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ASYMPTOTIC BEHAVIOR FOR SMALL MASS IN AN ATTRACTION-REPULSION CHEMOTAXIS SYSTEM

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Abstract. This article is concerned with the model

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), \quad x \in \Omega,\ t > 0, \\ 0 = \Delta v + \alpha u - \beta v, \quad x \in \Omega,\ t > 0, \\ 0 = \Delta w + \gamma u - \delta w, \quad x \in \Omega,\ t > 0 \end{gathered}$$

with homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$. Under the critical condition $\chi\alpha - \xi\gamma = 0$, we show that the system possesses a unique global solution that is uniformly bounded in time. Moreover, when $n = 2$, by some appropriate smallness conditions on the initial data, we assert that this solution converges to $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\delta}\bar{u}_0)$ exponentially, where $\bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0$.

1. Introduction

Chemotaxis is a phenomenon of the directed movement of cells in response to the concentration gradient of the chemical which is produced by cells. A well-known chemotaxis model was proposed by Keller and Segel [15] in the 1970s, which describes the aggregation of cellular slime molds Dictyostelium discoideum. A simple classical Keller-Segel model reads as follows

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (\chi u \nabla v), \quad x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v + \alpha u - \beta v, \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \tau v(x,0) = \tau v_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote the density of the cells and the concentration of the chemical, respectively. Here $\alpha > 0$, $\beta > 0$, $\tau = 0, 1$ are constants, and $\chi > 0$ (resp. $\chi < 0$) is a constant referred to as the attractive (resp. repulsive) chemotaxis.

Mathematical study of (1.1) has been extensively developed in the past four decades, see [8-10] and the references therein. In the case $\chi > 0$, the outcome in [26] states that a globally bounded solution of (1.1) with $\tau = 1$ exists when $n = 1$. When $n = 2$, it is shown that there exists a critical constant C such that if

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$\int_\Omega u_0 < C$, then the solutions of (1.1) are bounded [6, 25] and if $\int_\Omega u_0 > C$, then blow-up happens [10, 24, 28]. When $n \geq 3$, it is insufficient to rule out blow up in (1.1) even if $\int_\Omega u_0$ is sufficiently small [3, 31, 32]. On the other hand, the results of repulsive chemotaxis (i.e., $\chi < 0$) were much less. For $\tau = 0$, it is well known that the solutions of (1.1) are uniformly bounded and converge to some stationary solutions exponentially as time tends to infinity [22, 23]. In [4] the system (1.1) with $\tau = 1$ has been studied based on a Lyapunov function. It is asserted that (1.1) possesses a unique classical bounded solution in two dimensions and a global weak solution exists if $n = 3, 4$.

Taking into account attraction and repulsion together, we can get the following attraction-repulsion system

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), \quad x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v + \alpha u - \beta v, \quad x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w + \gamma u - \delta w, \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \tau v(x,0) = \tau v_0(x),\ \tau w(x,0) = \tau w_0(x), \quad x \in \Omega \end{gathered} \tag{1.2}$$

for cell density u, concentration of an attractive signal v, and concentration of a repulsive signal w, respectively, where $\chi, \xi, \alpha, \beta, \gamma$ and δ are positive and $\tau = 0, 1$.

Model (1.2) with $\tau = 1$ was proposed in [27] to describe the quorum effect in the chemotaxis process, and in [21] to describe the aggregation of microglia in Alzheimer's disease. In the one-dimensional framework, the resulting variant of (1.2) with $\tau = 1$ was proved to have global solutions in [18], and large time behavior was obtained in [14] for all $\alpha > 0$ and $\beta > 0$. Moreover, the time-periodic solution of (1.2) was studied in [19] for various ranges of parameter values. Since chemical diffuses faster than cells, it is valuable to consider (1.2) with $\tau = 0$. Especially in [29], by using the following transformation

$$s := \chi v - \xi w, \tag{1.3}$$

Equation (1.2) can be changed into the general classical Keller-Segel model (1.1) for the special case $\beta = \delta$. Thus under some additional assumptions on the parameters, the global existence, blow-up, stationary solutions and large-time behavior of (1.2) with $\tau = 0, 1$ were considered in [29] by using a number of mathematical techniques. But for the case of $\beta \neq \delta$ in higher dimensions, it becomes more challenging because there does not exist a Lyapunov functional for (1.2). The first result of this case has been also found in [29], where global existence was asserted in any bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ if $\chi\alpha - \xi\gamma < 0$ and $\tau = 0$. When $\tau = 1$, global existence of weak solutions to (1.2) was obtained in three dimensions [13]. Recently, some further information on the existence of bounded solutions or on the occurrence of blow-up has been explored in [4, 20, 16, 17] in a bounded domain $\Omega \subset \mathbb{R}^2$.

In this article we focus on (1.2) with $\tau = 0$ for the cases $\chi\alpha = \xi\gamma$ and $\beta \not\equiv \delta$. As for the initial data u_0, we may assume that

$$u_0 \in C^0(\bar{\Omega}), \quad u_0 > 0 \quad \text{in } \bar{\Omega}. \tag{1.4}$$

To study (1.2) directly, we turn (1.2) into the initial-boundary value problem

$$
\begin{gathered}
u_t = \Delta u - \nabla\cdot(u\nabla s), \quad x\in\Omega,\ t>0,\\
0 = \Delta s - \delta s + (\chi\alpha - \xi\gamma)u + \chi(\delta-\beta)v, \quad x\in\Omega,\ t>0,\\
0 = \Delta v + \alpha u - \beta v, \quad x\in\Omega, ; t>0,\\
\frac{\partial u}{\partial\nu} = \frac{\partial s}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\
u(x,0) = u_0(x), \quad x\in\Omega\,,
\end{gathered} \tag{1.5}
$$

by using the same transformation (1.3) given in [29]. Firstly, our result involving global existence is stated as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^n (n=2,3)$ be a bounded domain with smooth boundary $\partial\Omega$ and $\tau=0$. Assume that*

$$
\chi\alpha - \xi\gamma = 0. \tag{1.6}
$$

Then for all u_0 satisfying (1.4), (1.2) possesses a unique classical solution (u,v,w) which is global in time and uniformly bounded in $\Omega\times(0,\infty)$.

Secondly, for all positive β and δ, inspired by [33], under some suitable smallness on u_0, we have the following result.

Theorem 1.2. *Let $n=2$, and let $\tau=0$. Suppose that (1.6) holds. Given some u_0 fulfilling (1.4), one can find some $\epsilon_0>0$ such that if*

$$
m := \int_\Omega u_0 \le \epsilon \tag{1.7}
$$

holds for all $0<\epsilon<\epsilon_0$, then the unique global solution of (1.2) satisfies

$$
\begin{gathered}
\|u(\cdot,t) - \bar u_0\|_{L^\infty(\Omega)} \to 0,\\
\|v(\cdot,t) - \frac{\alpha}{\beta}\bar u_0\|_{L^\infty(\Omega)} \to 0,\\
\|w(\cdot,t) - \frac{\gamma}{\delta}\bar u_0\|_{L^\infty(\Omega)} \to 0,
\end{gathered} \tag{1.8}
$$

as $t\to\infty$, where $\bar u_0 := \frac{1}{|\Omega|}\int_\Omega u_0$.

Remark 1.3. Theorem 1.2 shows that the asymptotic behavior of solutions to (1.2) are very similar to the special case $\beta=\delta$ in [29]. Unfortunately, the question of global dynamics for arbitrarily large m has to be left as an open problem here.

2. Preliminaries

Before proving the main results in this article, we state some basic and useful properties in this section. We start with the local-in-time existence of a classical solution to (1.2) with $\tau=0$ which has been proved in [29].

Lemma 2.1. *For any nonnegative function $u_0\in C^0(\bar\Omega)$, there exist $T_{\max}\in(0,\infty]$ and a unique triple $(u,v,w)\in C^0(\bar\Omega\times[0,T_{\max}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$ solving (1.2) with $\tau=0$ classically. Moreover, if $T_{\max}<\infty$, then*

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t\to T_{\max}. \tag{2.1}
$$

The following properties immediately result from an integration of each equation in (1.2) with respect to $x\in\bar\Omega$, and from the maximum principle.

Lemma 2.2. *Suppose u_0 satisfies (1.4). Then the solution (u,v,w) of (1.2) with $\tau=0$ satisfies*

$$\begin{aligned} \|u(\cdot,t)\|_{L^1(\Omega)} &= \|u_0\|_{L^1(\Omega)} \quad \text{for all } t\in(0,T_{\max}),\\ \|v(\cdot,t)\|_{L^1(\Omega)} &= \frac{\alpha}{\beta}\|u_0\|_{L^1(\Omega)} \quad \text{for all } t\in(0,T_{\max}),\\ \|w(\cdot,t)\|_{L^1(\Omega)} &= \frac{\gamma}{\delta}\|u_0\|_{L^1(\Omega)} \quad \text{for all } t\in(0,T_{\max}). \end{aligned} \tag{2.2}$$

Moreover,

$$u>0,\quad v>0,\quad w>0 \quad \text{in } \bar\Omega\times(0,T_{\max}). \tag{2.3}$$

3. Proof of Theorem 1.1

A crucial step towards our boundedness proof will be provided by the following lemma.

Lemma 3.1. *Assume that (1.6) holds, and that Ω is a bounded domain in $\mathbb{R}^n$ ($n=2,3$). For any $r>10/3$, there exists some constant $C>0$ such that*

$$\int_\Omega u^r(x,t)dx \le C \quad \text{for all } t\in(0,T_{\max}). \tag{3.1}$$

Proof. Multiplying u^{r-1} to the first equation in (1.5) and integrating by parts, we have

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r = -\frac{4(r-1)}{r^2}\int_\Omega |\nabla u^{r/2}|^2 + \frac{r-1}{r}\int_\Omega \nabla u^r\cdot\nabla s \tag{3.2}$$

for all $t\in(0,T_{\max})$. On the other hand, multiplying the second equation in (1.5) by u^r, we get that

$$\begin{aligned} \int_\Omega \nabla u^r\cdot\nabla s &= -\delta\int_\Omega u^r s + \chi(\delta-\beta)\int_\Omega u^r v.\\ &= -\delta\int_\Omega u^r(\chi v-\xi w) + \chi(\delta-\beta)\int_\Omega u^r v\\ &= \xi\delta\int_\Omega u^r w - \chi\beta\int_\Omega u^r v \quad \text{for all } t\in(0,T_{\max}). \end{aligned} \tag{3.3}$$

Noting that $u(x,t)>0$ and $v(x,t)>0$ for all $x\in\bar\Omega$ and $t\in(0,T_{\max})$, then combining (3.2) and (3.3) yields

$$\frac{d}{dt}\int_\Omega u^r + \frac{4(r-1)}{r}\int_\Omega |\nabla u^{r/2}|^2 \le \xi\delta(r-1)\int_\Omega u^r w \quad \text{for all } t\in(0,T_{\max}). \tag{3.4}$$

By the Gagliardo-Nirenberg inequality, there exist some constants $C_1>0$ and $C_2>0$ satisfying

$$\begin{aligned} \int_\Omega u^{\frac{rn+2}{n}} &= \|u^{r/2}\|_{L^{\frac{2rn+4}{rn}}(\Omega)}^{\frac{2rn+4}{rn}}\\ &\le C_1\|\nabla u^{r/2}\|_{L^2(\Omega)}^{\frac{2rn+4}{rn}a_1}\|u^{r/2}\|_{L^{\frac{2}{r}}(\Omega)}^{\frac{2rn+4}{rn}\cdot(1-a_1)} + C_1\|u^{r/2}\|_{L^{\frac{2}{r}}(\Omega)}^{\frac{2rn+4}{rn}}\\ &\le C_2\|\nabla u^{r/2}\|_{L^2(\Omega)}^2 + C_2 \quad \text{for all } t\in(0,T_{\max}), \end{aligned} \tag{3.5}$$

where

$$a_1 = \frac{rn}{rn+2}\in(0,1).$$

On the other hand, applying Young's inequality to (3.4), we infer that

$$\frac{d}{dt}\int_\Omega u^r + \frac{4(r-1)}{r}\int_\Omega |\nabla u^{r/2}|^2 \le \frac{r-1}{rC_2}\int_\Omega u^{\frac{rn+2}{n}} + C_3\int_\Omega w^{\frac{rn+2}{2}} \tag{3.6}$$

for all $t \in (0, T_{\max})$, where

$$C_3 = \xi\delta(r-1)\Big(\frac{1}{\xi\delta rC_2}\cdot\frac{rn+2}{rn}\Big)^{-rn/2}\Big(\frac{rn+2}{2}\Big)^{-1}.$$

To estimate the second term on the right-hand side of (3.6), noting that w satisfies

$$\begin{gathered} 0 = \Delta w + \gamma u - \delta w, \quad x \in \Omega,\ t \in (0, T_{\max}),\\ \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t \in (0, T_{\max}), \end{gathered} \tag{3.7}$$

then testing (3.7) by $w^{\frac{rn}{2}}$ and applying Young's inequality again, we immediately obtain

$$\begin{aligned} &\frac{8rn}{(rn+2)^2}\int_\Omega |\nabla w^{\frac{rn+2}{4}}|^2 + \delta\int_\Omega w^{\frac{rn+2}{2}}\\ &= \gamma\int_\Omega u w^{\frac{rn}{2}}\\ &\le \frac{\delta(r-1)}{rC_2C_3}\int_\Omega u^{\frac{rn+2}{n}} + C_4\int_\Omega w^{\frac{rn(rn+2)}{2(rn-n+2)}}, \end{aligned} \tag{3.8}$$

where

$$C_4 = \gamma\Big(\frac{\delta(r-1)}{\gamma rC_2C_3}\frac{rn+2}{n}\Big)^{-\frac{n}{rn-n+2}}\Big(\frac{rn+2}{rn-n+2}\Big)^{-1}.$$

We use the Gagliardo-Nirenberg inequality to estimate

$$\begin{aligned} \int_\Omega w^{\frac{rn(rn+2)}{2(rn-n+2)}} &= \|w^{\frac{rn+2}{4}}\|^{\frac{2rn}{rn-n+2}}_{L^{\frac{2rn}{rn-n+2}}(\Omega)}\\ &\le C_5\|\nabla w^{\frac{rn+2}{4}}\|^{\frac{2rn}{rn-n+2}a_2}_{L^2(\Omega)}\|w^{\frac{rn+2}{4}}\|^{\frac{2rn}{rn-n+2}(1-a_2)}_{L^{\frac{4}{rn+2}}(\Omega)}\\ &\quad + C_5\|w^{\frac{rn+2}{4}}\|^{\frac{2rn}{rn-n+2}}_{L^{\frac{4}{rn+2}}(\Omega)} \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \tag{3.9}$$

with some constant $C_5 > 0$ and a_2 determined by

$$\frac{rn-n+2}{2rn} = (\frac{1}{2} - \frac{1}{n})a_2 + \frac{rn+2}{4}(1-a_2).$$

Thus a_2 satisfies

$$\begin{gathered} a_2 = \frac{r^2n^2 + 2n - 4}{(rn^2+4)r} \in (0,1),\\ \frac{2rn}{rn-n+2}a_2 = \frac{2rn}{rn-n+2}\frac{r^2n^2+2n-4}{(rn^2+4)r} < 2 \end{gathered}$$

because $r > \frac{10}{3}$ and $n = 2, 3$. By Young's inequality, (3.9) becomes

$$\begin{aligned} \int_\Omega w^{\frac{rn(rn+2)}{2(rn-n+2)}} &\le C_6\Big(\int_\Omega |\nabla w^{\frac{rn+2}{4}}|^2\Big)^{\frac{rn}{rn-n+2}\cdot\frac{r^2n^2+2n-4}{(rn^2+4)r}} + C_6\\ &\le \epsilon\int_\Omega |\nabla w^{\frac{rn+2}{4}}|^2 + C_7(\epsilon) \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \tag{3.10}$$

with constants $C_6 > 0$ and $C_7(\epsilon) > 0$, where we take $\epsilon = \frac{4rn}{(rn+2)^2 C_4}$. Inserting (3.10) into (3.8), we find some constant $C_8 > 0$ satisfying

$$\int_\Omega w^{\frac{rn+2}{2}} \le \frac{r-1}{rC_2C_3} \int_\Omega u^{\frac{rn+2}{n}} + C_8 \quad \text{for all } t \in (0, T_{\max}). \tag{3.11}$$

As a consequence of (3.11) and (3.5), (3.6) can be turned into the inequality

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^r + \frac{4(r-1)}{r}\int_\Omega |\nabla u^{r/2}|^2 &\le \frac{2(r-1)}{rC_2}\int_\Omega u^{\frac{rn+2}{n}} + C_9 \\ &\le \frac{2(r-1)}{rC_2}\Big(C_2\int_\Omega |\nabla u^{r/2}|^2 + C_2\Big) + C_9 \end{aligned}$$

for all $t \in (0, T_{\max})$ with $C_9 > 0$. Therefore, we can pick $C_{10} > 0$ to obtain

$$\frac{d}{dt}\int_\Omega u^r + \int_\Omega u^r \le -\frac{2(r-1)}{r}\int_\Omega |\nabla u^{r/2}|^2 + \int_\Omega u^r + C_{10} \tag{3.12}$$

for all $t \in (0, T_{\max})$. It follows from the Gagliardo-Nirenberg inequality that

$$\begin{aligned} \int_\Omega u^r &= \|u^{r/2}\|^2_{L^2(\Omega)} \\ &\le C_{11}\|\nabla u^{r/2}\|^{2a_3}_{L^2(\Omega)}\|u^{r/2}\|^{2(1-a_3)}_{L^{\frac{2}{r}}(\Omega)} + C_{11}\|u^{r/2}\|^2_{L^{\frac{2}{r}}(\Omega)} \\ &\le C_{12}\|\nabla u^{r/2}\|^{2a_3}_{L^2(\Omega)} + C_{12} \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \tag{3.13}$$

for some constants $C_{11} > 0$ and $C_{12} > 0$, where

$$a_3 = \frac{rn-n}{rn-n+2} \in (0,1).$$

Inserting (3.13) in (3.12) and by Young's inequality, there exists some constant $C_{13} > 0$ such that

$$\frac{d}{dt}\int_\Omega u^r + \int_\Omega u^r \le C_{13} \quad \text{for all } t \in (0, T_{\max}),$$

which leads to

$$\|u(\cdot,t)\|_{L^r(\Omega)} \le C_{14} \quad \text{for all } t \in (0, T_{\max})$$

with some constant $C_{14} > 0$. The proof is complete. $\square$

Proof of Theorem 1.1. Since v satisfies

$$\begin{gathered} 0 = \Delta v - \beta v + \alpha u, \quad x \in \Omega,\ t \in (0, T_{\max}), \\ \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t \in (0, T_{\max}), \end{gathered}$$

then applying the Agmon-Douglis-Nirenberg L^r estimates [1, 2] on linear elliptic equations with homogeneous Neumann boundary condition, there provides some constant $C_1 > 0$ satisfying

$$\|v(\cdot,t)\|_{W^{2,r}(\Omega)} \le C_1\|u(\cdot,t)\|_{L^r(\Omega)} \quad \text{for all } t \in (0, T_{\max}).$$

Now from Lemma 3.1 and using the Sobolev embedding: $W^{2,r}(\Omega) \hookrightarrow C^1_B(\Omega) := \{u \in C^1(\Omega) | Du \in L^\infty(\Omega)\}$ if $r > n$ [7], we find

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all } t \in (0, T_{\max}) \tag{3.14}$$

with some constant $C_2 > 0$. Similarly, we can pick some constant $C_3 > 0$ such that

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le C_3 \quad \text{for all } t \in (0, T_{\max}).$$

In view of the variation-of-constants formula to the first equation in (1.2), we can see that

$$\begin{aligned} u(\cdot,t) &= e^{t\Delta}u_0 - \chi\int_0^t e^{(t-\sigma)\Delta}\nabla\cdot(u(\cdot,\sigma)\nabla v(\cdot,\sigma))d\sigma \\ &\quad + \xi\int_0^t e^{(t-\sigma)\Delta}\nabla\cdot(u(\cdot,\sigma)\nabla w(\cdot,\sigma))d\sigma \\ &=: I_1(\cdot,t) + I_2(\cdot,t) + I_3(\cdot,t) \quad \text{for all } t\in(0,T_{\max}). \end{aligned}$$

As an easy consequence of the smoothing estimates for the Neumann heat semigroup, we immediately obtain

$$\|I_1(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \quad \text{for all } t\in(0,T_{\max}).$$

Applying the known smoothing estimates from [32] (see also [3]), for some $C_4>0$ we have

$$\begin{aligned} \|I_2(\cdot,t)\|_{L^\infty(\Omega)} &\le \chi\int_0^t \|e^{(t-\sigma)\Delta}\nabla\cdot(u(\cdot,\sigma)\nabla v(\cdot,\sigma))\|_{L^\infty(\Omega)}d\sigma \\ &\le C_4\int_0^t \Big(1+(t-\sigma)^{-\frac12-\frac{n}{2r}}\Big)e^{-\lambda_1(t-\sigma)}\|u(\cdot,\sigma)\nabla v(\cdot,\sigma)\|_{L^r(\Omega)}d\sigma \end{aligned}$$

for all $t\in(0,T_{\max})$, where $\lambda_1>0$ denotes the first eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. For any $r>n$, according to (3.14) and the boundedness of $u(\cdot,t)$ in $L^r(\Omega)$ asserted by Lemma 3.1, this yields $C_5>0$ such that

$$\begin{aligned} &\|I_2(\cdot,t)\|_{L^\infty(\Omega)} \\ &\le C_4\int_0^t \Big(1+(t-\sigma)^{-\frac12-\frac{n}{2r}}\Big)e^{-\lambda_1(t-\sigma)}\|u(\cdot,\sigma)\|_{L^r(\Omega)}\|\nabla v(\cdot,\sigma)\|_{L^\infty(\Omega)}d\sigma \\ &\le C_5\int_0^t \Big(1+\upsilon^{-\frac12-\frac{n}{2r}}\Big)e^{-\lambda_1\upsilon}d\upsilon \\ &\le C_6 \quad \text{for all } t\in(0,T_{\max}). \end{aligned}$$

It is similar to deal with I_3, that means

$$\|I_3(\cdot,t)\|_{L^\infty(\Omega)} \le C_7 \quad \text{for all } t\in(0,T_{\max})$$

holds with some constant $C_7>0$. Therefore, the maximal existence time $T_{\max}$ of solutions to (1.2) must be infinite by means of Lemma 2.1 and we finish our proof. □

4. Proof of Theorem 1.2

4.1. **A bound for u.** To avoid confusion, through this section, we should state that the constants c_i and C_i ($i=1,2,\dots$) are independent of the total mass $\int_\Omega u_0$.

Lemma 4.1. *Suppose $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. Then for all $\alpha>0$ and $\beta>0$, the solution v of*

$$\begin{gathered} 0=\Delta v-\beta v+\alpha u, \quad x\in\Omega, \\ \frac{\partial v}{\partial\nu}=0, \quad x\in\partial\Omega \end{gathered} \tag{4.1}$$

satisfies

$$\|v\|_{L^p(\Omega)} \le \alpha C_p \|u\|_{L^1(\Omega)} \quad \textit{for all } p \in (1,\infty), \tag{4.2}$$

$$\|\nabla v\|_{L^q(\Omega)} \le \alpha C_q \|u\|_{L^2(\Omega)} \quad \textit{for all } q \in (1,\infty), \tag{4.3}$$

where C_p (resp. C_q) is a positive constant depending on p (resp. q).

Proof. From (4.1), v can be represented as

$$v(x) = \alpha \int_\Omega G(x,y)u(y)dy, \quad \text{a.e. } x \in \Omega,$$

where $G(x,y)$ is the Green function of $-\Delta + \beta$ in Ω subject to homogeneous Neumann boundary conditions (see [24, 12, 30]). Noting that $G(x,y)$ satisfies

$$|G(x,y)| \le C\Big(1 + \ln \frac{1}{|x-y|}\Big), \quad |\nabla_x G(x,y)| \le \frac{C}{|x-y|} \quad \text{for all } x,y \in \Omega \text{ with } x \ne y$$

with some constant $C > 0$, by means of Young's inequality for convolutions we easily arrive at (4.2)-(4.3). $\square$

Lemma 4.2. *Assume that the assumptions in Theorem 1.2 are satisfied. Then for all $r > 1$ there exists some constant $C > 0$ satisfying*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^r(\Omega)} \le Cm\big(1 + m^{\frac{2}{r}+2}\big), \tag{4.4}$$

where $m := \int_\Omega u_0$.

Proof. In light of the third equation in (1.5) and the inequality (4.2), for all $p \in (1,\infty)$ we obtain that

$$\|v(\cdot,t)\|_{L^p(\Omega)} \le C_1 m \quad \text{for all } t > 0 \tag{4.5}$$

with some constant $C_1 > 0$. Observing that s solves

$$0 = \Delta s - \delta s + \chi(\delta - \beta)v, \quad x \in \Omega,\ t > 0,$$

$$\frac{\partial s}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0,$$

for all $q \in (1,\infty)$ we use (4.3) and (4.5) to find some $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned} \|\nabla s(\cdot,t)\|_{L^q(\Omega)} &\le \chi|\delta-\beta| C_2 \cdot \|v(\cdot,t)\|_{L^2(\Omega)} \\ &\le C_3 m \quad \text{for all } t > 0. \end{aligned} \tag{4.6}$$

Testing the first equation of (1.5) by u^{r-1} and integrating by parts, we see that

$$\frac{d}{dt}\int_\Omega u^r + \int_\Omega u^r + \frac{2(r-1)}{r}\int_\Omega |\nabla u^{r/2}|^2 \le \frac{r(r-1)}{2}\int_\Omega u^r|\nabla s|^2 + \int_\Omega u^r \tag{4.7}$$

for all $t > 0$. To deal with the right-hand side of (4.7), since

$$\begin{aligned} \|u\|_{L^{r+1}(\Omega)}^{r+1} &= \|u^{r/2}\|_{L^{\frac{2(r+1)}{r}}(\Omega)}^{\frac{2(r+1)}{r}} \\ &\le C_4 \|\nabla u^{r/2}\|_{L^2(\Omega)}^2 \|u^{r/2}\|_{L^{\frac{2}{r}}(\Omega)}^{\frac{2}{r}} + C_4 \|u^{r/2}\|_{L^{\frac{2}{r}}(\Omega)}^{\frac{2(r+1)}{r}} \\ &= C_4 m \|\nabla u^{r/2}\|_{L^2(\Omega)}^2 + C_4 m^{r+1} \end{aligned}$$

holds for some constant $C_4 > 0$, and

$$\|u\|_{L^r(\Omega)}^r = \|u^{r/2}\|_{L^2(\Omega)}^2 \le C_5 \|\nabla u^{r/2}\|_{L^2(\Omega)}^{\frac{2(r-1)}{r}} \|u^{r/2}\|_{L^{\frac{2}{r}}(\Omega)}^{\frac{2}{r}} + C_5 \|u^{r/2}\|_{L^{\frac{2}{r}}(\Omega)}^2$$

$$= C_5 m \|\nabla u^{r/2}\|_{L^2(\Omega)}^{\frac{2(r-1)}{r}} + C_5 m^r$$

holds for $C_5 > 0$ by means of the Gagliardo-Nirenberg inequality. Then Young's inequality implies

$$\begin{aligned}
\frac{r(r-1)}{2}\int_\Omega u^r|\nabla s|^2 &\le \frac{r(r-1)}{2}\Big(\epsilon_1\int_\Omega u^{r+1} + \epsilon_1^{-r}\frac{r^r}{(r+1)^{r+1}}\int_\Omega |\nabla s|^{2(r+1)}\Big)\\
&\le \epsilon_1\frac{r(r-1)}{2}C_4 m\|\nabla u^{r/2}\|_{L^2(\Omega)}^2 + \epsilon_1\frac{r(r-1)}{2}C_4 m^{r+1}\\
&\quad + \epsilon_1^{-r}\frac{(r-1)}{2}\big(\frac{r}{r+1}\big)^{r+1}\int_\Omega |\nabla s|^{2(r+1)} \quad \text{for all } t>0
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega u^r &= \|u\|_{L^r(\Omega)}^r\\
&\le C_5 m\|\nabla u^{r/2}\|_{L^2(\Omega)}^{\frac{2(r-1)}{r}} + C_5 m^r\\
&\le \epsilon_2 C_5\|\nabla u^{r/2}\|_{L^2(\Omega)}^2 + \Big(\epsilon_2^{-(r-1)}\cdot\frac{(r-1)^{(r-1)}}{r^r}+1\Big)C_5 m^r \quad \text{for all } t>0.
\end{aligned}$$

Taking $\epsilon_1 = 2r^{-2}C_4^{-1}m^{-1}$ and $\epsilon_2 = \frac{r-1}{r}C_5^{-1}$, inequality (4.7) becomes

$$\frac{d}{dt}\int_\Omega u^r + \int_\Omega u^r \le C_6 m^r\Big(1+\int_\Omega |\nabla s|^{2(r+1)}\Big) \quad \text{for all } t>0. \tag{4.8}$$

Recalling (4.6), integrating (4.8) over $(0,t)$, we find that

$$\int_\Omega u^r \le e^{-t}\|u_0\|_{L^r(\Omega)}^r + C_6 m^r\Big(1+m^{2(r+1)}\Big) \quad \text{for all } t>0,$$

which yields (4.4). □

Proof of Theorem 1.2. With Lemma 4.2 at hand, the most important step towards global behavior of the case $\chi\alpha - \xi\gamma = 0$ is to drive a bound for $U := u - \overline{u}_0$ in this section (Lemma 4.3). The later will enforce $\|U(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t\to\infty$ under a smallness condition on the initial data u_0 by using a fixed-point type argument (see also [33]). Let us introduce

$$\begin{gathered}
U(x,t) := u(x,t) - \bar u_0,\\
S(x,t) := s(x,t) - \chi\alpha\Big(\frac{1}{\beta}-\frac{1}{\delta}\Big)\bar u_0,\\
V(x,t) := v(x,t) - \frac{\alpha}{\beta}\bar u_0
\end{gathered}$$

for all $x \in \bar{\Omega}$ and $t > 0$. Then if (1.6) holds, (U, S, V) solves the initial-value problem

$$
\begin{gathered}
U_t = \Delta U - \nabla \cdot (u \nabla S), \quad x \in \Omega,\ t > 0, \\
0 = \Delta S - \delta S + \chi(\delta - \beta) V, \quad x \in \Omega,\ t > 0, \\
0 = \Delta V - \beta V + \alpha U, \quad x \in \Omega,\ t > 0, \\
\frac{\partial U}{\partial \nu} = \frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
U(x,0) = u_0(x) - \bar{u}_0, \quad V(x,0) = v_0(x) - \frac{\alpha}{\beta}\bar{u}_0, \\
S(x,0) = \chi\big(v_0(x) - \frac{\alpha}{\beta}\bar{u}_0\big) - \xi\big(w_0(x) - \frac{\gamma}{\delta}\bar{u}_0\big), \quad x \in \Omega.
\end{gathered}
\tag{4.9}
$$

By a straightforward adaptation of the ideas in [17], we proceed to derive an estimate for U with respect to the norm in $L^\infty(\Omega)$. □

Lemma 4.3. *Let $n = 2$. For some $r > 1$, the solution (U, S, V) of (4.9) satisfies*

$$
\limsup_{t \to \infty} \|U(\cdot, t)\|_{L^\infty(\Omega)} \le C m^2 \big(1 + m^{\frac{2}{r}+2}\big). \tag{4.10}
$$

Proof. Since $\nabla S = \nabla s$, we first apply (4.6) and Lemma 4.2 to pick $t_1 = t_1(u, v, w) > 0$ such that

$$
\|\nabla S(\cdot, t)\|_{L^q(\Omega)} \le C_1 m \quad \text{for all } t \ge t_1,\ q \in (1, \infty), \tag{4.11}
$$

$$
\|u(\cdot, t)\|_{L^r(\Omega)} \le C_2 m \big(1 + m^{\frac{2}{r}+2}\big) \quad \text{for all } t \ge t_1,\ r \in (1, \infty), \tag{4.12}
$$

where C_1 and C_2 are positive constant. By means of the variation-of-constants formula to the first equation in (4.9), we have

$$
U(\cdot, t) = e^{(t - t_1)\Delta} U(\cdot, t_1) - \int_{t_1}^{t} e^{(t-\sigma)\Delta} \nabla \cdot (u(\cdot, \sigma) \nabla S(\cdot, \sigma))\, d\sigma \quad \text{for all } t > t_1.
$$

This in conjunction with some arguments on the asymptotic behavior of the heat semigroup [17, 31] yields (4.10) by using (4.11)–(4.12). □

Now, invoking the upper estimate for U in Lemma 4.3, we can pick $t_2 = t_2(u, v, w) > 0$ such that

$$
\|U(\cdot, t)\|_{L^\infty(\Omega)} \le c_1 m^2 (1 + m^{\frac{2}{r}+2}) \quad \text{for all } t \ge t_2 \tag{4.13}
$$

with some constant $c_1 > 0$. With $\epsilon_0 > 0$ to be specified below, we fix the total mass $m := \int_\Omega u_0$ small enough such that $0 < m \le \epsilon$ for $0 < \epsilon < \epsilon_0$.

Suppose that ϵ_0 satisfies

$$
2 c_1 \epsilon_0 (1 + \epsilon_0^{\frac{2}{r}+2}) \le 1. \tag{4.14}
$$

Then (4.13) implies

$$
\|U(\cdot, t)\|_{L^\infty(\Omega)} \le \frac{1}{2}\epsilon \quad \text{for all } t \ge t_2.
$$

Now let $\lambda_1 > 0$ denote the first eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions, and let some κ satisfy

$$
\kappa \in \big(0, \frac{\lambda_1}{2}\big). \tag{4.15}
$$

Then since $\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \epsilon/2$ holds for all $t \ge t_2$, the set

$$S^* := \big\{T^* \ge t_2 \mid \|U(\cdot,t)\|_{L^\infty(\Omega)} \le \epsilon e^{-\kappa(t-t_2)} \text{ for all } t \in [t_2,T^*]\big\}$$

is well-defined.

The following lemma provides $T=\infty$, where $T := \sup S^* \in (t_2,\infty]$. Therefore we obtain our goal that the component u of (1.2) actually converges to $\overline{u}_0$, at an exponential rate.

Lemma 4.4. *Suppose that κ satisfies* (4.15) *and that $n=2$. Then one can find some constant $C>0$ such that*

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\kappa(t-t_2)} \quad \textit{for all } t > t_2. \tag{4.16}$$

Proof. Given any $p \in (1,\infty)$, since V solves the third equation in (4.9), from (4.2) we can find some positive C_1 and C_2 such that

$$\begin{aligned}
\|V(\cdot,t)\|_{L^p(\Omega)} &\le \alpha C_1 \cdot \|U(\cdot,t)\|_{L^1(\Omega)} \\
&\le \alpha|\Omega|C_1 \cdot \|U(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le C_2\epsilon e^{-\kappa(t-t_2)} \quad \text{for all } t \in (t_2,T).
\end{aligned} \tag{4.17}$$

Moreover, given any $q \in (1,\infty)$, employing the inequality (4.3), we can pick some constants $C_3>0$ and $C_4>0$ satisfying

$$\|\nabla S(\cdot,t)\|_{L^q(\Omega)} \le \chi|\delta-\beta|C_3 \cdot \|V(\cdot,t)\|_{L^2(\Omega)} \le C_4\epsilon e^{-\kappa(t-t_2)}, \quad \forall t \in (t_2,T). \tag{4.18}$$

Observing that $U = u - \overline{u}_0$, u can be easily controlled as

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \epsilon\Big(e^{-\kappa(t-t_2)} + \frac{1}{|\Omega|}\Big) \quad \text{for all } t \in (t_2,T).$$

In view of (4.18) and Lemma 4.3, applying some L^p-L^q estimates for the Neumann heat semigroup (see [31, Lemma 1.3] or [17, Lemma 5.4]) to the representation of $U(\cdot,t)$, for some $k>n$ we have

$$\begin{aligned}
&\|U(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le \|e^{(t-t_2)\Delta}U(\cdot,t_2)\|_{L^\infty(\Omega)} + \int_{t_2}^t \|e^{(t-\sigma)\Delta}\nabla\cdot(u(\cdot,\sigma)\nabla S(\cdot,\sigma))\|_{L^\infty(\Omega)}d\sigma \\
&\le C_5e^{-\lambda_1(t-t_2)}\|U(\cdot,t_2)\|_{L^\infty(\Omega)} \\
&\quad + C_5\int_{t_2}^t \Big(1+(t-\sigma)^{-\frac12-\frac{n}{2k}}\Big)e^{-\lambda_1(t-\sigma)}\|u(\cdot,\sigma)\nabla S(\cdot,\sigma)\|_{L^k(\Omega)}d\sigma \\
&\le C_6\epsilon^2\Big(1+\epsilon^{\frac{2}{r}+2}\Big)e^{-\lambda_1(t-t_2)} \\
&\quad + C_6\epsilon^2\int_{t_2}^t \Big(1+(t-\sigma)^{-\frac12-\frac{n}{2k}}\Big)e^{-\lambda_1(t-\sigma)}\Big(e^{-\kappa(\sigma-t_2)}+e^{-2\kappa(\sigma-t_2)}\Big)d\sigma
\end{aligned}$$

for all $t \in (t_2,T)$. For any $0<\kappa<\frac{\lambda_1}{2}$ and given some $r>1$, we may use [31, Lemma 1.2] to find some constant $C_7>0$ such that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le C_7\epsilon^2\big(1+\epsilon^{\frac{2}{r}+2}\big)e^{-\kappa(t-t_2)} \quad \text{for all } t \in (t_2,T).$$

Thus fixing $\epsilon_0>0$ small enough such that

$$C_7\epsilon_0\Big(1+\epsilon_0^{\frac{2}{r}+2}\Big) < 1$$

and (4.14), and in view of the continuity of U, we find that $T = \infty$. This implies (4.16) and hence completes the proof. $\square$

Proof of Theorem 1.2. Applying the maximum principle to the second equation in (1.2) we have

$$\frac{\alpha}{\beta}\min_{x\in\bar{\Omega}} u(x,t) \le v(x,t) \le \frac{\alpha}{\beta}\max_{x\in\bar{\Omega}} u(x,t) \quad \text{for all } t > 0.$$

In light of Lemma 4.4, there exists $C > 0$ satisfying

$$\|v(\cdot,t) - \frac{\alpha}{\beta}\overline{u}_0\|_{L^\infty(\Omega)} \le \frac{\alpha}{\beta}\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\kappa t} \quad \text{for all } t > 0.$$

The convergence of w can be similarly proved. $\square$

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