int_{\mathbb{R}}Q^2},\quad A_{1,0}=a_{1,0}\Big(\frac{3}{2}Q+\frac{3-\lambda}{2}xQ'\Big)-Q^2.$$

Proof. Recall that from Proposition 2.3, $F_{1,0}=(3Q^2)'$ and $G_{1,0}=3Q^2$. Thus from Lemma 2.14 we obtain

$$a_{1,0}=\frac{12}{(\lambda-3)(\lambda-7)}\frac{1}{\int_{\mathbb{R}}Q^2}\Big(3\int_{\mathbb{R}}Q^3-3\int_{\mathbb{R}}Q^3P_\lambda\Big)=\frac{-6\lambda}{(\lambda-3)(\lambda-7)}\frac{\int_{\mathbb{R}}Q^3}{\int_{\mathbb{R}}Q^2}. \tag{2.25}$$

Integrating (2.21) with $(k,l)=(1,0)$ gives

$$LA_{1,0}=a_{1,0}((\lambda-3)Q''-3Q^3)+3Q^2, \tag{2.26}$$

$$(LB_{1,0})'=(3-\lambda)A''_{1,0}+3Q^2A_{1,0}+a_{1,0}(2\lambda-3)Q''+3Q^2. \tag{2.27}$$

Since $L(-Q^2)=3Q^2$ and $LP_\lambda=(\lambda-3)Q''-3Q^3$, we have

$$A_{1,0}=a_{1,0}P_\lambda-Q^2, \tag{2.28}$$

where $P_\lambda=(\frac{3}{2}Q+\frac{(3-\lambda)}{2})xQ'$. □

2.4. **Resolution of Systems** (2.21) **for** $2\le k+l\le 4$**.**

Proposition 2.16. *Let $F\in\mathcal{M}$ be odd and $G\in\mathcal{M}$ be even. Let $\gamma\in\mathbb{R}$. Then, there exists $a,b\in\mathbb{R}$, $\tilde{A}\in\mathcal{M}$ being even, and $\tilde{B}\in\mathcal{M}$ being odd such that*

$$A=\tilde{A}+\gamma,\quad B=\tilde{B}+b\phi$$

and satisfy

$$\begin{gathered}(LA)'+a((3-\lambda)Q''+3Q^3)'=F\\ (LB)'+a(3-2\lambda)Q''-(3-\lambda)A''-3Q^2A=G.\end{gathered}\tag{2.29}$$

Proof. Using $(L1)'=(1-3Q^2)'=-3(Q^2)'$, we obtain $\tilde{A}$ and $\tilde{B}$, respectively. Since $F\in\mathcal{M}$ is odd, and $H(x)=\int_{-\infty}^x F(z)dz+3\gamma Q^2$ belongs to M and is even, We have

$$\begin{gathered}L\tilde{A}+a((3-\lambda)Q''+3Q^3)=H,\\ (L\tilde{B})'+a(3-2\lambda)Q''-(3-\lambda)\tilde{A}''-3Q^2\tilde{A}=G+3\gamma Q^2-b(L\phi)'.\end{gathered}$$

Since $\int_{\mathbb{R}}HQ'=0$ and $H\in\mathcal{M}$, by Lemma 2.5, there exists $\bar{H}\in\mathcal{M}$ such that $L\bar{H}=H$. It follows that $\tilde{A}=-aP_\lambda+\bar{H}$ is even, and belongs to M. □

We need to find $\tilde{B}\in\mathcal{M}$ be odd, such that $(L\tilde{B})'=-aZ_0+D-b(L\phi)'$, where

$$\begin{gathered}D=(3-\lambda)\bar{H}''+3Q^2\bar{H}+G+3\gamma Q^2\in\mathcal{M},\\ Z_0=(3-2\lambda)Q''+(3-\lambda)P_\lambda''+3Q^2P_\lambda\in\mathcal{M}.\end{gathered}$$

Let

$$E=\int_0^x(D-aZ_0)(z)dz-bL\phi.$$

Since

$$\int_{\mathbb{R}}Z_0Q=(2\lambda-3)\int_{\mathbb{R}}(Q')^2-\int_{\mathbb{R}}(LP_\lambda)P_\lambda=\frac{(\lambda-3)(\lambda-7)}{12}\int_{\mathbb{R}}Q^2\neq 0,$$

if we choose $a=\int_{\mathbb{R}}DQ/\int_{\mathbb{R}}Z_0Q$, and $b=\int_0^{+\infty}(D-aZ_0)(z)dz$, then we have the following lemma.

Lemma 2.17. *There exist a and b such that $E\in\mathcal{M}$ and $\int_{\mathbb{R}}EQ'=0$.*

2.5. **Recomposition of the approximate solution after the collision.** Let $1<c_2<c_1$, where $0<c_2-1<\varepsilon_0$ is small and set

$$\lambda=\frac{c_1-1}{c_1},\quad \sigma=\frac{c_2-1}{c_2\lambda}.$$

Consider the function $z(t,x)$ defined by (2.10)-(2.13), where for all $(k,l)\in\Sigma_0$, $a_{k,l}$, $A_{k,l}$, $B_{k,l}$ are chosen as in Lemmas 2.15 and 2.17. Set

$$\tau_\sigma=\sigma^{-\frac12-\frac{1}{100}}=\big(\frac{c-1}{c\lambda}\big)^{-\frac12-\frac{1}{100}},\quad d(\lambda)=b_{3,0}(\lambda)+\frac{1}{6(1-\lambda)}b_{1,0}^3(\lambda). \tag{2.30}$$

Lemma 2.18. *There holds: for all* t*,,* $z(t,x) = z(-t,-x)$,

$$\|(1-\lambda\partial_x^2)\partial_t z + \partial_x(\partial_x^2 z - z + z^3)\|_{H^1(\mathbb{R})} \le C\sigma^{\frac{11}{4}} \quad \forall t, \tag{2.31}$$

$$\begin{aligned} &\|z(\tau_\sigma) - \Big\{Q(x - \frac{1}{2}\delta) + \tilde{Q}_\sigma(x + \mu_\sigma\tau_\sigma - \frac{1}{2}\delta_\sigma) \\ &\quad - d(\lambda)(\tilde{Q}_\sigma^3)'(x + \mu_\sigma\tau_\sigma - \frac{1}{2}\delta_\sigma)\Big\}\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}, \end{aligned} \tag{2.32}$$

$$\begin{aligned} &\|z(-\tau_\sigma) - \Big\{Q(x + \frac{1}{2}\delta) + \tilde{Q}_\sigma(x - \mu_\sigma\tau_\sigma + \frac{1}{2}\delta_\sigma) \\ &\quad + d(\lambda)(\tilde{Q}_\sigma^3)'(x - \mu_\sigma\tau_\sigma + \frac{1}{2}\delta_\sigma)\Big\}\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}, \end{aligned} \tag{2.33}$$

where

$$\delta = \sum_{(k,l)\in\Sigma_0} a_{k,l}\sigma^l \int_{\mathbb{R}} \tilde{Q}_\sigma^k, \quad \tilde{b}_{1,1} = b_{1,1} - (1/6)b_{1,0}^3, \quad \delta_\sigma = 2(b_{1,0} + \sigma\tilde{b}_{1,1}). \tag{2.34}$$

Proof. The symmetry property $z(t,x) = z(-t,-x)$ follows from (2.10)–(2.13), since the transformation $x \to -x$, $t \to -t$ gives $y_\sigma \to -y_\sigma$ and $y \to -y$. Note that $a_{k,l}$, $A_{k,l}$, $B_{k,l} \in \Sigma_0$ solve $\Omega_{k,l}$, and $S(z) = \varepsilon(t,x)$. It follows that

$$\|S(t)\|_{H^1(\mathbb{R})} \le C\sigma^{5/2}\|\tilde{Q}_\sigma\|_{H^1(\mathbb{R})} \le C\sigma^{\frac{11}{4}}.$$

To prove (2.32), we begin with some preliminary estimates

$$\|\alpha(s)\|_{L^\infty(\mathbb{R})} \le C, \quad \|\alpha'(s)\|_{L^\infty(\mathbb{R})} \le C\sqrt{\sigma}. \tag{2.35}$$

For $t = \tau_\sigma$, $f \in \mathcal{M}$ and the small $\sigma > 0$, we have

$$\|f(y)\tilde{Q}_\sigma(y_\sigma)\|_{H^1(\mathbb{R})} \le C\sigma^{10}, \tag{2.36}$$

$$\|Q(y) - Q(x - \frac{1}{2}\delta)\|_{H^1(\mathbb{R})} \le C\sigma^{10}. \tag{2.37}$$

To prove (2.35), by the definition of $\tilde{Q}_\sigma$ (see Lemma 2.1), we have

$$0 \le \tilde{Q}_\sigma(x) \le C\sqrt{\sigma}e^{-\sqrt{\sigma}|x|}, \quad \forall x \in \mathbb{R}. \tag{2.38}$$

Let $f \in \mathcal{M}$, so that $|f(y)| \le C|y|e^{-|y|}$ on $\mathbb{R}$. Note that $t = \tau_\sigma$, since $\mu_\sigma > 1/2$, we have

$$\sqrt{\sigma}|y_\sigma| \ge \sqrt{\sigma}(\mu_\sigma\tau_\sigma - |y| - |\alpha(y_\sigma)|) \ge \frac{1}{2}\sigma^{-\frac{1}{100}} - \sqrt{\sigma}|y| - 1.$$

Thus, by (2.38),

$$|\tilde{Q}_\sigma(y_\sigma)f(y)|^2 \le C\sigma e^{-\sigma^{-\frac{1}{100}}}|y|^{2r}e^{-2(1-\sqrt{\sigma})|y|} \le Ce^{-\sigma^{-\frac{1}{100}}}e^{-|y|}.$$

Using $\int_{\mathbb{R}} e^{-|y|}dx = \int_{\mathbb{R}} e^{-|y|}\frac{dy}{1-\alpha'(y_\sigma)} \le C$, we obtain

$$\|\tilde{Q}_\sigma(y_\sigma)f(y)\|_{L^2(\mathbb{R})} \le Ce^{-\frac{1}{2}\sigma^{-1/100}} \le C\sigma^{10}.$$

For $t = \tau_\sigma$ and $x > -\tau_\sigma/2$, we have $|\alpha(y_\sigma) - \frac{1}{2}\delta| \le K\sigma^{10}$. Indeed, $|\alpha(y_\sigma) - \frac{1}{2}\delta| \le C\int_{y_\sigma}^{+\infty}\tilde{Q}_\sigma \le Ce^{-\sqrt{\sigma}y_\sigma}$ holds for $t = \tau_\sigma$ and $x > -\frac{1}{2}\tau_\sigma$, so we have $y_\sigma \ge \frac{1}{4}\tau_\sigma$ and so $e^{-\sqrt{\sigma}y_\sigma} \le e^{-\frac{1}{4}\sigma^{-\frac{1}{100}}} \le C\sigma^{-10}$. For $t = \tau_\sigma$, we get

$$\|Q(y) - Q(x - \frac{1}{2}\delta)\|_{H^1(x > -\frac{1}{2}\tau_\sigma)} \le C\sigma^{10}.$$

To prove (2.37), it suffices to use the decay of Q. Note that if $x < -\frac{1}{2}\tau_\sigma$, since $|\alpha(y_\sigma)| \le 1$, we have $y < -\frac{1}{2}\tau_\sigma + 1$ and

$$\begin{aligned}
&\|Q(y) - Q(x - \frac{1}{2}\delta)\|_{H^1(x<-\frac{1}{2}\tau_\sigma)} \\
&\le \|Q(y)\|_{H^1(y<-\frac{1}{2}\tau_\sigma+1)} + \|Q(x - \frac{1}{2}\delta)\|_{H^1(x<-\frac{1}{2}\tau_\sigma)} \le C\sigma^{10}.
\end{aligned}$$

From the expression of $z(\tau_\sigma)$, the structure of the functions $A_{k,l}$, $B_{k,l}$, as well as $\lim_{y\to-\infty}\phi(y) = -1$, we have

$$\begin{aligned}
&\big\|z(\tau_\sigma) - \Big\{Q(y) + \tilde{Q}_\sigma - b_{1,0}\tilde{Q}_\sigma' + \gamma_{2,0}\tilde{Q}_\sigma^2 - b_{2,0}(\tilde{Q}_\sigma^2)' + \gamma_{1,1}\sigma\tilde{Q}_\sigma \\
&\quad - b_{1,1}\sigma\tilde{Q}_\sigma' - b_{3,0}(\tilde{Q}_\sigma^3)' + \gamma_{3,0}\tilde{Q}_\sigma^3 + \gamma_{2,1}\sigma\tilde{Q}_\sigma^2 + \gamma_{4,0}\tilde{Q}_\sigma^4\Big\}\big\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}.
\end{aligned} \tag{2.39}$$

Note that $\sigma^{9/4}$ corresponds to the sizes of $b_{2,1}\sigma(\tilde{Q}_\sigma^2)'$ and $b_{4,0}(\tilde{Q}_\sigma^4)'$ in $H^1(\mathbb{R})$, where $b_{2,1}$ and $b_{4,0}$ are bounded.

Let us expand $\tilde{Q}_\sigma(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1})$ and $(\tilde{Q}_\sigma^3)'(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1})$ up to the order $\sigma^{\frac{13}{4}}$ in $H^1(\mathbb{R})$ as:

$$\begin{aligned}
&\big\|\tilde{Q}_\sigma(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1}) - \Big\{\tilde{Q}_\sigma - b_{1,0}\tilde{Q}_\sigma' - \tilde{b}_{1,1}\sigma\tilde{Q}_\sigma' \\
&+ \frac{1}{2}b_{1,0}^2\tilde{Q}_\sigma'' - \frac{1}{6}b_{1,0}^3\tilde{Q}_\sigma'''\Big\}\big\|_{H^1(\mathbb{R})} \le C\sigma^{9/4},
\end{aligned} \tag{2.40}$$

$$\|(\tilde{Q}_\sigma^3)'(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1}) - (\tilde{Q}_\sigma^3)'\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}. \tag{2.41}$$

By (2.40) and (2.41), we have

$$\tilde{Q}_\sigma'' = \sigma\tilde{Q}_\sigma - \frac{1}{1-\lambda}\tilde{Q}_\sigma^3 + \frac{\lambda}{1-\lambda}\sigma\tilde{Q}_\sigma^3, \quad \tilde{Q}_\sigma''' = \sigma\tilde{Q}_\sigma' - \frac{1}{1-\lambda}(\tilde{Q}_\sigma^3)' + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma),$$

and

$$\begin{aligned}
&\big\|\Big\{\tilde{Q}_\sigma(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1}) - d(\lambda)(\tilde{Q}_\sigma^3)'(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1})\Big\} \\
&- \Big\{\tilde{Q}_\sigma - b_{1,0}\tilde{Q}_\sigma' + (1/2)b_{1,0}^2\sigma\tilde{Q}_\sigma - (\tilde{b}_{1,1} + (1/6)b_{1,0}^3)\sigma\tilde{Q}_\sigma' \\
&- (b_{1,0}^2/2(1-\lambda))\tilde{Q}_\sigma^3 - b_{3,0}(\tilde{Q}_\sigma^3)'\Big\}\big\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}.
\end{aligned} \tag{2.42}$$

Combing (2.38), (2.40) with (2.41) yields

$$\begin{aligned}
&\big\|z(\tau_\sigma) - \Big\{Q(y) + \tilde{Q}_\sigma - b_{1,0}\tilde{Q}_\sigma' + \gamma_{2,0}\tilde{Q}_\sigma^2 - b_{2,0}(\tilde{Q}_\sigma^2)' + \gamma_{1,1}\sigma\tilde{Q}_\sigma - b_{1,1}\sigma\tilde{Q}_\sigma' \\
&- b_{3,0}(\tilde{Q}_\sigma^3)' + \gamma_{3,0}\tilde{Q}_\sigma^3 + \gamma_{2,1}\sigma\tilde{Q}_\sigma^2 + \gamma_{4,0}\tilde{Q}_\sigma^4\Big\}\big\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}, \\
&\big\|z(\tau_\sigma) - \Big\{Q(y) + \tilde{Q}_\sigma(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1}) - d(\lambda)(\tilde{Q}_\sigma^3)'(y_\sigma - b_{1,0} - \sigma\tilde{b}_{1,1}) \\
&+ (\gamma_{1,1} - (1/2)b_{1,0}^2)\sigma\tilde{Q}_\sigma + (-b_{1,1} + \tilde{b}_{1,1} + (1/6)b_{1,0}^3)\sigma\tilde{Q}_\sigma' + \gamma_{2,0}\tilde{Q}_\sigma^2 - b_{2,0}(\tilde{Q}_\sigma^2)' \\
&+ \gamma_{2,1}\sigma\tilde{Q}_\sigma^2 + (\gamma_{3,0} + (b_{1,0}^2/2(1-\lambda))\tilde{Q}_\sigma^3 + \gamma_{4,0}\tilde{Q}_\sigma^4\Big\}\big\|_{H^1(\mathbb{R})} \le C\sigma^{9/4}.
\end{aligned}$$

By choosing

$$\gamma_{1,1} = \frac{1}{2}b_{1,0}^2, \quad \tilde{b}_{1,1} = b_{1,1} - \frac{1}{6}b_{1,0}^3, \quad \gamma_{2,0} = 0,$$
$$b_{2,0} = 0, \quad \gamma_{2,1} = 0, \quad \gamma_{3,0} = -\frac{b_{1,0}^2}{2(1-\lambda)}, \quad \gamma_{4,0} = 0,$$

together with (2.35), we arrive at (2.32). $\square$

2.6. **Existence of approximate 2-soliton solutions.**

Proposition 2.19. *There exists a function $z_\#$ of the form* (2.10)-(2.11) *such that for all $t \in [-\tau_\sigma, \tau_\sigma]$,*

$$\|(1-\lambda\partial_x^2)\partial_t z_\# + \partial_x(\partial_x^2 z_\# - z_\# + z_\#^3)\|_{H^1(\mathbb{R})} \le C\sigma^{7/4}, \tag{2.43}$$

$$\begin{aligned}
&\Big\|z_\#(\tau_\sigma) - \Big\{Q(x-(1/2)\delta) + \tilde{Q}_\sigma(x+\mu_\sigma\tau_\sigma - (1/2)\delta_\sigma) \\
&- 2d(\lambda)(\tilde{Q}_\sigma^3)'(x+\mu_\sigma\tau_\sigma-(1/2)\delta_\sigma)\Big\}\Big\|_{H^1} \\
&+ \|z_\#(-\tau_\sigma) - \big\{Q(x+(1/2)\delta) + \tilde{Q}_\sigma(x-\mu_\sigma\tau_\sigma+(1/2)\delta_\sigma)\big\}\|_{H^1(\mathbb{R})} \le C\sigma^{7/4},
\end{aligned} \tag{2.44}$$

where for all $\lambda \in (0,1)$ and $d(\lambda) \neq 0$,

$$\big|\delta - \frac{-6\lambda(1-\lambda)^{1/2}}{(\lambda-3)(\lambda-7)}\frac{(\int Q)^2}{\int Q^2}\big| \le C\sigma^{1/2}, \quad |\delta_\sigma - 2b_{1,0}| \le C\sigma. \tag{2.45}$$

Making the change of variable (2.2), *we define*

$$v(t,x) = \sqrt{\frac{\lambda}{1-\lambda}}z_\#(\hat{t},\hat{x}), \quad D = \frac{(1-\lambda)^{3/2}}{\lambda^{5/2}}d(\lambda),, \tag{2.46}$$

$$T = \frac{1-\lambda}{\lambda^{3/2}}\tau_\sigma = (\frac{c_2-1}{c_2})^{-\frac{1}{2}-\frac{1}{100}}(\frac{1-\lambda}{\lambda})\lambda^{\frac{1}{100}}. \tag{2.47}$$

From the above proposition, we have the following result.

Proposition 2.20. *For a positive constant C, the function v defined by* (2.46) *satisfies*

$$\begin{aligned}
&\|(1-\partial_x^2)\partial_t v + \partial_x(v+v^3)\|_{H^1(\mathbb{R})} \le C(c_2-1)^{5/2}, \quad \forall t \in [-T,T], \\
&\big\|v(T) - \big\{\phi_{c_1}(x-c_1T-\frac{1}{2}\Delta_1) + \phi_{c_2}(x-c_2T-\frac{1}{2}\Delta_2) \\
&- 2D(\phi_{c_2}^3)'(x-c_2T-\frac{1}{2}\Delta_2)\big\}\big\|_{H^1(\mathbb{R})} + \big\|v(-T) - \big\{\phi_{c_1}(x+c_1T+\frac{1}{2}\Delta_1) \\
&+ \phi_{c_2}(x+c_2T+\frac{1}{2}\Delta_2)\big\}\big\|_{H^1(\mathbb{R})} \le K(c_2-1)^{9/4},
\end{aligned}$$

where for all $c_1 > 1$, $D = D(c_1) \neq 0$, and

$$\big|\Delta_1 - \frac{-6\lambda(1-\lambda)^{1/2}}{(\lambda-3)(\lambda-7)}\frac{(\int_{\mathbb{R}} Q)^2}{\int_{\mathbb{R}} Q^2}\big| \le C(c_2-1)^{1/2}, \quad |\Delta_2 - 2b_{1,0}| \le C(c_2-1). \tag{2.48}$$

To prove Propositions 2.19 and 2.20, we see that the first estimate is a consequence of

$$(1-\partial_x^2)\partial_t v + \partial_x(v+v^3) = \frac{\lambda^2}{(1-\lambda)^{3/2}}\{(1-\lambda\partial_{\hat{x}}^2)\partial_{\hat{t}} z + \partial_{\hat{x}}(\partial_{\hat{x}}^2 z - z + z^3)\}.$$

Since $\sqrt{\frac{\lambda}{1-\lambda}}\tilde{Q}_\sigma(y_\sigma+\delta)=\phi_c(x-ct+\frac{1}{\sqrt{\lambda}}\delta)$ for $c=c_1$ or c_2, we have

$$
\begin{aligned}
&Q(\hat{x}-\frac{\delta}{2})+\tilde{Q}_\sigma(\hat{x}+\mu_\sigma\tau_\sigma-\frac{\delta_\sigma}{2})-2d(\lambda)(\tilde{Q}_\sigma^3)'(\hat{x}+\mu_\sigma\tau_\sigma-\frac{\delta_\sigma}{2})\\
&=\sqrt{\frac{1-\lambda}{\lambda}}\Big\{\phi_{c_1}(x-c_1T-\frac{\delta}{2\sqrt{\lambda}})+\phi_{c_2}(x-c_2T-\frac{\delta_\sigma}{2\sqrt{\lambda}})\\
&\quad -2\frac{(1-\lambda)^{3/2}}{\lambda^{5/2}}(\phi_{c_2}^3)'(x-c_2T-\frac{\delta_\sigma}{2\sqrt{\lambda}})\Big\},\\
&Q(\hat{x}+\frac{\delta}{2})+\tilde{Q}_\sigma(\hat{x}-\mu_\sigma\tau_\sigma+\frac{1}{2}\delta_\sigma)\\
&=\sqrt{\frac{1-\lambda}{\lambda}}\Big\{\phi_{c_1}(x+c_1T+\frac{\delta}{2\sqrt{\lambda}})+\phi_{c_2}(x+c_2T+\frac{\delta_\sigma}{2\sqrt{\lambda}})\Big\}.
\end{aligned}
$$

Using these identities and the estimates for $z_\#$, we complete the proof of Proposition 2.20. Let

$$
z_\#(t,x)=z(t,x)+w_\#(t,x),\quad w_\#(t,x)=-d(\lambda)(\tilde{Q}_\sigma^3)'(y_\sigma)(1-P(y))
$$

where P is defined in (2.23).

To prove (2.44), we replace $z=z_\#-w_\#$ in (2.32) and obtain

$$
\begin{aligned}
&\|z_\#(\tau_\sigma)-\big\{Q(x-\frac{1}{2}\delta)+\tilde{Q}_\sigma(x+\mu_\sigma\tau_\sigma-\frac{1}{2}\delta_\sigma)\\
&-d(\lambda)(\tilde{Q}_\sigma^3)'(x+\mu_\sigma\tau_\sigma-\frac{1}{2}\delta_\sigma)\big\}-w_\#(\tau_\sigma)\|_{H^1(\mathbb{R})}\le C\sigma^{9/4}.
\end{aligned}
$$

Using (2.36) ($P\in\mathcal{M}$) gives

$$
\begin{aligned}
&\big\|z_\#(\tau_\sigma)-\big\{Q(x-\frac{1}{2}\delta)+\tilde{Q}_\sigma(x+\mu_\sigma\tau_\sigma-\frac{1}{2}\delta_\sigma)\\
&-2d(\lambda)(\tilde{Q}_\sigma^3)'(x+\mu_\sigma\tau_\sigma-\frac{1}{2}\delta_\sigma)\big\}\big\|_{H^1(\mathbb{R})}\\
&\le C\sigma^{9/4}+\|d(\lambda)(\tilde{Q}_\sigma^3)'(x+\mu_\sigma\tau_\sigma-\frac{1}{2}\delta_\sigma)-w_\#(\tau_\sigma)\|_{H^1(\mathbb{R})}\\
&\le C\sigma^{9/4}+C\|(\tilde{Q}_\sigma^3)'(x-\frac{1}{2}\delta_\sigma)-(\tilde{Q}_\sigma^3)'\|_{H^1(\mathbb{R})}\le C\sigma^{7/4}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|z_\#(-\tau_\sigma)-\{Q(x+\frac{1}{2}\delta)+\tilde{Q}_\sigma(x-\mu_\sigma\tau_\sigma+\frac{1}{2}\delta_\sigma)+d(\lambda)(\tilde{Q}_\sigma^3)'(x-\mu_\sigma\tau_\sigma+\frac{1}{2}\delta_\sigma)\}\\
&-w_\#(-\tau_\sigma)\|_{H^1(\mathbb{R})}\le C\sigma^{9/4},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\|z_\#(-\tau_\sigma)-\{Q(x+\frac{1}{2}\delta)+\tilde{Q}_\sigma(x-\mu_\sigma\tau_\sigma+\frac{1}{2}\delta_\sigma)\}\|_{H^1(\mathbb{R})}\\
&\le C\sigma^{9/4}+C\|(\tilde{Q}_\sigma^3)'(x+\frac{1}{2}\delta_\sigma)-(\tilde{Q}_\sigma^3)'\|_{H^1(\mathbb{R})}\le C\sigma^{7/4}.
\end{aligned}
$$

Note that (2.45) is a consequence of (2.34).

To prove (2.43), we let

$$
\begin{aligned}
S_\#(t,x)&=(1-\lambda\partial_x^2)\partial_t z_\#+\partial_x(\partial_x^2 z_\#-z_\#+z_\#^3)\\
&=S(z(t,x))+\delta S(w_\#)+\partial_x((z+w_\#)^3-z^3-3Q^2w_\#).
\end{aligned}
$$

We claim that

$$\|\delta S(w_\#)\|_{H^1(\mathbb{R})} \le C\sigma^{7/4}. \tag{2.49}$$

It follows from Lemmas 2.11 and 2.12 that, the lower order term in $\delta S(w_\#) = \delta S_{\mathrm{mKdV}}(w_\#) + S_{gBBM}(w_\#)$ is $d(\lambda)(\tilde{Q}_\sigma^3)'(y_\sigma)\ (L(1-P))'$. This term is controlled in $H^1(\mathbb{R})$ by $\sigma^{7/4}$. Note that

$$\|\partial_x((z+w_\#)^3 - z^3 - 3Q^2 w_\#)\|_{H^1(\mathbb{R})} \le C\sigma^{\frac{25}{12}}. \tag{2.50}$$

This follows from the expressions of z and $w_\#$.

3. Stability of 2-soliton structure

3.1. **Dynamic stability in the interaction region.**

For any $c > 1$ sufficiently close to 1, we consider the function $z_\#(t)$ of the form

$$z_\#(\hat{t},\hat{x}) = Q(y) + Q(y_\sigma) + \sum_{(k,l)\in\Sigma_0} \sigma^l(\tilde{Q}_\sigma^k(y_\sigma)A_{k,l}(y) + (\tilde{Q}_\sigma^k)'(y_\sigma)B_{k,l}(y)),$$

which is used in Proposition 2.19 (recall that y, y_σ are defined in (2.10)). As in Proposition 2.20, we set

$$\begin{aligned} v(t,x) &= \sqrt{\frac{\lambda}{1-\lambda}}\, z_\#(\hat{t},\hat{x}) \\ &= \phi_{c_1}(y_1) + \phi_{c_2}(y_2) \\ &\quad + \sum_{(k,l)\in\Sigma_0} \sigma^l\big\{\phi_{c_2}^k(y_2)\tilde{A}_{k,l}(y_1) + (\phi_{c_2}^k)'(y_2)\tilde{B}_{k,l}(y_1)\big\}, \end{aligned} \tag{3.1}$$

where

$$\begin{gathered} y_2 = x - c_2 t, \quad y_1 = \frac{y}{\sqrt{\lambda}} = x - c_1 t - \tilde{\alpha}(y_2), \quad \tilde{\alpha}(y_2) = \frac{1}{\sqrt{\lambda}}\alpha(\sqrt{\lambda}y_2), \\ \tilde{A}_{k,l}(y_1) = (\frac{1-\lambda}{\lambda})^{k-\frac12} A_{k,l}(\sqrt{\lambda}y_1), \quad \tilde{B}_{k,l}(y_1) = (\frac{1-\lambda}{\lambda})^{k-\frac12}\frac{1}{\sqrt{\lambda}} B_{k,l}(\sqrt{\lambda}y_1). \end{gathered} \tag{3.2}$$

Now, we set

$$S(t) = (1-\partial_x^2)\partial_t v + \partial_x(v + v^3).$$

Proposition 3.1. *Let $\theta > 1/2$. Suppose that there exists $\varepsilon_0 > 0$ such that the following holds for $0 < c_2 - 1 < \varepsilon_0$ and all $t \in [-T,T]$,*

$$\|S(t)\|_{H^1(\mathbb{R})} \le K\frac{(c_2-1)^\theta}{T}, \tag{3.3}$$

and for some $T_0 \in [-T,T]$,

$$\|u(T_0) - v(T_0)\|_{H^1(\mathbb{R})} \le K(c_2-1)^\theta, \tag{3.4}$$

where $u(t)$ is a $H^1(\mathbb{R})$ solution of (1.1). Then there exist $K_0 = K_0(\theta, K, \lambda)$ and a function $\rho : [-T,T] \to \mathbb{R}$ such that, for all $t \in [-T,T]$,

$$\|u(t) - v(t, x-\rho(t))\|_{H^1(\mathbb{R})} \le K_0(c_2-1)^\theta, \quad |\rho'(t)| \le K_0(c_2-1)^\theta. \tag{3.5}$$

Proof. We prove the result on $[T_0,T]$. Using the transformation $x \to -x,\ t \to -t$, then the proof is the same on $[T_0,T]$. Let $K^* > K$ be a constant to be fixed. Since $\|u(T_0) - v(T_0)\|_{H^1(\mathbb{R})} \le K(c_2-1)^\theta$, by continuity in time in $H^1(\mathbb{R})$, there exists $T^* > T_0$ such that

$$T^* = \sup\Big\{T_1 \in [T_0,T]\ |\exists r \in C^1([T_0,T_1]) :$$

$$\sup_{t\in[T_0,T_1]} \|u(t)-v(t,x-r(t))\|_{H^1(\mathbb{R})} \le K^*(c_2-1)^\theta\Big\}.$$

□

Next we give some estimates related to v.

Lemma 3.2. *There holds*

$$\begin{aligned}
&\|(1-\partial_x^2)(\partial_t v+c_1\partial_x v)(t)\|_{L^\infty(\mathbb{R})} + \|\partial_t^2\partial_x^2 v(t)+c_1\partial_t\partial_x^3 v(t)\|_{L^\infty(\mathbb{R})}\\
&\le K(c_2-1)^{1/2},
\end{aligned}\tag{3.6}$$

$$\|\partial_t v(t)+c_1\partial_x v(t)+(c_1-c_2)\tilde{\alpha}'(y_2)\phi'_{c_1}(y_1)\|_{L^2(\mathbb{R})} \le K(c_2-1)^{\frac{3}{4}},\tag{3.7}$$

$$\|\partial_t v(t)+c_1\partial_x v(t)+(c_1-c_2)\tilde{\alpha}'(y_2)\phi'_{c_1}(y_1)\|_{L^\infty(\mathbb{R})} \le K(c_2-1),\tag{3.8}$$

$$\|\partial_x v-\phi'_{c_1}(y_1)\|_{L^2(\mathbb{R})} \le K(c_2-1)^{1/2},\tag{3.9}$$

$$\|\tilde{\alpha}''(y_2)\|_{L^\infty(\mathbb{R})} + \frac{1}{c_2-1}\|\tilde{\alpha}^{(4)}(y_2)\|_{L^\infty(\mathbb{R})} \le K(c_2-1).\tag{3.10}$$

Note that these estimates are consequence of the (3.1).

Lemma 3.3 (Modulation)**.** *There exists a C^1 function $\rho:[T_0,T^*]\to\mathbb{R}$ such that, for all $t\in[T_0,T^*]$, the function $\varepsilon(t,x)$ defined by $\varepsilon(t,x)=u(t,x+\rho(t))-v(t,x)$ satisfies*

$$\int_{\mathbb{R}} \varepsilon(t,x)(1-\partial_x^2)(\phi'_{c_1}(y_1))dx=0,\quad \forall t\in[T_0,T^*],$$

and for K independent of K^*,

$$\begin{aligned}
\|\varepsilon(t)\|_{H^1(\mathbb{R})} &\le 2K^*(c_2-1)^\theta,\\
\rho(T_0)| + \|\varepsilon(T_0)\|_{H^1(\mathbb{R})} &\le K(c_2-1)^\theta,\\
|\rho'(t)| &\le K\|\varepsilon(t)\|_{H^1(\mathbb{R})} + K\|S(t)\|_{H^1(\mathbb{R})}.
\end{aligned}\tag{3.11}$$

Proof. Let

$$\zeta(U,r)=\int_{\mathbb{R}} (U(x+r)-v(t,x))(1-\partial_x^2)(\phi'_{c_1}(y_1))dx.$$

Then

$$\frac{\partial\zeta}{\partial r}(U,r)=\int_{\mathbb{R}} U'(x+r)(1-\partial_x^2)(\phi'_{c_1}(y_1))dx,$$

so that from Lemma 3.2 (see(3.9)), for (c_2-1) small enough, we get

$$\begin{aligned}
\frac{\partial\zeta}{\partial r}(v(t),0) &= \int_{\mathbb{R}} (\partial_x v)(t,x)(1-\partial_x^2)(\phi'_{c_1}(y_1))dx\\
&> \int_{\mathbb{R}} [(\phi''_{c_1})^2+(\phi'_{c_1})^2]dx - K(c_2-1)^{1/2}\\
&> \frac{1}{2}\int_{\mathbb{R}} [(\phi''_{c_1})^2+(\phi'_{c_1})^2]dx.
\end{aligned}$$

Since $\zeta(v,0)=0$, for U is near $v(t)$ in $L^2(\mathbb{R})$ norm, the existence of a unique $\rho(U)$ satisfying $\zeta(U(x-\rho(U)),\rho(U))=0$ can be seen by the Implicit Function Theorem.

From the definition of T^*, it follows that there exists $\rho(t)=\rho(u(t))$, such that $\zeta(U(x-\rho(t)),\rho(t))=0$. We set

$$\varepsilon(t,x)=u(t,x+\rho(t))-v(t,x),\tag{3.12}$$

then $\int_{\mathbb{R}} \varepsilon(t)(1-\partial_x^2)(\phi'_{c_1}(y_1))dx = 0$ follows from the definition of $\rho(t)$. Estimate (3.11) follows from the Implicit Function Theorem and the definition of K^*. Moreover, since (3.4), we have $|\rho(T_0)| + \|\varepsilon(T_0)\|_{H^1(\mathbb{R})} \le K(c_2-1)^\theta$, where K is independent of K^*.

To prove

$$|\rho'(t)| \le K\|\varepsilon(t)\|_{H^1(\mathbb{R})} + K\|S(t)\|_{H^1(\mathbb{R})}, \tag{3.13}$$

by the definition of $\varepsilon(t)$, we have

$$\begin{aligned}
&(1-\partial_x^2)\partial_t\varepsilon + \partial_x(\varepsilon + (\varepsilon+v)^3 - v^3)\\
&= -[(1-\partial_x^2)\partial_t v + \partial_x(v+v^3)] + \rho'(t)(1-\partial_x^2)\partial_x(v+\varepsilon)\\
&= -S(t) + \rho'(t)(1-\partial_x^2)\partial_x(v+\varepsilon).
\end{aligned} \tag{3.14}$$

Since $\int_{\mathbb{R}} \varepsilon(t,x)(1-\partial_x^2)(\phi'_{c_1}(y_1))dx = 0$, we have

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{\mathbb{R}} \varepsilon(t)(1-\partial_x^2)(\phi'_{c_1}(y_1))dx\\
&= \int_{\mathbb{R}} [(1-\partial_x^2)\partial_t\varepsilon(t)](\phi'_{c_1}(y_1)) + \int_{\mathbb{R}} \varepsilon(t)(1-\partial_x^2)[\partial_t\phi'_{c_1}(y_1)]\\
&= -\int_{\mathbb{R}} \partial_x(\varepsilon+(\varepsilon+v)^3-v^3)\phi'_{c_1}(y_1) - \int_{\mathbb{R}} S(t)\phi'_{c_1}(y_1)\\
&\quad + \rho'(t)\int_{\mathbb{R}} (1-\partial_x^2)\partial_x(v+\varepsilon)\phi'_{c_1}(y_1)\\
&\quad + \int_{\mathbb{R}} \varepsilon(1-\partial_x^2)[-c_1\phi''_{c_1}(y_1) + c_2\tilde{\alpha}'(y_2)\phi''_{c_1}(y_1)].
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&\rho'(t)\int_{\mathbb{R}} (v+\varepsilon)[(1-\partial_x^2)\partial_x(\phi'_{c_1}(y_1))]\\
&= -\int_{\mathbb{R}} S(t)\phi'_{c_1}(y_1) + \int_{\mathbb{R}} \varepsilon((1+\varepsilon^2+3\varepsilon v+3v^2)[\partial_x\phi'_{c_1}(y_1)]\\
&\quad + (1-\partial_x^2)[-c_1\phi''_{c_1}(y_1) + c_2\tilde{\alpha}'(y_2)\phi''_{c_1}(y_1)]),
\end{aligned} \tag{3.15}$$

and so

$$\big|\rho'(t)\int_{\mathbb{R}} (v+\varepsilon)[(1-\partial_x^2)\partial_x(\phi'_{c_1}(y_1))]\big| \le C(\|\varepsilon(t)\|_{L^2(\mathbb{R})} + \|S(t)\|_{L^2(\mathbb{R})}). \tag{3.16}$$

it is not difficult to check that

$$\begin{aligned}
&\int_{\mathbb{R}} (v+\varepsilon)[(1-\partial_x^2)\partial_x(\phi'_{c_1}(y_1))]\\
&= \int_{\mathbb{R}} [(1-\partial_x^2)(\phi_{c_1}(y_1))][\partial_x(\phi'_{c_1}(y_1))] + \int_{\mathbb{R}} (v-\phi_{c_1}(y_1)+\varepsilon)[(1-\partial_x^2)\partial_x(\phi'_{c_1}(y_1))],
\end{aligned}$$

and for $c_2 - 1 < \varepsilon_0$ small enough,

$$\begin{aligned}
-\int_{\mathbb{R}} [(1-\partial_x^2)(\phi_{c_1}(y_1))][\partial_x(\phi'_{c_1}(y_1))] &\ge -\frac{3}{4}\int_{\mathbb{R}} (\phi_{c_1} - \phi''_{c_1})\phi''_{c_1}\\
&= \frac{3}{4}\int_{\mathbb{R}} (\phi'_{c_1})^2 + (\phi''_{c_1})^2 > 0,
\end{aligned}$$

so that

$$|\int_{\mathbb{R}} (v+\varepsilon)(1-\partial_x^2)\partial_x(\phi'_{c_1}(y_1))| \ge \frac{1}{2}\int_{\mathbb{R}} (\phi'_{c_1})^2 + (\phi''_{c_1})^2$$

holds for $c_2 - 1 < \varepsilon_0$ small, and (3.13) follows from (3.16). □

Lemma 3.4 (Control of the negative direction)**.** *For all $t \in [T_0, T^*]$, there holds*

$$\begin{aligned}&|\int_{\mathbb{R}} \varepsilon(t)(1-\partial_x^2)\phi_{c_1}(y_1)dx| \\ &\le K(c_2-1)^{\theta} + K(c_2-1)^{1/4}\|\varepsilon(t)\|_{L^2(\mathbb{R})} + K\|\varepsilon(t)\|^2_{H^1(\mathbb{R})}.\end{aligned} \tag{3.17}$$

Proof. Since $v(t)$ is an approximate solution of (1.1), $m(v(t))$ has a small variation. Indeed, by multiplying the equation $S(t)$ by $v(t)$ and integrating, we obtain

$$|\frac{d}{dt}m(v(t))| = |\int_{\mathbb{R}} S(t,x)v(t,x)dx| \le K\|S(t)\|_{L^2(\mathbb{R})}.$$

Thus for all $t \in [T_0, T^*]$,

$$|m(v(t)) - m(v(T_0))| \le KT \sup_{t\in[-T,T]} \|S(t)\|_{H^1(\mathbb{R})} \le K(c_2-1)^{\theta}. \tag{3.18}$$

Since $u(t)$ is a solution of (1.1), we have

$$m(u(t)) = m(v(t)+\varepsilon(t)) = m(u(T_0)) = m(v(T_0)+\varepsilon(T_0)). \tag{3.19}$$

By expanding (3.19), and using (3.18) and (3.11), we derive

$$\begin{aligned}&2|\int_{\mathbb{R}}((1-\partial_x^2)v(t))\varepsilon(t)| \\ &\le K(c_2-1)^{\theta} + 2|\int_{\mathbb{R}}((1-\partial_x^2)v(T_0))\varepsilon(T_0)| + \|\varepsilon(T_0)\|^2_{H^1(\mathbb{R})} + \|\varepsilon(t)\|^2_{H^1(\mathbb{R})} \\ &\le K(c_2-1)^{\theta} + \|\varepsilon(t)\|^2_{H^1(\mathbb{R})}.\end{aligned}$$

Using this and $\|(1-\partial_x^2)(v(t)-\phi_{c_1}(y_1))\|_{L^2(\mathbb{R})} \le K(c_2-1)^{1/4}$, so we obtain

$$\begin{aligned}&|\int_{\mathbb{R}} \varepsilon(t)((1-\partial_x^2)\phi_{c_1}(y_1))dx| \\ &\le |\int_{\mathbb{R}} \varepsilon(t)[(1-\partial_x^2)(v(t)-\phi_{c_1}(y_1))]| + |\int_{\mathbb{R}} \varepsilon(t)((1-\partial_x^2)v(t))| \\ &\le K(c_2-1)^{\theta} + K(c_2-1)^{1/4}\|\varepsilon(t)\|_{L^2(\mathbb{R})} + K\|\varepsilon(t)\|^2_{H^1(\mathbb{R})}.\end{aligned}$$

Set

$$\begin{aligned}F(t) = \frac{1}{2}\int_{\mathbb{R}}&((c_1-1)\varepsilon^2 + c_1(\partial_x\varepsilon)^2 - \frac{1}{2}((v+\varepsilon)^4 - v^4 - 4v^3\varepsilon)) \\ &+ \frac{1}{2}(c_1-c_2)\int_{\mathbb{R}} \tilde{\alpha}'(y_2)((\partial_x\varepsilon)^2+\varepsilon^2).\end{aligned}$$

□

Lemma 3.5 (Coercivity of F)**.** *There exists $k_0 > 0$ such that for $c_2 - 1$ small enough, there holds*

$$\|\varepsilon(t)\|^2_{H^1(\mathbb{R})} \le k_0F(t) + k_0\Big|\int_{\mathbb{R}} \varepsilon(t)(1-\partial_x^2)\phi_{c_1}(y_1)dx\Big|^2. \tag{3.20}$$

The proof of the above lemma can be found in [6, 14].

Lemma 3.6 (Control of the variation of the energy functional)**.** *There holds*

$$|F'(t)| \le K(c_2-1)\|\varepsilon(t)\|^2_{H^1(\mathbb{R})} + K\|\varepsilon(t)\|_{H^1(\mathbb{R})}\|S(t)\|_{H^1(\mathbb{R})}, \tag{3.21}$$

where K is independent of c_2.

Proof. First, we compute

$$\begin{aligned}
F'(t) &= \int_{\mathbb{R}} (\partial_t \varepsilon)\big((c_1-1)\varepsilon - c_1\varepsilon_{xx} - ((v+\varepsilon)^3 - v^3)\big) \\
&\quad - \int_{\mathbb{R}} (\partial_t v)(\varepsilon^3 + 3v\varepsilon^2) + \frac{1}{2}(c_1-c_2)\Big\{-c_2\int_{\mathbb{R}} \tilde{\alpha}''(y_2)(\varepsilon_x^2+\varepsilon^2) \\
&\quad + \int_{\mathbb{R}} \tilde{\alpha}'(y_2)\partial_t(\varepsilon_x^2+\varepsilon^2)\Big\} \\
&= F_1 + F_2 + F_3.
\end{aligned}$$

We claim that

$$\begin{aligned}
&\Big|F_1 + F_2 - \Big(\rho'(t)\int_{\mathbb{R}} \varepsilon[(1-\partial_x^2)(\partial_t v + c_1\partial_x v)] - \int_{\mathbb{R}} (\varepsilon^3+3v\varepsilon^2)(\partial_t v + c_1\partial_x v)\Big)\Big| \\
&\le K\|\varepsilon(t)\|_{L^2(\mathbb{R})}\|S(t)\|_{H^1(\mathbb{R})},
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
&|F_3 - (c_1-c_2)\Big\{\rho'(t)\int_{\mathbb{R}} \varepsilon[(1-\partial_x^2)(\tilde{\alpha}'(y_2)\phi_{c_1}')] - \int_{\mathbb{R}} (\varepsilon^3+3v\varepsilon^2)\tilde{\alpha}'(y_2)\phi_{c_1}'\Big\}| \\
&\le K(c_2-1)\|\varepsilon\|_{H^1(\mathbb{R})}^2 + K\|\varepsilon\|_{H^1(\mathbb{R})}\|S(t)\|_{H^1(\mathbb{R})}.
\end{aligned} \tag{3.23}$$

To prove (3.22), using the equation of $\varepsilon(t)$ (that is (3.14)), we find

$$\begin{aligned}
F_1 &= c_1\int_{\mathbb{R}} \varepsilon((1-\partial_x^2)\partial_t\varepsilon) - \int_{\mathbb{R}} (\partial_t\varepsilon)(\varepsilon + ((v+\varepsilon)^3 - v^3)) \\
&= c_1\Big\{\int_{\mathbb{R}} (-\partial_x(\varepsilon + (\varepsilon+v)^3 - v^3))\varepsilon - \int_{\mathbb{R}} S(t)\varepsilon + \rho'(t)\int_{\mathbb{R}} [(1-\partial_x^2)\partial_x(v+\varepsilon)]\varepsilon\Big\} \\
&\quad + \int_{\mathbb{R}} [(1-\partial_x^2)^{-1}\partial_x(\varepsilon + (\varepsilon+v)^3 - v^3)](\varepsilon + ((v+\varepsilon)^3 - v^3)) \\
&\quad + \int_{\mathbb{R}} [(1-\partial_x^2)^{-1}S(t)](\varepsilon + ((v+\varepsilon)^3 - v^3)) \\
&\quad - \rho'(t)\int_{\mathbb{R}} [\partial_x(v+\varepsilon)](\varepsilon + ((v+\varepsilon)^3 - v^3)),
\end{aligned}$$

$$\begin{aligned}
F_1 &= -c_1\int_{\mathbb{R}} \varepsilon^3(\partial_x v) - \frac{3}{2}c_1\int_{\mathbb{R}} \varepsilon^2(\partial_x v^2) - c_1\int_{\mathbb{R}} S(t)\varepsilon + c_1\rho'(t)\int_{\mathbb{R}} [(1-\partial_x^2)\partial_x(v)]\varepsilon \\
&\quad + \int_{\mathbb{R}} [(1-\partial_x^2)^{-1}S(t)](\varepsilon + ((v+\varepsilon)^3 - v^3)) - \rho'(t)\int_{\mathbb{R}} (\partial_x v)\varepsilon + \rho'(t)\int_{\mathbb{R}} (\partial_x\varepsilon)v^3.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&|F_1 - \Big(-c_1\int_{\mathbb{R}} \varepsilon^3(\partial_x v) - \frac{3}{2}c_1\int_{\mathbb{R}} \varepsilon^2\partial_x(v^2)\Big) + \rho'(t)\int_{\mathbb{R}} \varepsilon\partial_x(c_1(1-\partial_x^2)v - v - v^3)| \\
&\le K\|\varepsilon(t)\|_{L^2(\mathbb{R})}\|S(t)\|_{H^1(\mathbb{R})}.
\end{aligned}$$

Using $S = (1-\partial_x^2)\partial_t v + \partial_x(v+v^3)$, we find that

$$\begin{aligned}
&\Big|F_1 - \Big(-c_1\int_{\mathbb{R}} \varepsilon^3(\partial_x v) - \frac{3}{2}c_1\int_{\mathbb{R}} \varepsilon^2\partial_x(v^2)\Big) + \rho'(t)\int_{\mathbb{R}} \varepsilon[(1-\partial_x^2)(\partial_t v + c_1\partial_x v)]\Big| \\
&\le K\|\varepsilon(t)\|_{L^2(\mathbb{R})}\|S(t)\|_{H^1(\mathbb{R})}.
\end{aligned}$$

So (3.22) follows from the definition of F_2.

To prove (3.23), from (3.10) we have

$$\Big|\int_{\mathbb{R}} \tilde{\alpha}''(y_2)(\varepsilon_x^2+\varepsilon^2)\Big| \le \|\tilde{\alpha}''\|_{L^\infty}\|\varepsilon\|_{H^1(\mathbb{R})}^2 \le K(c_2-1)\|\varepsilon\|_{H^1(\mathbb{R})}^2,$$

$$\frac{1}{2}\int_{\mathbb{R}} \tilde{\alpha}'(y_2)\partial_t(\varepsilon_x^2+\varepsilon^2) = \int_{\mathbb{R}} \tilde{\alpha}'(y_2)(\partial_t(\varepsilon-\partial_x^2\varepsilon))\varepsilon - \int_{\mathbb{R}} \tilde{\alpha}''(y_2)(\partial_x\varepsilon)(\partial_t\varepsilon).$$

Using Lemma 3.2, we have

$$\Big|\int_{\mathbb{R}} \tilde{\alpha}''(y_2)(\partial_x\varepsilon)(\partial_t\varepsilon)\Big| \le C(c_2-1)\|\varepsilon\|_{H^1(\mathbb{R})}(\|\varepsilon\|_{H^1(\mathbb{R})}+\|S\|_{H^1(\mathbb{R})}).$$

Using the equation of ε gives

$$\begin{aligned}
&\int_{\mathbb{R}} \tilde{\alpha}'(y_2)(\partial_t(\varepsilon-\partial_x^2\varepsilon))\varepsilon \\
&= \int_{\mathbb{R}} \tilde{\alpha}'(y_2)(-\varepsilon_x\varepsilon-(\varepsilon^3)_x\varepsilon-3(v^2\varepsilon)_x\varepsilon-3(v\varepsilon^2)_x\varepsilon \\
&\quad - S(t)\varepsilon+\rho'(t)\varepsilon((1-\partial_x^2)\partial_x(v+\varepsilon))) \\
&= \int_{\mathbb{R}} \tilde{\alpha}''(y_2)(\frac{1}{2}\varepsilon^2+\frac{3}{4}\varepsilon^4+\frac{3}{2}v^2\varepsilon^2+2v\varepsilon^3-\rho'(t)\frac{1}{2}(\varepsilon^2+3\varepsilon_x^2))+\rho'(t)\int_{\mathbb{R}} \tilde{\alpha}^{(4)}(y_2)\frac{1}{2}\varepsilon^2 \\
&\quad + \int_{\mathbb{R}} \tilde{\alpha}'(y_2)(-\varepsilon^3 v_x-\frac{3}{2}\varepsilon^2(\partial_x v^2)+\rho'(t)\varepsilon[(1-\partial_x^2)v_x]-S(t)\varepsilon).
\end{aligned}$$

The coefficients of $\tilde{\alpha}''(y_2)$ and $\tilde{\alpha}^{(4)}(y_2)$ are controlled, so we get (3.23).

By Lemma 3.4 and (3.11), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} \varepsilon(T^*)(1-\partial_x^2)\phi_{c_1}(y_1)dx\Big| \\
&\le K(c_2-1)^\theta + K(c_2-1)^{1/4}\|\varepsilon(T^*)\|_{L^2(\mathbb{R})} + K\|\varepsilon(T^*)\|_{H^1(\mathbb{R})}^2 \\
&\le (K+1)(c_2-1)^\theta,
\end{aligned}$$

for $0<c_2-1<\varepsilon_0$ small enough. Thus, by Lemma 3.5, we obtain

$$\|\varepsilon(T^*)\|_{H^1(\mathbb{R})}^2 \le k_0F(T^*)+K(c_2-1)^{2\theta}.$$

Integrating (3.21) on $[T_0,T^*]$, by (3.11) and (3.3), there exists $K_1>0$ independent of K^* such that

$$\begin{aligned}
|F(T^*)| &\le |F(T_0)| + K(c_2-1)^{\frac{3}{4}}T\sup_{t\in[T_0,T^*]}\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \\
&\quad + KT\sup_{t\in[T_0,T^*]}(\|\varepsilon(t)\|_{H^1(\mathbb{R})}\|S(t)\|_{H^1(\mathbb{R})}) \\
&\le K_1(c_2-1)^{2\theta}+K(K^*)^2(c_2-1)^{2\theta+\frac{3}{4}}+K_1K^*(c_2-1)^{2\theta}.
\end{aligned}$$

Thus, for $0<c_2-1<\varepsilon_0$ small enough, we obtain

$$\|\varepsilon(T^*)\|_{H^1(\mathbb{R})}^2 \le C(c_2-1)^{2\theta}(2+K^*).$$

By fixing K^* such that $C(2+K^*)<\frac{1}{2}(K^*)^2$, we see $\|\varepsilon(T^*)\|_{H^1(\mathbb{R})}^2 \le \frac{1}{2}(K^*)^2(c_2-1)^{2\theta}$. This contradicts the definition of T^*, thus we have $T^*=T$, and arrive at (3.5). $\square$

3.2. **Stability and asymptotic stability for large time.** In this section, we consider the stability of the 2-soliton structure after the collision. For $v \in H^1(\mathbb{R})$, we denote $\|v\|_{H^1_{c_2}(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} ((v'(x))^2 + (c_2-1)v^2(x))\,dx}$, which corresponds to the natural norm to study the stability of ϕ_{c_2}.

Proposition 3.7 (Stability of the two decoupled solitons)**.** *Let $c_1 > 1$. Let $u(t)$ be an $H^1(\mathbb{R})$ solution of (1.1) such that for some $w > 0$, $X_0 \geq \frac{1}{2}(c_1 - c_2)T$, there holds*

$$\|u(0) - \phi_{c_1} - \phi_{c_2}(x + X_0)\|_{H^1(\mathbb{R})} \leq (c_2-1)^{\frac{3}{4}+w}. \tag{3.24}$$

There exists $\varepsilon_0 > 0$ such that $1 < c_2 - 1 < 1 + \varepsilon_0$. Then there exist the C^1 functions $\rho_1(t), \rho_2(t)$ defined on $[0, +\infty)$ and $K > 0$ such that

(i) (stability)

$$\sup_{t\geq 0} \|u(t) - (\phi_{c_1}(x-\rho_1(t)) + \phi_{c_2}(x-\rho_2(t)))\|_{H^1_{c_2}(\mathbb{R})} \leq K(c_2-1)^{\frac{3}{4}+w}, \tag{3.25}$$

$$\frac{c_1}{2} \leq \rho_1'(t) - \rho_2'(t) \leq \frac{3c_1}{2} \quad \forall t \geq 0,$$

$$|\rho_1(t_0)| \leq K(c_2-1)^{\frac{3}{4}+w}, \quad |\rho_2(t_0) + X_0| \leq K(c_2-1)^{w}. \tag{3.26}$$

(ii) (asymptotic stability) There exist c_1^+, $c_2^+ > 1$ such that

$$\lim_{t\to+\infty} \|u(t) - (\phi_{c_1^+}(x-\rho_1(t)) + \phi_{c_2^+}(x-\rho_2(t)))\|_{H^1(x > \frac{1}{2}(1+c_2)t)} = 0, \tag{3.27}$$

$$|c_1^+ - c_1| \leq K(c_2-1)^{\frac{3}{4}+w}, \quad |c_2^+ - c_2| \leq K(c_2-1)^{1+w+\min(\frac{1}{2},w)}. \tag{3.28}$$

The proof of the above proposition is similar to the one of [6, Theorem 1.1], so we omit it.

4. Proof of Theorem 1.1

Proposition 4.1. *Let $c_1 > 1$ and $1 < c_2 < 1 + \varepsilon_0$, for $\varepsilon_0 > 0$ small enough.*

1. Existence and exponential decay: Let $x_1, x_2 \in \mathbb{R}$. There exists a unique $H^1(\mathbb{R})$ solution $u(t) = u_{c_1,c_2,x_1,x_2}(t)$ of (1.1) such that

$$\lim_{t\to-\infty} \|u(t) - \phi_{c_1}(x - c_1t - x_1) - \phi_{c_2}(x - c_2t - x_2)\|_{H^1(\mathbb{R})} = 0. \tag{4.1}$$

Moreover, for all $t \leq -\frac{T}{32}$,

$$\|u(t) - \phi_{c_1}(x - c_1t - x_1) - \phi_{c_2}(x - c_2t - x_2)\|_{H^1(\mathbb{R})} \leq Ke^{\frac{1}{4}\sqrt{c_2-1}(c_1-1)t}. \tag{4.2}$$

2. Uniqueness of the asymptotic 2-soliton solution at $-\infty$: if $w(t)$ is an $H^1(\mathbb{R})$ solution of (1.1) satisfying

$$\lim_{t\to-\infty} \|w(t) - \phi_{c_1}(x-\rho_1(t)) - \phi_{c_2}(x-\rho_2(t))\|_{H^1(\mathbb{R})} = 0, \tag{4.3}$$

for $\rho_1(t)$ and $\rho_2(t)$, then there exist $x_1, x_2 \in \mathbb{R}$ such that $w(t) = u_{c_1,c_2,x_1,x_2}(t)$.

The above proposition is essentially the same as [6, Theorem 1.3]. Recall that such a result was first proved for the generalized KdV equations in [19], and refined techniques were introduced in [13, 21].

Lemma 4.2. *Let $c_1 > 1$. Let $1 < c_2 < 1 + \varepsilon_0$ and $\varepsilon_0 = \varepsilon_0(c_1) > 0$. We suppose that $u(t)$ is a solution of (1.1) satisfying: for some $\rho_1(t)$, $\rho_2(t)$,*

$$\lim_{t\to-\infty} \|u(t) - \phi_{c_1}(x-\rho_1(t)) - \phi_{c_2}(x-\rho_2(t))\|_{H^1(\mathbb{R})} = 0, \tag{4.4}$$

$$\lim_{t\to+\infty} \|u(t) - \phi_{c_1}(x-\rho_1(t)) - \phi_{c_2}(x-\rho_2(t)) - w^+(t)\|_{H^1(\mathbb{R})} = 0, \tag{4.5}$$

where $|c_j^+ - c_j| \le \varepsilon_0 |c_j - 1|$ *and*

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(x>\frac{1}{2}(1+c_2)t)} = 0, \quad \limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} \le \varepsilon_0 |c_2-1|^{1/2}. \tag{4.6}$$

Then, there exist $C = C(c_1)$ *such that*

$$\begin{gathered} \frac{1}{C}\limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1_{c_2}(\mathbb{R})} \le c_1^+ - c_1 \le C \liminf_{t\to+\infty} \|w^+(t)\|^2_{H^1_{c_2}(\mathbb{R})}, \\ \begin{aligned} \frac{1}{C}(c_2-1)^{-1/2} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} &\le c_2 - c_2^+ \\ &\le C(c_2-1)^{-1/2} \liminf_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})}. \end{aligned} \end{gathered} \tag{4.7}$$

Proof. By (1.3), (1.4), (4.4), (4.5) and (4.6), we have that for the large t,

$$m(u(0)) = m(\phi_{c_1}) + m(\phi_{c_2}) = m(\phi_{c_1^+}) + m(\phi_{c_2^+}) + m(w^+(t)) + o(1), \tag{4.8}$$

$$E(u(0)) = E(\phi_{c_1}) + E(\phi_{c_2}) = E(\phi_{c_1^+}) + E(\phi_{c_2^+}) + E(w^+(t)) + o(1). \tag{4.9}$$

Let $(j=1,2)$,

$$\bar a_j = \frac{E(\phi_{c_j^+}) - E(\phi_{c_j})}{m(\phi_{c_j^+}) - m(\phi_{c_j})},$$

so that

$$|\bar a_j - c_j| \le C|c_j^+ - c_j|. \tag{4.10}$$

Indeed, by (2.19), one has

$$\frac{E(\phi_{c_j^+}) - E(\phi_{c_j})}{m(\phi_{c_j^+}) - m(\phi_{c_j})} = \frac{\frac{d}{dc}E(\phi_c)}{\frac{d}{dc}N(\phi_c)}|_{c=c_j} + O(|c_j^+ - c_j|) = c_j + O(|c_j^+ - c_j|).$$

Considering $\bar a_2 E$ times (4.8) - (4.9) and then $\bar a_1$ times (4.8) - (4.9), we find that for the large t

$$[E(\phi_{c_1^+}) - \bar a_2 m(\phi_{c_1^+})] - [E(\phi_{c_1}) - \bar a_2 m(\phi_{c_1})] = \bar a_2 m(w^+(t)) - E(w^+(t)) + o(1), \tag{4.11}$$

$$[\bar a_1 m(\phi_{c_2}) - E(\phi_{c_2})] - [\bar a_1 m(\phi_{c_2^+}) - E(\phi_{c_2^+})] = \bar a_1 m(w^+(t)) - E(w^+(t)) + o(1). \tag{4.12}$$

Note that

$$\int_{\mathbb{R}} |w^+|^4 \le C\|w^+\|^2_{H^1(\mathbb{R})} \int_{\mathbb{R}} (w^+)^2 \le C\varepsilon_0 |c_2-1| \int_{\mathbb{R}} (w^+)^2$$

so that

$$\bar a_2 m(w^+(t)) - E(w^+(t)) > \frac{1}{4}((c_2-1)\int_{\mathbb{R}} (w^+)^2 + \int_{\mathbb{R}} (w_x^+)^2).$$

Now, let $\beta_1 = \frac{d}{dc} m(\phi_c)|_{c=c_1} > 0$, by (2.19), we have

$$(\frac{d}{dc}E(\phi_c) - \bar a_2 \frac{d}{dc} m(\phi_c))|_{c=c_1} = (c_1 - \bar a_2)\frac{d}{dc} m(\phi_c)|_{c=c_1},$$

and so

$$\frac{1}{2}(c_1-1)\beta_1 < (\frac{d}{dc}E(\phi_c) - \bar a_2 \frac{d}{dc} m(\phi_c))|_{c=c_1} < (c_1-1)\beta_1.$$

Thus, from (4.11), we obtain that for the large t,

$$c_1^+ - c_1 \geq C[(c_2-1)\int_{\mathbb{R}} (w^+(t))^2 + \int_{\mathbb{R}} (w_x^+(t))^2] + o(1) \geq C\|w^+(t)\|^2_{H^1_{c_2}(\mathbb{R})} + o(1),$$
$$c_1^+ - c_1 \leq C'\|w^+(t)\|^2_{H^1_{c_2}(\mathbb{R})} + o(1).$$

Similarly, we have

$$\begin{aligned} c_2 - c_2^+ &\geq \frac{C}{(c_2-1)^{1/2}}[(c_1-1)\int_{\mathbb{R}} (w^+(t))^2 + \int_{\mathbb{R}} (w_x^+(t))^2] + o(1) \\ &\geq \frac{C}{(c_2-1)^{1/2}}\|w^+(t)\|^2_{H^1(\mathbb{R})} + o(1), \\ c_2 - c_2^+ &\leq \frac{C'}{(c_2-1)^{1/2}}\|w^+(t)\|^2_{H^1(\mathbb{R})} + o(1). \end{aligned}$$

Estimate (4.7) follows. □

Proof of Theorem 1.1. Let $c_1 > 1$ and $\varepsilon_0 = \varepsilon_0(c_1)$ be small enough. Let $1 < c_2 < 1 + \varepsilon_0$ and T be defined by (2.47). Let $\tilde{u}(t)$ be the unique solution of (1.1) such that

$$\lim_{t\to-\infty} \|\tilde{u}(t) - \phi_{c_1}(x - c_1 t) - \phi_{c_2}(x - c_2 t)\|_{H^1(\mathbb{R})} = 0.$$

(1) By Proposition 4.1, for all $t \leq -\frac{T}{32}$,

$$\|\tilde{u}(t) - \phi_{c_1}(x - c_1 t) - \phi_{c_2}(x - c_2 t)\|_{H^1(\mathbb{R})} \leq K e^{\frac{1}{4}\sqrt{c_2-1}(c_1-1)t}. \tag{4.13}$$

Let Δ_1, Δ_2 be defined in Proposition 2.20, and let

$$T^- = T + \frac{1}{2}\frac{\Delta_1 - \Delta_2}{c_1 - c_2}.$$

Since $|\Delta_1| + |\Delta_2| \leq C = C(c_1)$, and $c_1 - c_2 > c_1 - 1 - \varepsilon_0 \geq \frac{1}{2}(c_1 - 1)$, we have $-T^- < -\frac{1}{32}T$, for small $c_2 - 1$ and so

$$\begin{aligned} \|\tilde{u}(-T^-) - \phi_{c_1}(x + c_1 T^-) - \phi_{c_2}(x + c_2 T^-)\|_{H^1(\mathbb{R})} &\leq K e^{-\frac{1}{4}\sqrt{c_2-1}(c_1-1)T^-} \\ &\leq (c_2-1)^{10}, \end{aligned} \tag{4.14}$$

for ε_0 small enough. Let

$$u(t,x) = \tilde{u}(t + T - T^-,\ x + \frac{1}{2}\Delta_1 + c_1(T - T^-)). \tag{4.15}$$

Then, $u(t)$ is a solution of (1.1) and satisfies

$$\|u(-T) - \phi_{c_1}(x + c_1 T + \frac{1}{2}\Delta_1) - \phi_{c_2}(x + c_2 T + \frac{1}{2}\Delta_2)\|_{H^1(\mathbb{R})} \leq (c_2-1)^{10}. \tag{4.16}$$

It is easily checked that the results obtained for $u(t)$ imply the desired results on $\tilde{u}(t)$.

(2) By Proposition 2.20 and (4.16), we have

$$\|u(-T) - v(-T)\|_{H^1(\mathbb{R})} \leq K(c_2-1)^{9/4}.$$

By Proposition 2.19 and the above estimate, we can apply Proposition 3.1 with $\theta = \frac{5}{2} - \frac{1}{2} - \frac{1}{100} = 2 - \frac{1}{100}$. There exists $\rho(t)$ such that for all $t \in [-T, T]$,

$$\|u(t) - v(t, x - \rho(t))\|_{H^1(\mathbb{R})} + |\rho'(t)| \leq C(c_2-1)^{2-\frac{1}{100}}.$$

In particular, for $r = \rho(T),\ |r| \le C(c_2-1)^{2-\frac{1}{50}}$, we have

$$\|u(T) - v(t, x-r)\|_{H^1(\mathbb{R})} \le C(c_2-1)^{2-\frac{1}{100}},$$

and using Proposition 2.20, we obtain

$$\begin{aligned}&\|u(T) - \{\phi_{c_1}(x-r_1) + \phi_{c_2}(x-r_2) - 2D(\phi_{c_2}^3)'(x-r_2)\}\|_{H^1(\mathbb{R})}\\ &\le C(c_2-1)^{2-\frac{1}{100}},\end{aligned} \tag{4.17}$$

where $r_1 = c_1T + \frac{1}{2}\Delta_1 + r$ and $r_2 = c_2T + \frac{1}{2}\Delta_2 + r$, so that

$$\frac{1}{2}(c_1-c_2)T \le r_1 - r_2 \le \frac{3}{2}(c_1-c_2)T.$$

Moreover, since $\|(\phi_{c_2}^3)'\|_{H^1(\mathbb{R})} \le C(c_2-1)^{7/4}$, we also obtain

$$\|u(T) - \{\phi_{c_1}(x-r_1) + \phi_{c_2}(x-r_2)\}\|_{H^1(\mathbb{R})} \le C(c_2-1)^{7/4}. \tag{4.18}$$

In what follows, (4.18) will serve us to prove that $u(t)$ is close to the sum of two solitons for $t > T$, whereas (4.17) will allow us to prove that $u(t)$ is not a pure 2-soliton solution at $+\infty$.

(3) By using Proposition 3.7 with $w = 1$, it follows from (3.25), (3.27) and (3.28) that there exists $\rho_1(t), \rho_2(t), c_1^+, c_2^+$ such that

$$\begin{gathered} c_1^+ = \lim_{t\to+\infty} \bar{c}_1(t), \quad c_2^+ = \lim_{t\to+\infty} \bar{c}_2(t), \quad |c_1^+ - c_1| \le C(c_2-1)^{7/4},\\ |c_2^+ - c_2| \le C(c_2-1)^{5/2}, \quad w^+(t,x) = u(t,x) - \{\phi_{c_1^+}(x-\rho_1(t)) + \phi_{c_2^+}(x-\rho_2(t))\},\\ \sup_{t\ge T} \|w^+(t)\|_{H^1_{c_2}(\mathbb{R})} \le C(c_2-1)^{7/4}, \quad \lim_{t\to+\infty} \|w^+(t)\|_{H^1(x>\frac{1}{2}(1+c_2)t)} = 0. \end{gathered} \tag{4.19}$$

From Lemma 4.2, we obtain

$$0 \le c_1^+ - c_1 \le C(c_2-1)^{\frac{7}{2}}, \quad 0 \le c_2^+ - c_2 \le C(c_2-1)^4.$$

(4) There exists $K_0 > 0$ such that

$$\liminf_{t\to+\infty} \|w^+(t)\|_{H^1_{c_2}(\mathbb{R})} \ge K_0(c_2-1)^{9/4}. \tag{4.20}$$

This follows the proof of the BBM case in [20] immediately, so we omit it.

Based on Cases (1) through (4), The proof of Theorem 1.1 is complete □

Acknowledgements. This research is supported by the National Natural Science Foundation of China (No. 11171135,51276081), and by the Natural Science Foundation of Jiangsu (No. BK20140525), the Major Project of Natural Science Foundation of Jiangsu Province Colleges and Universities (14KJA110001), and the Advantages of Jiangsu Province and the Innovation Project for Graduate Student Research of Jiangsu Province KYLX_1052.

References

[1] K. O. Abdulloev, I. L. Bogolubsky, V. G. Makhankov; *One more example of inelastic soliton interaction*. Phys. Lett. A, 56 (1976), 427–428.

[2] T. B. Benjamin, J. L. Bona, J. J. Mahony; *Model equations for long waves in nonlinear dispersion systems*, Philos. Trans. Roy. Soc. London Ser. A, 272 (1972), 47–78.

[3] J. L. Bona, W. G. Pritchard, L. R. Scott; *Solitary-wave interaction*, Phys. Fluids, 23, 438, (1980).

[4] W. Craig, P. Guyenne, J. Hammack, D. Henderson, C. Sulem; *Solitary water wave interactions*. Phys. Fluids, 18 (2006), 057106.

[5] J. C. Eilbeck, G. R. McGuire; *Numerical study of the regularized long-wave equation II.* Interaction of solitary waves, J. Computational Phys., 23 (1977), 63–73.
[6] K. El Dika, Y. Martel; *Stability of solitary waves for the generalized BBM equations*, Dynamics Par. Diff. Equ., 4 (2004), 401–437.
[7] K. El Dika; *Asymptotic stability of solitary waves for the Benjamin-Bona-Mahony equation*, Discrete Contin. Dyn. Syst., 13 (2005), 583–622.
[8] E. Fermi, J. Pasta, S. Ulam; *Studies of nonlinear problems I*, Los Alamos Report LA1940 (1955); reproduced in Nonlinear Wave Motion, A.C. Newell, ed., American Mathematical Society, Providence, RI, (1974), 143–156.
[9] R. Hirota; *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett., 27 (1971), 1192–1194.
[10] H. Kalisch, J. L. Bona; *Models for internal waves in deep water*, Discrete Contin. Dyn. Syst., 6 (2000), 1–20.
[11] R. LeVeque; *On the interaction of nearly equal solitons in the KdV equations*, SIAM J. Appl. Math., 47 (1987), 254–262.
[12] J. H. Maddocks, R. L Sachs; *On the stability of KdV multi-solitons*, Commun. Pure. Appl. Math., 46 (1993), 867–901.
[13] Y. Martel, F. Merle; *Asymptotic stability of the solitons for the generalized KdV equations*, Arch. Ration. Mech. Anal., 38 (2001), 219–254.
[14] Y. Martel, F. Merle; *Description of two soliton collision for the quartic gKdV equation.* Ann. Math., 174 (2011), 757–857.
[15] Y. Martel, F. Merle; *Inelastic interaction of nearly equal solitons for the BBM equation*, Discrete Contin. Dyn. Syst., 27 (2010), 487–532.
[16] Y. Martel, F. Merle; *Inelastic interaction of nearly equal solitons for the quartic gKdV equation*, Invent. Math., 183 (2011), 563–648.
[17] Y. Martel, F. Merle; *Refined asymptotics around solitons for the gKdV equations with a general nonlinearity*, Discrete Contin. Dyn. Syst., 20 (2008), 177–218.
[18] Y. Martel, F. Merle; *Stability of two soliton collision for nonintegrable gKdV equations*, Commun. Math. Phys., 286 (2009), 39–79.
[19] Y. Martel; *Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations*, Amer. J. Math., 127 (2005), 1103–1140.
[20] Y. Martel, F. Merle, T. Mizumachi; *Description of the inelastic collision of two solitary waves for the BBM equation*, Arch. Ration. Mech. Anal., 196 (2010), 517–574.
[21] Y. Martel, F. Merle, Tai-Peng Tsai; *Stability and asymptotic stability in the energy space of the sum of N solitons for the subcritical gKdV equations*, Commun. Math. Phys., 231 (2002), 347–373.
[22] Y. Martel; *Linear problems related to asymptotic stability of solitons of the generalized KdV equations*, SIAM J. Math. Anal., 38 (2006), 759–781.
[23] J. B. Mcleod, P. J. Olver; *The connection between completely integrable partial differential equations and ordinary differential equation of Painleve type*, SIAM J. Math. Anal., 14 (1983), 56–75.
[24] R. M. Miura; *The Korteweg-de Vries equation: a survey of results*, SIAM Review, 18, (1976), 412–459.
[25] T. Mizumachi; *Asymptotic stability of the solitary wave solutions to the regularized-long-wave equation*, J. Diff. Equ., 200 (2004), 312–341.
[26] T. Mizumachi; *Weak interaction between solitary waves of the generalized KdV equations*, SIAM J. Math. Anal., 35 (2003), 1042–1080.
[27] P. J. Morrison, J. D. Meiss, J. R. Cary; *Scattering of regularized-long-wave solitary waves*, Phys. D, 11 (1984), 324–336.
[28] C. Muñoz; *On the inelastic two-soliton collision for gKdV equations with general nonlinearity*, Inter. Math. Res. Not., 9 (2010), 1624–1719.
[29] P. J. Olver; *Euler operators and conservation laws of the BBM equation*, Math. Proc. Camb. Phil. Soc., 85 (1979), 143–160.
[30] D. H. Peregrine; *Long waves on a beach*, J. Fluid Mech., 27 (1967), 815–827.
[31] L. Y. Shih; *Soliton-like interaction governed by the generalized Korteweg-de Vries equation*, Wave motion, 2 (1980), 197–206.
[32] M. Wadati, M. Toda; *The exact N-soliton solution of the Korteweg-de Vries equation*, J. Phy. Soc. Japan, 32 (1972), 1403–1411.

[33] P. D. Weidman, T. Maxworthy; *Experiments on strong interactions between solitary waves*, J. Fluids Mech., 85 (1978), 417–431.
[34] N. J. Zabusky, M. D. Kruskal; *Interaction of solitons in a collisionless plasma and recurrence of initial states*, Phys. Rev. Lett., 15 (1965), 240–243.
[35] N. J. Zabusky; *Solitons and energy transport in nonlinear lattices*, Computer Phys. Commun., 5 (1973), 1–10.

Jingdong Wei
Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu 212013, China
E-mail address: wjd19871022@126.com

Lixin Tian (corresponding author)
Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu 212013, China.
School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210023, China
E-mail address: tianlx@ujs.edu.cn, tianlixin@njnu.edu.cn

Zaili Zhen
Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu 212013, China
E-mail address: lddcb@126.com

Weiwei Gao
Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu 212013, China
E-mail address: yeahgaoweiwei@163.com