Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 144, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE, UNIQUENESS AND STABILITY OF TRAVELING WAVEFRONTS FOR NONLOCAL DISPERSAL EQUATIONS WITH CONVOLUTION TYPE BISTABLE NONLINEARITY

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Abstract. This article concerns the bistable traveling wavefronts of a nonlocal dispersal equation with convolution type bistable nonlinearity. Applying a homotopy method, we establish the existence of traveling wavefronts. If the wave speed does not vanish, i.e. $c \neq 0$, then the uniqueness (up to translation) and the globally asymptotical stability of traveling wavefronts are proved by the comparison principle and squeezing technique.

1. Introduction

In this article, we consider the traveling wave solutions of the delayed nonlocal dispersal equation

$$\frac{\partial u}{\partial t} = J * u - u - du + \int_{\mathbb{R}} K(y) b(u(x-y, t-\tau)) dy, \tag{1.1}$$

in which $x \in \mathbb{R}$, $t > 0$. Equation (1.1) represents the dynamical population model of a single-species with age-structure in ecology [14, 35, 36]. Here $u(x,t)$ is the density of population at location x and at time t, $d > 0$ is the death rate, and $b(\cdot)$ is the birth function. The parameter $\tau > 0$ is the maturation time, we call it the time-delay. $J * u - u$ is a nonlocal dispersal operator, which can be interpreted as the net rate of increase due to dispersal, where, $J(x)$ is a non-negative, unit and symmetric kernel, and $J * u$ is a spatial convolution defined by

$$(J * u)(x,t) = \int_{\mathbb{R}} J(x-y) u(y,t) dy.$$

As stated in [3, 12, 15], if $J(x-y)$ is considered to be the probability distribution of jumping from location y to location x, then $(J*u)(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to location x from all other places, while, the term $-u(x,t) = -\int_{\mathbb{R}} J(x-y)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other places.

Throughout this article, we assume that the kernel functions $J \in C^1(\mathbb{R})$ and $K \in C^2(\mathbb{R})$, and the birth function $b \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy:

2010 *Mathematics Subject Classification.* 35K57, 35R20, 92D25.
Key words and phrases. Nonlocal dispersal; traveling wavefronts; bistable nonlinearity; continuation method; squeezing technique.

Submitted August 8, 2014. Published June 6, 2015.

(H1) $J(x) = J(-x) \geq 0$, $K(x) = K(-x) \geq 0$ for $x \in \mathbb{R}$.
(H2) $\int_{\mathbb{R}} J(x)dx = 1$, $\int_{\mathbb{R}} K(x)dx = 1$.
(H3) $\int_{\mathbb{R}}\{|x|J(x) + |J'(x)|\}dx < +\infty$, $\int_{\mathbb{R}}\{|x|K(x) + |K'(x)| + |K''(x)|\}dx < +\infty$.
(H4) $b(0) = d\alpha - b(\alpha) = d - b(1) = 0$ for some $0 < \alpha < 1$.
(H5) $b'(u) > 0$ for $u \in (0, 1)$, $d > \max\{b'(0), b'(1)\}$.
(H6) $b'(\alpha) > d$.

A specific function $b(u) = pu^2e^{-\beta u}$ with $p > 0$ and $\beta > 0$, which has been widely used in the mathematical biology literature, satisfies the above conditions for a wide range of parameters p and β. From (H4)–(H6), we can see that 0, α and 1 are constant equilibria of (1.1), and the equilibria 0 and 1 are stable and α is unstable for the spatially homogeneous equation associated with (1.1). We are interested in bistable waves of nonlocal dispersal equation (1.1), i.e., traveling wave solutions connecting the two stable equilibria 0 and 1. A *traveling wave solution* of (1.1) always refers to a pair (U, c), where $U = U(\xi)$ is a function on $\mathbb{R}$ and c is a constant, such that $u(x, t) := U(\xi)$, $\xi = x + ct$ is a solution of (1.1) and satisfies the following asymptotic boundary conditions

$$U(-\infty) = e_1, \quad U(+\infty) = e_2,$$

where e_1 and e_2 are two equilibria of (1.1). Since we are interested in traveling waves connecting 0 and 1, in this paper, $e_1 = 0$ and $e_2 = 1$. We call c the traveling wave speed and U the profile of the wave solution. If $c = 0$, we say U is a standing wave. Moreover, If $U(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called a traveling wavefront.

For some special cases of the equation (1.1), many well-known results have been obtained. Some of them can be summarized as follows:

(i) If $K(x) = \delta(x)$, $\tau = 0$ and $-du + b(u) =: f(u)$, then (1.1) reduces to

$$\frac{\partial u}{\partial t} = J * u - u + f(u). \tag{1.2}$$

Equation (1.2) has been extensively studied recently due to its wide applications in material science [1], population dynamics [3, 7, 8], epidemiology [24] and neural network [41]. Many excellent results about traveling wave solutions of (1.2) are obtained, see Bates et al. [1], Chen [4] and Yajisita [34] for the bistable equations; Coville et al. [7, 8], Carr and Chmaj [2], Pan et al. [27, 28] for monostable equations; Zhang et al. [37, 38] for degenerate monostable equations and references cited therein.

(ii) If $K(x) = \delta(x)$, then (1.1) becomes

$$\frac{\partial u}{\partial t} = J * u - u - du + b(u(x, t - \tau)). \tag{1.3}$$

Pan et al. [28] considered the equation (1.3) with monostable nonlinearity. They established the existence and asymptotic behavior of traveling wavefronts by constructing proper upper and lower solutions, and proved the asymptotic stability and uniqueness of traveling wavefronts by applying the idea of squeezing technique. In particular, when $b(u) = \alpha e^{-\gamma\tau}u(x, t - \tau)$ and du is replaced by βu^2, Li and Lin [17] gave the existence of traveling wavefronts in view of a pair of admissible upper-lower solutions. Recently, Zhang and Li [39] further proved that the traveling wavefronts with large speed are globally exponentially stable by using the weighted energy method together with the comparison principle.

Note that the birth rate function $b(\cdot)$ in (1.3) is considered to be local. In ecological context, there is no real justification for assuming that the birth of individuals of the population is a local behavior (see [13, 21, 25, 26, 32, 42]). The species' activities always involve the whole space and they move and marry in all region but not isolated in one spot. For that reason, many people begin to generalize the equation (1.3) by incorporating nonlocal effects in birth rate function. Inspired by the nonlocal reaction-diffusion model [31, 16], by introducing the nonlocal dispersal into an age-structured population model

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u,$$

and integrating along characteristics, Zhang [36] and Yu and Yuan [35] independently derived the model (1.1), which has a nonlocal nonlinearity term. Although the nonlocal dispersal operator $J * u - u$ lacks compactness and regularity, it maintains a maximum principle, which consequently enables a comparison principle, see [12]. Zhang [36] combined Schauder's fixed point theorem with upper-lower solutions to investigate the existence of traveling wave solutions when the nonlinearity function is monostable and crossing-monostable. Very recently, Zhang and Ma [40] further proved that the minimal wave speed of traveling waves is also the spreading speed for the solutions of (1.1) with initial functions having compact supports. In [14], Huang et al. proved that the planar traveling waves of (1.1) with monostable nonlinearity are globally asymptotically stable. We should point out that the above authors only considered (1.1) with monostable nonlinearity. When the nonlinearity is of bistable type, the existence, uniqueness and stability of traveling wavefronts of (1.1) remain an open problem. As we know, such a problem is also very significant, see [20] for the corresponding local diffusion case. Hence, the aim of this paper is to solve this problem.

In view of the existence of traveling wavefronts for both the nonlocal monostable equation (1.1) and the bistable non-local delayed diffusion equation [20], it is then expected that the nonlocal bistable equation (1.1) supports the existence of traveling wavefronts. Typically for bistable dynamics, the existence of a traveling wave solution is proven by a homotopy method or vanishing viscosity techniques [1, 6, 9, 18], or a recursive method for abstract monotone dynamical systems [34], or by taking the asymptotical limit, as $t \to +\infty$, of a solution to (1.1) with an appropriate initial data [4]. In this paper, we shall take the first method to prove the existence of traveling wavefronts of (1.1). Although our method is based on the work of [1, 6, 18], the technical details are quite different, due to the combination of nonlocal dispersal and nonlocal nonlinearity. In order to study the uniqueness and asymptotic stability of traveling wavefronts with nonzero speed, we shall construct various pairs of super- and subsolutions and utilize the comparison principle and the squeezing technique, which is introduced in [4, 10], and applied in many other papers [19, 20, 30, 33].

Now, we state the main result as follows.

Theorem 1.1 (Existence)**.** *Assume that* (H1)–(H6) *hold. Then there exists a non-decreasing traveling wavefront* (U, c) *to* (1.1) *connecting two equilibria* 0 *and* 1*.*

Theorem 1.2 (Uniqueness)**.** *Assume that* (H1)–(H6) *hold. Let* (U, c) *be a traveling wavefront with* $c \neq 0$ *as given in Theorem 1.1. Then the traveling wavefronts of* (1.1) *are unique up to a translation in the sense that for any traveling wavefront* $\tilde{U}(x + \tilde{c}t)$ *with* $0 \leq \tilde{U}(\xi) \leq 1$, $\xi \in \mathbb{R}$*, we have* $\tilde{c} = c$ *and* $\tilde{U}(\cdot) = U(\cdot + \xi_0)$ *for some* $\xi_0 \in \mathbb{R}$*.*

Theorem 1.3 (Stability). *Assume that* (H1)–(H6) *hold. Let* (U, c) *be a traveling wavefront with* $c \neq 0$ *as given in Theorem 1.1. Then* $U(x + ct)$ *is globally asymptotically stable with phase shift in the sense that there exists* $k > 0$ *such that for any* $\varphi \in [0, 1]_C$ *with*

$$\limsup_{x\to-\infty} \max_{s\in[-\tau,0]} \varphi(x, s) < \alpha < \liminf_{x\to+\infty} \min_{s\in[-\tau,0]} \varphi(x, s),$$

the solution $u(x, t; \varphi)$ *of* (1.1) *with initial data* φ *satisfies*

$$|u(x, t; \varphi) - U(x + ct + \xi_0)| \leq Me^{-kt}, \quad x \in \mathbb{R}, t \geq 0,$$

for some $M = M(\varphi) > 0$ *and* $\xi_0 = \xi_0(\varphi) \in \mathbb{R}$.

We remark that by the continuity of b and the assumption (H5), we can obtain that $du > b(u)$ for $u \in (0, \alpha)$ and $du < b(u)$ for $u \in (\alpha, 1)$. Moreover, we can make an extension by choosing a positive constant $\delta_0 > 0$ such that $du < b(u) < 0$ for $u \in [-\delta_0, 0]$ and $du > b(u) > 0$ for $u \in (1, 1+\delta_0]$. We can also assume that $b'(u) \geq 0$ for $u \in [-\delta_0, 1+\delta_0]$. By the first part of (H5), this can be achieved by modifying (if necessary) the definition of b outside the closed interval $[0, 1]$ to a new C^1-smooth function and applying our results to the new function b.

The rest of this paper is organized as follows. In Section 2, we establish the existence of traveling wavefronts of (1.1). In Section 3, we give some results on the corresponding initial value problem of (1.1). In Sections 4 and 5, the uniqueness (up to translation) and stability of traveling wavefronts are proved by applying the elementary super- and subsolution comparison method and squeezing technique.

2. Existence of traveling wavefronts

Substituting $U(x+ct)$ into (1.1) and denoting $x+ct$ as ξ, we obtain the following wave profile equation

$$cU' = J * U - U - dU + \int_{\mathbb{R}} K(y)b(U(\xi - y - c\tau))dy \tag{2.1}$$

with the boundary conditions

$$U(-\infty) = 0, \quad U(+\infty) = 1. \tag{2.2}$$

In this section, we shall use a homotopy method, i.e., continuation method to prove the existence of traveling wavefronts of (1.1). The main ideas of this method can be described in the following three steps:

Step 1. We embed (2.1) into a family of equations continuously parameterized by $\theta \in [0, 1]$ as follows:

$$\theta(J * U - U) + (1 - \theta)U'' - cU' - dU + \int_{\mathbb{R}} K(y)b(U(\xi - y - c\tau))dy = 0. \tag{2.3}$$

When $\theta = 0$, the equation (2.3) is already known to admit a unique (up to translation) traveling wavefronts (see [20]), and when $\theta = 1$, the equation (2.3) becomes (2.1).

Step 2. Applying a continuation argument given by the Implicit Function Theorem, we pass in increments from 0 to 1 in θ, obtaining existence for all values in the process.

Step 3. We extract a converging sequence when θ goes to 1.

Lemma 2.1. *For $\theta = 0$, (2.3) has a unique non-decreasing solution U satisfying $0 < U'(\xi) \le \frac{b(1)}{2\sqrt{d}}$ for all $\xi \in \mathbb{R}$.*

The proof can be found in [20, Theorem 4.3, Lemma 2.5] and so is omitted.

Lemma 2.2. *Let $\theta \in (0,1)$ and U satisfy (2.3) and (2.2). Then $U(\xi) \in (0,1)$ for all $\xi \in \mathbb{R}$.*

Proof. Firstly, it is clear that any L^∞ solution of (2.3) is of class C^3 . If U reaches its global maximum at ξ_0 with $U(\xi_0) \ge 1$, then $U'(\xi_0) = 0$, $U''(\xi_0) \le 0$ and $U(\xi) \le U(\xi_0)$ for all $\xi \in \mathbb{R}$, which together with $\int_{\mathbb{R}} J(x)dx = 1$ and $\int_{\mathbb{R}} K(x)dx = 1$ imply that

$$(J * U - U)(\xi_0) \le 0, \tag{2.4}$$

$$\int_{\mathbb{R}} K(y)b(U(\xi_0 - y - c\tau))dy \le \int_{\mathbb{R}} K(y)b(U(\xi_0))dy = b(U(\xi_0)) \le dU(\xi_0), \tag{2.5}$$

and by (2.3), one has

$$\theta(J * U - U)(\xi_0) - dU(\xi_0) + \int_{\mathbb{R}} K(y)b(U(\xi_0 - y - c\tau))dy \ge 0. \tag{2.6}$$

Taking into account (2.5), we further get from (2.6) that

$$\theta(J * U - U)(\xi_0) \ge 0. \tag{2.7}$$

Combining (2.4) and (2.7), we obtain that $(J * U - U)(\xi_0) = 0$. That is,

$$(J * U - U)(\xi_0) = \int_{\mathbb{R}} J(y - \xi_0)(U(y) - U(\xi_0))dy = 0,$$

which implies that $U(y) = U(\xi_0)$ for all $y \in \xi_0 + \operatorname{supp}(J)$. By an iteration of this process, one can show that $U(y) \equiv U(\xi_0)$ for all $y \in \mathbb{R}$, which contradicts to the fact that U is not a constant. Hence, we obtain that $U(\xi) < 1$ for all $\xi \in \mathbb{R}$. A similar argument shows that $U(\xi) > 0$ for all $\xi \in \mathbb{R}$. The proof is complete. $\square$

Now assume that (U_0, c_0) is a solution of (2.3) and (2.2) for some $\theta_0 \in [0,1)$ and that $U_0'(\xi) > 0$ for all $\xi \in \mathbb{R}$. We shall apply the Implicit Function Theorem to obtain a solution for $\theta > \theta_0$.

We take perturbations in the space:

$$X_0 = \{\text{uniformly continuous functions on } \mathbb{R} \text{ which vanish at } \pm\infty\}.$$

Let $\mathcal{L} = \mathcal{L}(U_0, c_0; \theta_0)$ be the linear operator defined in X_0 by

$$\begin{aligned}\mathcal{L}v = {}& \theta_0(J * v - v) + (1 - \theta_0)v'' - c_0 v' - dv \\ & + \int_{\mathbb{R}} K(y)b'(U_0(\cdot - y - c_0\tau))v(\cdot - y - c_0\tau)dy,\end{aligned}$$

where

$$\operatorname{dom}(\mathcal{L}) = X_1 \equiv \{v \in X_0 : v'' \in X_0\}.$$

Lemma 2.3. *$\mathcal{L}$ has 0 as a simple eigenvalue.*

Proof. It is easy to see that $\mathcal{L}U_0' = 0$, which means that 0 is an eigenvalue of $\mathcal{L}$ with eigenfunction U_0'. Thus, we need only to prove the simplicity of the eigenvalue 0. Suppose that ϕ is another eigenfunction with eigenvalue 0 and assume also that ϕ

is positive at some points. We shall show that U_0' and ϕ are linearly dependent by considering the family of eigenfunctions

$$\phi_\beta = U_0' + \beta\phi, \quad \beta \in \mathbb{R}.$$

Let

$$\bar\beta = \sup\{\beta < 0 : \phi_\beta(\xi) < 0 \text{ for some } \xi \in \mathbb{R}\}.$$

Then $\bar\beta$ is well defined since ϕ is positive at some points and $U_0' > 0$ on $\mathbb{R}$. For $\beta < \bar\beta$, let ξ_β be a point where ϕ_β achieves its negative minimum. Thus, $(J*\phi_\beta - \phi_\beta)(\xi_\beta) \geq 0$ and $\phi_\beta''(\xi_\beta) \geq 0$ and $\phi_\beta'(\xi_\beta) = 0$. In fact, $(J*\phi_\beta - \phi_\beta)(\xi_\beta) > 0$, since otherwise ϕ_β becomes a constant. It then follows that

$$\begin{aligned}
&\theta_0(J*\phi_\beta - \phi_\beta)(\xi_\beta) + (1-\theta_0)\phi_\beta''(\xi_\beta) - d\phi_\beta(\xi_\beta)\\
&+ \int_{\mathbb{R}} K(y) b'(U_0(\xi_\beta - y - c_0\tau))\phi_\beta(\xi_\beta - y - c_0\tau)dy = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 \geq d\phi_\beta(\xi_\beta) &\geq \int_{\mathbb{R}} K(y) b'(U_0(\xi_\beta - y - c_0\tau))\phi_\beta(\xi_\beta - y - c_0\tau)dy\\
&\geq \phi_\beta(\xi_\beta)\int_{\mathbb{R}} K(y) b'(U_0(\xi_\beta - y - c_0\tau))dy,
\end{aligned}$$

which implies

$$\int_{\mathbb{R}} K(y) b'(U_0(\xi_\beta - y - c_0\tau))dy \geq d. \tag{2.8}$$

It is easy to verify that $\{\xi_\beta\}_{\beta<\bar\beta}$ is bounded. Indeed, suppose that there exists a sequence $\{\beta_n\}$ with $\beta_n < \bar\beta$ such that $|\xi_{\beta_n}| \to +\infty$ as $n \to \infty$. Then without loss of generality, we assume that $\xi_{\beta_n} \to +\infty$. By Lebesgue's dominated convergence theorem, we obtain from (2.8) that $b'(1) \geq d$, which contradicts to the assumption (H5).

Thus, we choose $\{\beta_n\}_{n\in\mathbb{N}}$, a sequence which converges to $\bar\beta$. Let $\{\xi_{\beta_n}\}_{n\in\mathbb{N}}$ be the corresponding sequence of negative minimum. Since $\{\xi_{\beta_n}\}_{n\in\mathbb{N}}$ is bounded sequence in $\mathbb{R}$, we can therefore extract a converging sub-sequence $\{\xi_{\beta_{n_k}}\}_{k\in\mathbb{N}}$ such that $\xi_{\beta_{n_k}}$ converges to some $\bar\xi$. Observe that $\phi_{\bar\beta}(\bar\xi) = 0 \leq \phi_{\bar\beta}(\xi)$ for all $\xi \in \mathbb{R}$ and $\phi_{\bar\beta}'(\bar\xi) = 0$. Thus, we obtain at $\bar\xi$ that

$$(J*\phi_{\bar\beta} - \phi_{\bar\beta})(\bar\xi) \geq 0, \quad \phi_{\bar\beta}''(\bar\xi) \geq 0,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} K(y) b'(U_0(\bar\xi - y - c_0\tau))\phi_{\bar\beta}(\bar\xi - y - c_0\tau)dy\\
&\geq \phi_{\bar\beta}(\bar\xi)\int_{\mathbb{R}} K(y) b'(U_0(\bar\xi - y - c_0\tau))dy = 0.
\end{aligned}$$

We also have

$$\begin{aligned}
&\theta_0(J*\phi_{\bar\beta} - \phi_{\bar\beta})(\bar\xi) + (1-\theta_0)\phi_{\bar\beta}''(\bar\xi)\\
&+ \int_{\mathbb{R}} K(y) b'(U_0(\bar\xi - y - c_0\tau))\phi_{\bar\beta}(\bar\xi - y - c_0\tau)dy = 0.
\end{aligned}$$

It then follows that

$$(J*\phi_{\bar\beta} - \phi_{\bar\beta})(\bar\xi) = 0.$$

By a similar argument as in the proof of Lemma 2.2, we obtain that $\phi_{\bar\beta} \equiv 0$. Hence, U_0' and ϕ are linearly dependent. The proof is complete. $\square$

The formal adjoint of $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}^* v &= \theta_0(J * v - v) + (1-\theta_0)v'' + c_0 v' - dv \\ &\quad + \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) v(\cdot - y - c_0\tau)dy.\end{aligned}$$

It is easy to show that 0 is also a simple eigenvalue of $\mathcal{L}^*$, and $U_0'(-\xi)$ is an eigenfunction corresponding to 0. Moreover, 0 is an isolated eigenvalue, since the same holds for the operator $\mathcal{M}$:

$$\mathcal{M}v = v'' + c_0 v' - dv + \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) v(\cdot - y - c_0\tau)dy$$

and the added term $\theta_0(J * v - v)$ leaves the essential spectrum unchanged (see [1]). By the Fredholm Alternative, for $f \in X_0$, $\mathcal{L}u = f$ has a solution in X_1 if and only if $\int_{\mathbb{R}} f\phi^* dx = 0$ where ϕ^* is the eigenfunction associated to the eigenvalue 0 of $\mathcal{L}^*$.

We now state the continuation result.

Lemma 2.4. *Let (U_0, c_0) be a solution of (2.3) and (2.2) such that $U_0' > 0$. Then there exists $\eta > 0$ such that for $\theta \in [\theta_0, \theta_0 + \eta)$, the problem (2.3) and (2.2) has a solution (U, c).*

Proof. We shall use the Implicit Function Theorem. Without loss of generality, we may assume $U_0(0) = \alpha$. For $(v, c) \in X_1 \times \mathbb{R}$ and $\theta \in \mathbb{R}$, we define

$$\begin{aligned}G(v, c, \theta) &= \Big(\theta(J * (U_0 + v) - (U_0 + v)) + (1-\theta)(U_0 + v)'' - (c_0 + c)(U_0 + v)' \\ &\quad - d(U_0 + v) + \int_{\mathbb{R}} K(y) b((U_0 + v)(\cdot - y - (c_0 + c)\tau))dy, \quad (U_0 + v)(0)\Big).\end{aligned}$$

Clearly, $G : X_1 \times \mathbb{R} \times \mathbb{R} \to X_0 \times \mathbb{R}$ is of class C^1. Also, we have $G(0, 0, \theta_0) = (0, U_0)$ and

$$\begin{aligned}DG :&= \frac{\partial G}{\partial(v, c)}(0, 0, \theta_0) \\ &= \begin{pmatrix} \mathcal{L} & -U_0' - \tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau)dy \\ \delta & 0 \end{pmatrix},\end{aligned}$$

where $\delta v = v(0)$.

If we can show that $DG : X_1 \times \mathbb{R} \to X_0 \times \mathbb{R}$ is invertible, then the lemma would follow from the Implicit Function Theorem. To this end, let $(g, b) \in X_0 \times \mathbb{R}$. We want to show the existence of a unique $(v, c) \in X_1 \times \mathbb{R}$ solving

$$DG(v, c) = (g, b).$$

That is,

$$\mathcal{L}v - cU_0' - c\tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau)dy = g, \tag{2.9}$$

$$v(0) = b. \tag{2.10}$$

As we observed above, (2.9) is solvable if and only if

$$-c\int_{\mathbb{R}} \Big(U_0' + \tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau)dy\Big)\phi^* = \int_{\mathbb{R}} g\phi^*. \tag{2.11}$$

We shall prove that the integral on the left of (2.11) is not zero. Suppose for the contrary that this is not true, then there exists $v_0 \in X_1$ such that

$$\mathcal{L}v_0 = U_0' + \tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau) dy. \tag{2.12}$$

Multiplying (2.12) by $U_0'(-\xi)$ and integrating over $\mathbb{R}$ yield

$$0 = \int_{\mathbb{R}} U_0' \mathcal{L}v_0 = \int_{\mathbb{R}} U_0' \Big\{ U_0' + \tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau) dy \Big\} > 0,$$

which leads to a contradiction. Hence, (2.11) holds.

Furthermore, (2.11) determines

$$c = -\frac{\int_{\mathbb{R}} g\phi^*}{\int_{\mathbb{R}} \left(U_0' + \tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau) dy \right) \phi^*}.$$

With this value of c, the solution of (2.9) is determined up to an additive term $\sigma U_0'$, where $\sigma \in \mathbb{R}$. That means, any solution of $\mathcal{L}\tilde{v} = g + cU_0' + c\tau \int_{\mathbb{R}} K(y) b'(U_0(\cdot - y - c_0\tau)) U_0'(\cdot - y - c_0\tau) dy$ can be written as $\tilde{v} = v + \sigma U_0'$, where v is the solution of (2.9). Now (2.10) is satisfied by a unique choice of σ since $U_0'(0) > 0$. Thus, DG is invertible. This completes the proof. □

Remark 2.5. We need to point out that the solution U_θ obtained by the Implicit Function Theorem also satisfies the boundary condition (2.2).

To prove Lemma 2.4, we needed the condition $U_0' > 0$. Thus, if we want to apply this lemma we must to show that for all $\theta \in [\theta_0, \theta_0 + \eta]$, any smooth solution U_θ of (2.3) previously constructed satisfies $U_\theta' > 0$.

Lemma 2.6. *Let $\theta \in [\theta_0, \theta_0 + \eta)$ and (U_θ, c_θ) be the solution given above. Then $U_\theta'(\xi) > 0$ for all $\xi \in \mathbb{R}$.*

Proof. We first prove that $U_\theta'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. By contradiction, we assume that there exists $\theta \in [\theta_0, \theta_0 + \eta)$ such that there exists $\xi \in \mathbb{R}$ with $U_\theta'(\xi) < 0$. Let

$$\bar{\theta} = \inf\{\theta > \theta_0; \quad U_\theta'(\xi) < 0 \text{ for some } \xi \in \mathbb{R}\}.$$

It is well defined, since $\theta_0 \leq \bar{\theta} < \theta_0 + \eta$. It also implies that $U_{\bar{\theta}}$ exists. From the definition of $\bar{\theta}$, there exists a decreasing sequence $\theta_n \to \bar{\theta}$ on which $U_{\theta_n}'(\xi)$ has a negative minimum at some point ξ_{θ_n}. At this minimum, U_{θ_n}' satisfies

$$\begin{aligned} &\theta_n (J * U_{\theta_n}' - U_{\theta_n}')(\xi_{\theta_n}) + (1 - \theta_n) U_{\theta_n}'''(\xi_{\theta_n}) \\ &+ \int_{\mathbb{R}} K(y) b'(U_{\theta_n}(\xi_{\theta_n} - y - c_{\theta_n}\tau)) U_{\theta_n}'(\xi_{\theta_n} - y - c_{\theta_n}\tau) dy = 0. \end{aligned} \tag{2.13}$$

From Lemma 2.4, we obtain that $U_{\theta_n} \to U_{\bar{\theta}}$ uniformly, and the sequence $\{\xi_{\theta_n}\}$ is bounded. Hence, we can extract a subsequence which converges to $\bar{\xi}$. It is easy to see that $U_{\bar{\theta}}'(\bar{\xi}) = 0 \leq U_{\bar{\theta}}'(\xi)$ for all $\xi \in \mathbb{R}$. Hence, $U_{\bar{\theta}}''(\bar{\xi}) = 0$ and $U_{\bar{\theta}}'''(\bar{\xi}) \geq 0$. By taking $n \to \infty$ in (2.13), we have

$$\begin{aligned} 0 = &\bar{\theta}(J * U_{\bar{\theta}}' - U_{\bar{\theta}}')(\bar{\xi}) + (1 - \bar{\theta}) U_{\bar{\theta}}'''(\bar{\xi}) \\ &+ \int_{\mathbb{R}} K(y) b'(U_{\bar{\theta}}(\bar{\xi} - y - c_{\bar{\theta}}\tau)) U_{\bar{\theta}}'(\bar{\xi} - y - c_{\bar{\theta}}\tau) dy. \end{aligned} \tag{2.14}$$

Since $U_{\bar{\theta}}(\xi) \in (0, 1)$ for all $\xi \in \mathbb{R}$, we have $b'(U_{\bar{\theta}}(\xi)) > 0$. Hence,

$$\int_{\mathbb{R}} K(y) b'(U_{\bar{\theta}}(\bar{\xi} - y - c_{\bar{\theta}}\tau)) U_{\bar{\theta}}'(\bar{\xi} - y - c_{\bar{\theta}}\tau) dy \geq 0.$$

It then follows from (2.14) that $(J * U'_{\bar{\theta}} - U'_{\bar{\theta}})(\bar{\xi}) = 0$. This implies that $U'_{\bar{\theta}}(\xi) \equiv 0$. That means that $U_{\bar{\theta}}$ is a constant. This is impossible. The proof is complete. $\square$

To continue the solution branch to $\theta \in [0, 1)$, we need some a priori estimates on the solution U_θ of (2.3) for $\theta \in [0, 1)$.

Lemma 2.7. *Suppose that for $\theta \in [0, \bar{\theta})$, there exists a solution (U_θ, c_θ) of (2.3) and (2.2). Then $\{c_\theta : \theta \in [0, \bar{\theta})\}$ is bounded.*

Proof. We show this by contradiction. Suppose that this set is unbounded. Then there would exist a sequence $\{\theta_n\}$ with $c_n \equiv c_{\theta_n} \to \pm\infty$ as $n \to \infty$. For the sake of convenience, we write $U_n \equiv U_{\theta_n}$. Since $U'_n(\xi) \to 0$ as $|\xi| \to +\infty$, $|U'_n(\xi)|$ achieves its maximum value at some point ξ_n of $\mathbb{R}$. At ξ_n, we have $U''(\xi_n) = 0$ and

$$\begin{aligned}
&\|c_n U'_n\|_{L^\infty(\mathbb{R})} = |c_n U'_n(\xi_n)| \\
&= \big|\theta_n (J * U_n - U_n)(\xi_n) - dU_n(\xi_n) + \int_{\mathbb{R}} K(y) b(U_n(\xi_n - y - c_n\tau)) dy\big| \qquad (2.15)\\
&\le 2 + d + b(1).
\end{aligned}$$

It then follows that

$$\|U'_n\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$

We now assert that for any $\epsilon > 0$ and any closed interval $I \subset (0, 1)$ of positive length there exists ξ_n such that $U_n(\xi_n) \in I$ and $|U''_n(\xi_n)| < \epsilon$. If this were not the case, there would exist such an interval I_0 and a number $\epsilon_0 > 0$ such that $|U''_n| \ge \epsilon_0$ on the interval $[a_n, b_n]$, where $U_n([a_n, b_n]) = I_0$. Then

$$2\|U'_n\|_{L^\infty(\mathbb{R})} \ge |U'_n(b_n) - U'_n(a_n)| = |U''_n(b_n - a_n)| \ge \epsilon(b_n - a_n), \qquad (2.16)$$

and by the Mean Value Theorem, the length of I_0 is

$$|I_0| = U_n(\bar{b}_n) - U_n(\bar{a}_n) \le \|U'_n\|_{L^\infty(\mathbb{R})}(\bar{b}_n - \bar{a}_n) \le \|U'_n\|_{L^\infty(\mathbb{R})}(b_n - a_n), \qquad (2.17)$$

where $\bar{a}_n, \bar{b}_n \in [a_n, b_n]$ with

$$U_n(\bar{a}_n) = \min_{\xi \in [a_n, b_n]} U_n(\xi) \quad \text{and} \quad U_n(\bar{b}_n) = \max_{\xi \in [a_n, b_n]} U_n(\xi).$$

Combining (2.16) and (2.17), we obtain that $2\|U'_n\|^2_{L^\infty(\mathbb{R})} \ge \epsilon_0 |I_0|$, which contradicts to the fact that $\|U'_n\|_{L^\infty(\mathbb{R})} \to 0$ as $n \to \infty$. Thus, the assertion is established.

Now take $r > 0$ small and let I be such that

$$du - b(u) \le -r \quad \text{for all } u \in I$$

in the case that $c_n \to -\infty$, and such that

$$du - b(u) \ge r \quad \text{for all } u \in I$$

in the case that $c_n \to +\infty$. Take $\epsilon = r/2$ and $\{\xi_n\}$ to be the sequence given by the assertion above. Without loss of generality, we assume that $c_n \to +\infty$, then (2.3) with $\theta = \theta_n$, $c = c_n$ and $U = U_n$ evaluated at ξ_n gives

$$\begin{aligned}
-r &\ge -c_n U'_n - dU_n(\xi_n) + b(U_n(\xi_n)) \\
&\ge -c_n U'_n - dU_n(\xi_n) + b(U_n(\xi_n - c_n\tau)) \\
&\ge -(J * U_n - U_n)(\xi_n) - (1 - \theta_n) U''_n(\xi_n) \\
&\quad - \int_{\mathbb{R}} [b(U_n(\xi_n - y - c_n\tau)) - b(U_n(\xi_n - c_n\tau))] K(y) dy
\end{aligned}$$

$$
\begin{aligned}
&\geq -|(J*U_n - U_n)(\xi_n)| - |U_n''(\xi_n)| - b'_{\max}\|U_n'\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|y|K(y)dy \\
&\geq -\|U_n'\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|y|J(y)dy - \epsilon - b'_{\max}\|U_n'\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|y|K(y)dy.
\end{aligned}
$$

Since $\|U_n'\|_{L^\infty(\mathbb{R})} \to 0$ as $n \to \infty$ and $\int_{\mathbb{R}}|y|J(y)dy < +\infty$, and $\int_{\mathbb{R}}|y|K(y)dy < +\infty$, taking $n \to \infty$, we have $r \leq \epsilon = r/2$, a contradiction. The proof is complete. $\square$

Lemma 2.8. *Suppose that for $\theta \in [0, \bar{\theta})$, there exists a solution (U_θ, c_θ) of (2.3) and (2.2). Then $\{U_\theta : \theta \in [0, \bar{\theta})\}$ is bounded in $C^3(\mathbb{R})$.*

Proof. It follows from (2.3) that $v_\theta \equiv U_\theta'$ satisfies

$$
\begin{aligned}
&\theta(J*v_\theta - v_\theta) + (1-\theta)v_\theta'' - c_\theta v_\theta' - dv_\theta \\
&+ \int_{\mathbb{R}} K(y)b'(U_\theta(\xi - y - c\tau))v_\theta(\xi - y - c\tau)dy = 0.
\end{aligned} \tag{2.18}
$$

Notice that

$$
\begin{aligned}
&\int_{\mathbb{R}} K(y)b'(U_\theta(\xi - y - c_\theta\tau))U_\theta'(\xi - y - c_\theta\tau)dy \\
&= -b(U_\theta(\xi - y - c_\theta\tau))K(y)|_{-\infty}^{+\infty} + \int_{\mathbb{R}} b(U_\theta(\xi - y - c_\theta\tau))K'(y)dy \\
&= \int_{\mathbb{R}} b(U_\theta(\xi - y - c_\theta\tau))K'(y)dy.
\end{aligned}
$$

Then equation (2.18) becomes

$$
\begin{aligned}
&\theta(J*v_\theta - v_\theta) + (1-\theta)v_\theta'' - c_\theta v_\theta' - dv_\theta \\
&+ \int_{\mathbb{R}} b(U_\theta(\xi - y - c_\theta\tau))K'(y)dy = 0.
\end{aligned} \tag{2.19}
$$

Since $U_\theta'(\xi) \to 0$ as $|\xi| \to \infty$, $v_\theta \equiv U_\theta'$ achieves its positive maximum at some point $\xi_\theta \in \mathbb{R}$, which implies that $v_\theta'(\xi_\theta) = 0$ and $v_\theta''(\xi_\theta) < 0$. Thus, we obtain from (2.19) that

$$
0 < dv_\theta(\xi_\theta) \leq \int_{\mathbb{R}} b(U_\theta(\xi_\theta - y - c_\theta\tau))K'(y)dy \leq b(1)\int_{\mathbb{R}}|K'(y)|dy.
$$

Hence,

$$
\|v_\theta\|_{L^\infty(\mathbb{R})} = v_\theta(\xi_\theta) \leq \frac{b(1)}{d}\int_{\mathbb{R}}|K'(y)|dy.
$$

Differentiating (2.19), one has

$$
\begin{aligned}
&\theta(J*v_\theta' - v_\theta') + (1-\theta)v_\theta''' - c_\theta v_\theta'' - dv_\theta' \\
&+ \int_{\mathbb{R}} K'(y)b'(U_\theta(\xi - y - c_\theta\tau))U_\theta'(\xi - y - c_\theta\tau)dy = 0.
\end{aligned} \tag{2.20}
$$

Since

$$
\begin{aligned}
&\int_{\mathbb{R}} K'(y)b'(U_\theta(\xi - y - c_\theta\tau))U_\theta'(\xi - y - c_\theta\tau)dy \\
&= -b(U_\theta(\xi - y - c_\theta\tau))K'(y)|_{-\infty}^{+\infty} + \int_{\mathbb{R}} b(U_\theta(\xi - y - c_\theta\tau))K''(y)dy \\
&= \int_{\mathbb{R}} b(U_\theta(\xi - y - c_\theta\tau))K''(y)dy,
\end{aligned}
$$

equation (2.20) reduces to

$$\begin{aligned}&\theta(J*v_\theta'-v_\theta')+(1-\theta)v_\theta'''-c_\theta v_\theta''-dv_\theta'\\&+\int_{\mathbb{R}}b(U_\theta(\xi-y-c_\theta\tau))K''(y)dy=0.\end{aligned}\tag{2.21}$$

It is easy to see that $v_\theta'(\xi)\to 0$ as $|\xi|\to\infty$. Thus, $|v_\theta'|$ achieves its maximum at some point $\chi_\theta\in\mathbb{R}$. Without loss of generality, we assume $v_\theta'(\chi)\geq 0$. Then $v_\theta''(\chi_\theta)=0$ and $v_\theta'''(\chi_\theta)\leq 0$. Thus, we have from (2.21) that

$$dv_\theta'(\chi_\theta)\leq\int_{\mathbb{R}}b(U_\theta(\xi_\theta-y-c_\theta\tau))K''(y)dy\leq b(1)\int_{\mathbb{R}}|K''(y)|dy,$$

which shows that

$$\|v_\theta'\|_{L^\infty(\mathbb{R})}=v_\theta'(\chi_\theta)\leq\frac{b(1)}{d}\int_{\mathbb{R}}|K''(y)|dy.$$

Furthermore, differentiating (2.21), we get

$$\begin{aligned}&\theta(J*v_\theta''-v_\theta'')+(1-\theta)v_\theta^{(4)}-c_\theta v_\theta'''-dv_\theta''\\&+\int_{\mathbb{R}}b'(U_\theta(\xi-y-c_\theta\tau))v_\theta(\xi-y-c_\theta\tau)K''(y)dy=0.\end{aligned}$$

By a similar argument, we obtain

$$\|v_\theta''\|_{L^\infty(\mathbb{R})}\leq\frac{1}{d}b'_{\max}\|v_\theta\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|K''(y)|dy.$$

The proof is complete. □

Proof of Theorem 1.1. Since for $\theta=0$ there exists a positive increasing solution U_0, we may apply Lemma 2.4 to get the existence of a solution U_θ of (2.3) and (2.2) for each $\theta\in(0,\upsilon)$ for some $\upsilon>0$. By Lemma 2.6, we further obtain that $U_\theta'(\xi)>0$ for $\xi\in\mathbb{R}$. Define

$$\bar\theta=\sup\{\theta>0:\ \text{there exists a positive increasing solution } U_\theta \text{ of (2.3)}\}.$$

Clearly, $\bar\theta\geq\upsilon$. We shall show that $\bar\theta\geq 1$. We argue by contradiction, assume that $\bar\theta<1$. Choose a sequence $(\theta_n)_{n\in\mathbb{N}}$ such that $\theta_n\to\bar\theta$, and for each n, (2.3) has a positive increasing solution denoted by (U_n,c_n). Recall that (U_n,c_n) satisfies

$$\begin{gathered}\theta_n(J*U_n-U_n)+(1-\theta_n)U_n''-c_nU_n'-dU_n\\+\int_{\mathbb{R}}K(y)b(U_n(\xi-y-c_n\tau))dy=0,\\U_n(-\infty)=0,\quad U_n(+\infty)=1.\end{gathered}\tag{2.22}$$

Without loss of generality we may also normalize U_n by $U_n(0)=\alpha$. By Lemmas 2.7 and 2.8, there exists a positive constant C independent of n such that for each n we have $\|U_n\|_{C^3(\mathbb{R})}\leq C$ and $|c_n|\leq C$. Since $\{c_n\}_{n\in\mathbb{N}}$ is bounded, we can extract a converging subsequence $\{c_{n_j}\}_{j\in\mathbb{N}}$, such that c_{n_j} converges to some real number $\bar c$. Let $\{U_{n_j}\}$ be the corresponding wave profile of wave speed c_{n_j}. Note that $\{U_{n_j}\}$ consists of positive uniformly bounded increasing functions. By Helly's theorem and C^3 estimates, it then follows that there exists a subsequence, still denoted by $\{U_{n_j}\}$, which converges pointwise and C^2_{loc} to a positive smooth function $\bar U$ as

$j \to +\infty$. Hence, $(U_{n_j}, c_{n_j}) \to (\bar{U}, \bar{c})$ as $j \to +\infty$. Therefore, by letting $j \to +\infty$ in (2.22) with $n = n_j$, we get

$$\bar{\theta}(J * \bar{U} - \bar{U}) + (1-\bar{\theta})\bar{U}'' - \bar{c}\bar{U}' - d\bar{U} + \int_{\mathbb{R}} K(y) b(\bar{U}(\xi - y - \bar{c}\tau)) dy = 0. \tag{2.23}$$

Clearly, this solution satisfies $\bar{U}' \geq 0$. Therefore, if $\bar{\theta} < 1$ and $\bar{U}$ satisfies (2.2), then by Lemma 2.2, $\bar{U}(\xi) \in (0,1)$, and hence the proof of Lemma 2.6 again shows that $\bar{U}' > 0$. Since we have assume that $\bar{\theta} < 1$, then it can be shown that there exists a positive increasing solution of (2.3) for $\theta \in [0, \bar{\theta} + \eta)$ for some positive η by applying Lemma 2.4 again with $\bar{U}$ instead of U_0. It leads to a contradiction with the definition of $\bar{\theta}$. Thus, $\bar{\theta} \geq 1$ and the equation (2.3) has a solution for every $\theta \in [0,1)$.

We can get a solution for $\theta = 1$ in the same way above. Let $(\theta_n)_{n\in\mathbb{N}}$ such that $\theta_n \to 1$ and $(U_n, c_n)_{n\in\mathbb{N}}$ be the corresponding normalized sequence of solution. From Lemmas 2.7 and 2.8, we have $\|U_n\|_{C_2(\mathbb{R})} \leq C$ and $|c_n| \leq C$ for some positive constant C. Thus, by Helly's theorem and a priori estimate, there exists a non-decreasing function $\hat{U}$ and a constant $\hat{c}$ such that $U_{\theta_n} \to \hat{U}$ pointwise and $c_n \to \hat{c}$. From the C^2 estimates, up to extraction, we have $U_{\theta_n} \to \hat{U}$ in C^1_{loc}. Therefore, $\hat{U}$ satisfies (2.1), i.e.,

$$J * \hat{U} - \bar{U} - \hat{c}\hat{U}' - d\hat{U} + \int_{\mathbb{R}} K(y) b(\hat{U}(\xi - y - \hat{c}\tau)) dy = 0. \tag{2.24}$$

It remains to prove that $\hat{U}$ satisfies the boundary condition (2.2). This will be done with the proof of the assumption below (2.23), i.e., $\bar{U}$ satisfies (2.2). Since $\bar{U}$ is positive bounded non-decreasing function, it admits limits as $\xi \to \pm\infty$. By Lebesgue's dominated convergence theorem, we see from (2.24) that these limits are zeros of the function $du - b(u)$, $u \in [0,1]$.

Suppose that $\bar{c} \geq 0$. Notice that α is the intermediate zero of $du - b(u)$. Take $\bar{\alpha} \in (0, \alpha)$ and translate U_θ so that $U_\theta(0) = \bar{\alpha}$ for each θ. We still may take a sequence of $\theta \to \bar{\theta}$, a subsequence of the original one, so that U_θ converges pointwise to some $\bar{U}$. Since c is independent of translations, we still have $c_\theta \to \bar{c}$. Then $\lim_{\xi\to-\infty} \bar{U}(\xi) = 0$ and $\lim_{\xi\to+\infty} \bar{U}(\xi) \in \{\alpha, 1\}$. If $\lim_{\xi\to+\infty} \bar{U}(\xi) = 1$, then we are done. Hence, we now assume that $\lim_{\xi\to+\infty} \bar{U}(\xi) = \alpha$. Due to the monotonicity of $\bar{U}$, it implies that $d\bar{U}(\xi) - b(\bar{U}(\xi)) > 0$ for $\xi \in \mathbb{R}$.

If $\bar{\theta} < 1$, from the above discussion, we can see that $\bar{U}$ is of class C^2 and satisfies (2.3). Hence,

$$\begin{aligned}
0 &< \int_{-M}^{M} (d\bar{U}(\xi) - b(\bar{U}(\xi)))\, d\xi \leq \int_{-M}^{M} (d\bar{U}(\xi) - b(\bar{U}(\xi - \bar{c}\tau)))\, d\xi \\
&= \int_{-M}^{M} \Big[\bar{\theta}(J * \bar{U} - \bar{U})(\xi) + (1-\bar{\theta})\bar{U}''(\xi) - \bar{c}\bar{U}'(\xi) \\
&\quad + \int_{\mathbb{R}} [b(U(\xi - y - \bar{c}\tau)) - b(U(\xi - \bar{c}\tau))] K(y) dy\Big]\, d\xi.
\end{aligned} \tag{2.25}$$

It is easy to show that

$$\int_{-M}^{M} (J * \bar{U} - \bar{U})(\xi)\, d\xi = \int_{-M}^{M} \int_{\mathbb{R}} J(y)[\bar{U}(\xi - y) - \bar{U}(\xi)] dy\, d\xi$$

$$
\begin{aligned}
&= -\int_{-M}^{M}\int_{\mathbb{R}} J(y)\int_0^1 y\bar{U}'(\xi - ty)\,dt\,dy\,d\xi \\
&= -\int_{\mathbb{R}} yJ(y)\int_0^1 \Big(\int_{-M}^{M} U'(\xi - ty)\,d\xi\Big)\,dt\,dy \\
&= -\int_{\mathbb{R}} yJ(y)\int_0^1 (\bar{U}(M - ty) - \bar{U}(-M - ty))\,dt\,dy
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-M}^{M}\int_{\mathbb{R}}[b(U(\xi - y - \bar{c}\tau)) - b(U(\xi - \bar{c}\tau))]K(y)dy\,d\xi \\
&= -\int_{-M}^{M}\int_{\mathbb{R}} K(y)\int_0^1 yb'(\bar{U}(\xi - ty - \bar{c}\tau))\bar{U}'(\xi - ty - \bar{c}\tau)\,dt\,dy\,d\xi \\
&= -\int_{\mathbb{R}} yK(y)\int_0^1 [b(\bar{U}(M - ty - \bar{c}\tau)) - b(\bar{U}(-M - ty - \bar{c}\tau))]\,dt\,dy.
\end{aligned}
$$

Hence, from (2.25) it follows that

$$
\begin{aligned}
0 &< \int_{-M}^{M}[d\bar{U}(\xi) - b(\bar{U}(\xi))]\,d\xi \\
&\le -\bar{\theta}\int_{\mathbb{R}} yJ(y)\int_0^1 (\bar{U}(M - ty) - \bar{U}(-M - ty))\,dt\,dy \\
&\quad + (1-\bar{\theta})(\bar{U}'(M) - \bar{U}'(-M)) \\
&\quad - \int_{\mathbb{R}} yK(y)\int_0^1 [b(\bar{U}(M - ty - \bar{c}\tau)) - b(\bar{U}(-M - ty - \bar{c}\tau))]\,dt\,dy.
\end{aligned}
$$

Taking $M \to +\infty$ in the above inequality and using Fubini's Theorem, Lebesgue's Theorem and the evenness of J and K, we obtain

$$
\begin{aligned}
0 &< \int_{\mathbb{R}}[d\bar{U}(\xi) - b(\bar{U}(\xi))]\,d\xi \\
&\le -\bar{\theta}(1-\alpha)\int_{\mathbb{R}} yJ(y)dy - (b(1) - b(\alpha))\int_{\mathbb{R}} yK(y)dy = 0,
\end{aligned}
$$

which leads to a contradiction.

If $\bar{\theta} = 1$, then $\bar{U}$ satisfies (2.1). Similarly, by using Lebesgue's Theorem and the evenness of J and K, we can see from (2.25) that

$$
\begin{aligned}
0 &< \int_{\mathbb{R}}[d\bar{U}(\xi) - b(\bar{U}(\xi))]\,d\xi \\
&\le -(1-\alpha)\int_{\mathbb{R}} yJ(y)dy - (b(1) - b(\alpha))\int_{\mathbb{R}} yK(y)dy = 0.
\end{aligned}
$$

It is a contradiction.

For the case $\bar{c} < 0$, we can use a similar argument by taking $\bar{\alpha} \in (\alpha, 1)$. The proof is complete. $\square$

3. Initial value problem

Consider the initial value problem

$$\begin{gathered}\frac{\partial u}{\partial t} = J * u - u - du + \int_{\mathbb{R}} K(y) b(u(x-y, t-\tau)) dy, \quad x \in \mathbb{R}, t > 0, \\ u(x,s) = \varphi(x,s), \quad x \in \mathbb{R}, s \in [-\tau, 0].\end{gathered} \tag{3.1}$$

It can be seen from [14] that for the initial value problem (3.1), we have the following result on the existence of solutions.

Lemma 3.1. *Assume $\varphi(x,s) \in C([-\tau,0]; C(\mathbb{R}))$ with $0 \le \varphi(x,s) \le 1$ for $(x,s) \in \mathbb{R} \times [-\tau, 0]$. Then the solution to (3.1) uniquely and globally exists, and satisfies that $u(x,t) \in C^1([0,+\infty); C(\mathbb{R}))$, and $0 \le u(x,t) \le 1$ for $(x,t) \in \mathbb{R} \times [0,+\infty)$.*

Let X be the Banach space defined by

$$X = \{\varphi(x) | \varphi(x) : \mathbb{R} \to \mathbb{R} \text{ is uniformly continuous and bounded}\}$$

with the usual supremum norm $|\cdot|_X$. Let

$$X^+ = \{\varphi(x) \in X : \varphi(x) \ge 0, x \in \mathbb{R}\}.$$

It is easily seen that X^+ is a closed cone of X and X is a Banach lattice under the partial ordering induced by X^+.

$\int_{\mathbb{R}} J(x-y)[u(y) - u(x)] dy : X \to X$ is bounded linear operator with respect to the norm $|\cdot|_X$. Then

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)] dy, \\ u(x,0) = \varphi(x) \in X\end{gathered} \tag{3.2}$$

generates a strongly continuous analytic semigroup $T(t)$ on X and $T(t)X^+ \subset X^+$, that is $T(t)u(x) \gg 0$ if $u(x) \ge 0$ has a nonempty support and $t > 0$. Moreover, the mild solution of (3.2) can be given by $u(x,t) = T(t)\varphi(x)$. For more details, we can refer to Pan et al. [28]. The theory of the operator semigroup can be seen in Pazy [29].

For any $\varphi \in [0,1]_C = \{\varphi \in C : \varphi(x,s) \in [0,1], x \in \mathbb{R}, s \in [-\tau, 0]\}$, define

$$F(\varphi)(x) = -d\varphi(x,0) + \int_{\mathbb{R}} K(x-y) b(\varphi(y,-\tau)) dy, \quad x \in \mathbb{R}.$$

Since $b \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, we can verify that $F(\varphi) \in X$ and $F : [0,1]_C \to X$ is globally Lipschitz continuous.

From (H5), it can be seen that for $\phi \le \psi$,

$$\begin{aligned}&F(\psi)(x) - F(\phi)(x) \\ &= -d(\psi(x,0) - \phi(x,0)) + \int_{\mathbb{R}} K(x-y)[b(\psi(y,-\tau)) - b(\phi(y,-\tau))] dy \\ &\ge -d(\psi(x,0) - \phi(x,0)).\end{aligned} \tag{3.3}$$

Definition 3.2. A continuous function $v : [-\tau, l] \to X$, $l > 0$ is called a supersolution (subsolution) of (1.1) on $[0,l)$ if

$$v(t) \ge (\le) T(t-s)v(s) + \int_s^t T(t-r) F(v_r) dr \tag{3.4}$$

for all $0 \le s < t < l$. If v is both a supersolution and a subsolution on $[0, l)$, then it is said to be a mild solution of (3.1).

Remark 3.3. Assume that $v : [-\tau, l) \times \mathbb{R} \to \mathbb{R}$ with $l > 0$ and v is C in $x \in \mathbb{R}$, C^1 in $t \in [0, l)$, and satisfies the following inequality

$$\frac{\partial v}{\partial t} \ge (\le) J * v - v - dv + \int_{\mathbb{R}} K(y) b(v(x-y, t-\tau)) dy, \quad x \in \mathbb{R}, t > 0,$$

Then by the positivity of $T(t) : X^+ \to X^+$ implies that (3.4) holds. Hence, v is a supersolution (subsolution) of (1.1) on $[0, l)$.

Lemma 3.4. *For any supersolution $u^+(x,t)$ and subsolution $u^-(x,t)$ of* (1.1) *on $\mathbb{R} \times [0 + \infty)$ with $0 \le u^+(x,t), u^-(x,t) \le 1$ for $x \in \mathbb{R}$, $t \in [-\tau, +\infty)$, and $u^+(x,s) \ge u^-(x,s)$ for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, there holds $u^+(x,t) \ge u^-(x,t)$ for $x \in \mathbb{R}, t \ge 0$, and there exists a positive continuous function $\Theta(x,t)$ defined on $[0,+\infty) \times [0,+\infty)$ such that*

$$u^+(x,t) - u^-(x,t) \ge \Theta(|x|, t) \int_0^1 [u^+(y,0) - u^-(y,0)] dy$$

for $x \in \mathbb{R}$, $t > 0$ and $z \in \mathbb{R}$.

The proof of the comparison principle in Lemma 3.4 strongly depends on the analyticity and positivity of semigroup $T(t)$. For more details, we can refer the readers to [22, 23]. Hence, we omit the proof here. It can also be seen in [14] for another type of proof. Due to (3.3), the proof of the last inequality in Lemma 3.4 is only a minor modification of the proof of Lemma 2.3 in [37], so we omit it.

4. Uniqueness of traveling wavefronts

It is well known that standing waves (that is, traveling waves with speed $c = 0$) are not necessarily unique, see [4, 5]. Hence, we consider the uniqueness of traveling wavefronts only when $c \ne 0$.

Lemma 4.1. *Let $U(x + ct)$ be a non-decreasing traveling wavefront of* (1.1)*. Then for $c \ne 0$,*

$$0 < U'(\xi) \le \frac{1}{|c|}(1 + b(1)) \quad \text{for all } \xi \in \mathbb{R}, \tag{4.1}$$

$$\lim_{\xi \to \pm\infty} U'(\xi) = 0. \tag{4.2}$$

Proof. By Lemma 3.4, we have that for $\xi = x + ct$ and every $h > 0$,

$$U(\xi + h) - U(\xi) \ge \Theta(|x|, t) \int_0^1 [U(y+h) - U(y)] dy > 0.$$

Then,

$$U'(\xi) \ge \Theta(|x|, t)(U(1) - U(0)) > 0.$$

It is easy to see that for $c \ne 0$,

$$U(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{d+1}{c}(\xi - s)} H(U)(s) ds,$$

where

$$H(U)(\xi) = J * U(\xi) + \int_{\mathbb{R}} K(y) b(U(\xi - y, t - \tau)) dy.$$

Hence,

$$U'(\xi)=\frac{1}{c}H(U)(\xi)+\frac{1}{c}\int_{-\infty}^{\xi}\Big(-\frac{d+1}{c}\Big)e^{-\frac{d+1}{c}(\xi-s)}H(U)(s)ds, \tag{4.3}$$

When $c>0$, we obtain

$$U'(\xi)\le\frac{1}{c}H(U)(\xi)\le\frac{1}{c}(1+b(1)).$$

When $c<0$, one has

$$U'(\xi)\le\frac{1}{c}\int_{-\infty}^{\xi}\Big(-\frac{d+1}{c}\Big)e^{-\frac{d+1}{c}(\xi-s)}H(U)(s)ds\le-\frac{1}{c}(1+b(1)).$$

Thus, (4.1) is obtained. Finally, (4.2) follows from (2.2) and (4.3). The proof is complete. □

Lemma 4.2. *Let $U(x+ct)$ be a non-decreasing traveling wavefront of* (1.1) *with $c\neq 0$. Then there exist three positive numbers β_0 (which is independent of U), σ_0 and $\bar{\delta}$ such that for any $\delta\in(0,\bar{\delta}]$ and every $\xi_0\in\mathbb{R}$, the function w^+ and w^- defined by*

$$w^{\pm}(x,t):=U(x+ct+\xi_0\pm\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0 t}))\pm\delta e^{-\beta_0 t}$$

are a supersolution and a subsolution of (1.1) *on $[0,+\infty)$, respectively.*

Proof. By (H5), we can choose $\beta_0>0$ and $\epsilon^*>0$ such that

$$d>\beta_0+e^{\beta_0\tau}(\max\{b'(0),b'(1)\}+\epsilon^*).$$

Since $b'(u)\ge 0$ for $u\in[0,1]$, there exists a sufficiently small number $\delta^*>0$ such that

$$\begin{gathered}0\le b'(\eta)\le b'(0)+\epsilon^*\quad\text{for all }\eta\in[-\delta^*,\delta^*],\\ 0\le b'(\eta)\le b'(1)+\epsilon^*\quad\text{for all }\eta\in[1-\delta^*,1+\delta^*].\end{gathered}\tag{4.4}$$

Let $c_0=|c|\tau+(e^{\beta_0\tau}-1)$. By the boundary condition (2.2), there exists $M_0=M_0(U,\beta_0,\delta^*,\epsilon^*)>0$ such that

$$\begin{gathered}U(\xi)\le\delta^*\quad\text{for all }\xi\le -M_0/2+c_0,\\ U(\xi)\ge 1-\delta^*\quad\text{for all }\xi\ge M_0/2-c_0\end{gathered}\tag{4.5}$$

and

$$d>\beta_0+e^{\beta_0\tau}(\max\{b'(0),b'(1)\}+\epsilon^*)+e^{\beta_0\tau}b'_{\max}\Big[\int_{\frac{M_0}{2}}^{+\infty}+\int_{-\infty}^{-\frac{M_0}{2}}K(y)dy\Big]. \tag{4.6}$$

Set

$$m_0:=m_0(U,\beta_0,\delta^*,\epsilon^*)=\min\{U'(\xi):|\xi|\le M_0\}>0, \tag{4.7}$$

and define

$$\sigma_0:=\frac{1}{m_0\beta_0}\big\{(e^{\beta_0\tau}b'_{\max}-d)+\beta_0\big\}>0, \tag{4.8}$$

$$\bar{\delta}=\min\big\{\frac{1}{\sigma_0},\delta^*e^{-\beta_0\tau}\big\}. \tag{4.9}$$

We only prove that $w^+(x,t)$ is a supersolution of (1.1), since the similar argument can be used for $w^-(x,t)$. By a translation, we can assume that $\xi_0=0$. For any given $\delta\in(0,\bar{\delta}]$, let $\xi(x,t)=x+ct+\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0 t})$. Then for any $t\ge 0$, we have

$S(w^+)(x,t)$

$$
\begin{aligned}
&:= \frac{\partial w^+}{\partial t} - J * w^+ + w^+ + dw^+ - \int_{\mathbb{R}} K(y) b(w^+(x-y,t-\tau))dy \\
&= U'(\xi(x,t))\left(c + \sigma_0\delta\beta_0 e^{-\beta_0 t}\right) - \beta_0\delta e^{-\beta_0 t} - \int_{\mathbb{R}} J(y)U(\xi(x,t)-y)dy \\
&\quad + U(\xi(x,t)) + dU(\xi(x,t)) + d\delta e^{-\beta_0 t} \\
&\quad - \int_{\mathbb{R}} K(y) b\Big(U\Big[\xi(x,t) - y - c\tau + \sigma_0\delta\big(e^{\beta_0\tau} - e^{-\beta_0(t-\tau)}\big) - \sigma_0\delta\big(e^{\beta_0\tau} - e^{-\beta_0 t}\big)\Big] \\
&\quad + \delta e^{-\beta_0(t-\tau)}\Big)dy \\
&= \Big\{cU'(\xi(x,t)) - \int_{\mathbb{R}} J(y)U(\xi(x,t)-y)dy + U(\xi(x,t)) + dU(\xi(x,t))\Big\} \\
&\quad + \sigma_0\delta\beta_0 e^{-\beta_0 t}U'(\xi(x,t)) - \beta_0\delta e^{-\beta_0 t} + d\delta e^{-\beta_0 t} \\
&\quad - \int_{\mathbb{R}} K(y) b\Big(U\Big[\xi(x,t) - y - c\tau + \sigma_0\delta\left(e^{\beta_0\tau} - e^{-\beta_0(t-\tau)}\right) - \sigma_0\delta\left(e^{\beta_0\tau} - e^{-\beta_0 t}\right)\Big] \\
&\quad + \delta e^{-\beta_0(t-\tau)}\Big)dy \\
&= \sigma_0\delta\beta_0 e^{-\beta_0 t}U'(\xi(x,t)) - \beta_0\delta e^{-\beta_0 t} + d\delta e^{-\beta_0 t} \int_{\mathbb{R}} K(y) b[U(\xi(x,t)-y-c\tau)]dy \\
&\quad - \int_{\mathbb{R}} K(y) b\left(U\left[\xi(x,t) - y - c\tau + \sigma_0\delta\left(1 - e^{-\beta_0\tau}\right)e^{-\beta_0 t}\right] + \delta e^{-\beta_0(t-\tau)}\right)dy \\
&= [\sigma_0\delta\beta_0 U'(\xi(x,t)) - \beta_0\delta + d\delta]e^{-\beta_0 t} \\
&\quad - \int_{\mathbb{R}} K(y) b'(\tilde\eta)\Big\{U[\xi(x,t) - y - c\tau + \sigma_0\delta(1 - e^{\beta_0\tau})e^{-\beta_0 t}] + \delta e^{-\beta_0(t-\tau)} \\
&\quad - U(\xi(x,t) - y - c\tau)\Big\}dy \\
&= \delta e^{-\beta_0 t}\Big[\sigma_0\beta_0 U'(\xi(x,t)) - \beta_0 + d + \int_{\mathbb{R}} K(y) b'(\tilde\eta)[U'(\tilde\xi)\sigma_0(e^{\beta_0\tau} - 1) - e^{\beta_0\tau}]dy\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde\eta &= \theta U[\xi(x,t) - y - c\tau + \sigma_0\delta(1 - e^{\beta_0\tau})e^{-\beta_0 t}] + \theta\delta e^{-\beta_0(t-\tau)} \\
&\quad + (1-\theta)U[\xi(x,t) - y - c\tau]
\end{aligned}
$$

and

$$\tilde\xi = \xi(x,t) - y - c\tau + \theta\sigma_0(1 - e^{\beta_0\tau})e^{-\beta_0 t}.$$

It is easy to see that $0 \le \tilde\eta \le 1 + \delta e^{-\beta\tau} \le 1 + \delta^*$. Thus, $b'(\tilde\eta) \ge 0$, and

$$S(w^+)(x,t) \ge \delta e^{-\beta_0 t}\Big\{\sigma_0\beta_0 U'(\xi(x,t)) - \beta_0 + d - e^{\beta_0\tau}\int_{\mathbb{R}} K(y) b'(\tilde\eta)dy\Big\}. \tag{4.10}$$

We need to consider three cases.

Case (i): $|\xi(x,t)| \le M_0$. In this case, by (4.7), (4.8) and (4.10), one has

$$S(w^+)(x,t) \ge \delta e^{-\beta_0 t}\{\sigma_0\beta_0 m_0 - \beta_0 + d - e^{\beta_0\tau}b'_{\max}\} = 0.$$

Case (ii): $\xi(x,t) \ge M_0$. For $y \in [-\frac{1}{2}\xi(x,t), \frac{1}{2}\xi(x,t)]$, we have

$$\frac{1}{2}M_0 \le \frac{1}{2}\xi(x,t) \le \xi(x,t) - y \le \frac{3}{2}\xi(x,t).$$

By the choice of $\bar\delta$, for any $\delta \in (0,\bar\delta]$, one has $\sigma_0\delta \le 1$, and hence,

$$\begin{aligned}
&\xi(x,t) - y - c\tau + \sigma_0\delta(1-e^{\beta_0\tau})e^{-\beta_0 t}\\
&\ge \frac{1}{2}M_0 - c\tau + \sigma_0\delta(1-e^{\beta_0\tau}) \ge \frac{1}{2}M_0 - c_0,
\end{aligned}$$

and

$$\xi(x,t) - y - c\tau \ge \frac{1}{2}M_0 - c\tau \ge \frac{1}{2}M_0 - c_0.$$

Then by (4.5) and (4.9), we have

$$1+\delta^* \ge 1+\delta e^{\beta_0\tau} \ge \tilde\eta \ge 1-\delta^*.$$

Furthermore, by (4.4), we get $b'(\tilde\eta) \le b'(1)+\epsilon^*$. Hence, by (4.6) and (4.10), we have

$$\begin{aligned}
S(w^+)(x,t) &\ge \delta e^{-\beta_0 t}\Big\{\sigma_0\beta_0 U'(\xi(x,t)) - \beta_0 + d - e^{\beta_0\tau}\int_{\mathbb{R}} K(y)b'(\tilde\eta)dy\Big\}\\
&\ge \delta e^{-\beta_0 t}\Big\{-\beta_0 + d - e^{\beta_0\tau}\int_{-\frac12\xi(x,t)}^{\frac12\xi(x,t)} K(y)b'(\tilde\eta)dy\\
&\quad - e^{\beta_0\tau}\int_{-\infty}^{-\frac12\xi(x,t)} K(y)b'(\tilde\eta)dy - e^{\beta_0\tau}\int_{\frac12\xi(x,t)}^{+\infty} K(y)b'(\tilde\eta)dy\Big\}\\
&\ge \delta e^{-\beta_0 t}\Big\{-\beta_0 + d - e^{\beta_0\tau}(b'(1)+\epsilon^*)\\
&\quad - e^{\beta_0\tau}b'_{\max}\Big[\int_{\frac12 M_0}^{+\infty} + \int_{-\infty}^{-\frac12 M_0} K(y)dy\Big]\Big\} \ge 0.
\end{aligned}$$

Case (iii): $\xi(x,t) \le -M_0$. The proof is similar to that for the Case (ii) and is omitted. The proof is complete. $\square$

Proof of Theorem 1.2. Since $\tilde U(\xi)$ and $U(\xi)$ have the same limits as $\xi \to \pm\infty$, there exist $\tilde\xi \in \mathbb{R}$ and a sufficiently large number $p>0$ such that for every $s \in [-\tau, 0]$ and $x \in \mathbb{R}$,

$$U(x+cs+\tilde\xi) - \bar\delta < \tilde U(x+\tilde s) < U(x+cs+\tilde\xi+p) + \bar\delta,$$

and hence,

$$\begin{aligned}
&U(x+cs+\tilde\xi - \sigma_0\bar\delta(e^{\beta_0\tau}-e^{-\beta_0 s})) - \bar\delta e^{-\beta_0 s}\\
&< \tilde U(x+\tilde c s) < U(x+cs+\tilde\xi+p+\sigma_0\bar\delta(e^{\beta_0\tau}-e^{-\beta_0 s})) + \bar\delta e^{-\beta_0 s},
\end{aligned}$$

where β_0, σ_0 and $\bar\delta$ are given in Lemma 4.2. By the comparison principle, we obtain that for all $x \in \mathbb{R}$ and $t \ge 0$,

$$\begin{aligned}
&U(x+ct+\tilde\xi - \sigma_0\bar\delta(e^{\beta_0\tau}-e^{-\beta_0 t})) - \bar\delta e^{-\beta_0 t}\\
&< \tilde U(x+\tilde c t) < U(x+ct+\tilde\xi+p+\sigma_0\bar\delta(e^{\beta_0\tau}-e^{-\beta_0 t})) + \bar\delta e^{-\beta_0 t}.
\end{aligned}$$

Keeping $\xi = x+ct$ fixed and letting $t\to\infty$, we then obtain from the first inequality that $c \le \tilde c$ and from the second inequality that $c \ge \tilde c$. Thus, $\tilde c = c$. In addition,

$$U(\xi+\tilde\xi-\sigma_0\bar\delta e^{\beta_0\tau}) - \bar\delta e^{-\beta_0 t} < \tilde U(\xi) < U(\xi+\tilde\xi+p+\sigma_0\bar\delta e^{\beta_0\tau}) \text{ for } \xi\in\mathbb{R}. \tag{4.11}$$

Define

$$\xi^* := \inf\{\xi : \tilde U(\cdot) \le U(\cdot+\xi)\} \text{ and } \xi_* := \sup\{\xi : \tilde U(\cdot) \ge U(\cdot+\xi)\}.$$

From (4.11), we can see that both ξ^* and ξ_* are well defined. Since $U(\cdot+\xi_*) \le \tilde{U}(\cdot) \le U(\cdot+\xi^*)$, we have $\xi_* \le \xi^*$.

We need only to prove that $\xi_* = \xi^*$. By contradiction, we assume that $\xi_* < \xi^*$ and $\tilde{U}(\cdot) \not\equiv U(\cdot+\xi^*)$. Since $\lim_{\xi\to\pm\infty} U'(\xi) = 0$, there exists a large positive constant $B = B(U) > 0$ such that

$$2\sigma_0 e^{\beta_0\tau} U'(\xi) \le 1 \text{ for } |\xi| \ge B.$$

Note that $\tilde{U}(\cdot) \le U(\cdot+\xi^*)$ and $\tilde{U}(\cdot) \not\equiv U(\cdot+\xi^*)$, by Lemma 3.4, it follows that $\tilde{U}(\cdot) < U(\cdot+\xi^*)$ on $\mathbb{R}$. Consequently, by the continuity of U and $\tilde{U}$, there exists a small constant $\rho \in [0,\bar{\delta}]$ with $\rho \le \frac{1}{2\sigma_0} e^{-\beta_0\tau}$, such that

$$\tilde{U}(\xi) < U(\xi+\xi^* - 2\sigma_0\rho e^{\beta_0\tau}), \text{ if } \xi \in [-B-1-\xi^*, B+1-\xi^*]. \tag{4.12}$$

When $|\xi+\xi^*| \ge B+1$, one has

$$\begin{aligned} U(\xi+\xi^*-2\sigma_0\rho e^{\beta_0\tau}) - \tilde{U}(\xi) &> U(\xi+\xi^*-2\sigma_0\rho e^{\beta_0\tau}) - \tilde{U}(\xi+\xi^*) \\ &= -2\sigma_0\rho e^{\beta_0\tau} U'(\xi+\xi^*-2\theta\sigma_0\rho e^{\beta_0\tau}) \ge -\rho. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we obtain that for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$,

$$\tilde{U}(x+cs) \le U(x+cs+\xi^* - 2\sigma_0\rho e^{\beta_0\tau} + \sigma_0\rho(e^{\beta_0\tau} - e^{-\beta_0 s})) + \rho e^{-\beta_0 s}.$$

Hence, by the comparison principle, for any $x \in \mathbb{R}$ and $t \ge 0$,

$$\tilde{U}(x+ct) \le U(x+ct+\xi^* - 2\sigma_0\rho e^{\beta_0\tau} + \sigma_0\rho(e^{\beta_0\tau} - e^{-\beta_0 t})) + \rho e^{-\beta_0 t}. \tag{4.14}$$

Keep $\xi = x+ct$ fixed and let $t \to \infty$ in (4.14). Then we have $\tilde{U}(\xi) \le U(\xi+\xi^* - \sigma_0\rho e^{\beta_0\tau})$ for all $\xi \in \mathbb{R}$. This contradicts the definition of ξ^*. Hence, $\xi_* = \xi^*$. The proof is complete. □

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5. Stability of traveling wavefronts

Let $\zeta \in C^\infty(\mathbb{R})$ be a fixed function with the following properties:

$$\zeta(s) = 0, \text{ if } s \le -2; \quad \zeta(s) = 1, \text{ if } s \ge 2;$$
$$0 < \zeta'(s) < 1, \text{ if } s \in (-2,2).$$

Lemma 5.1. *For any $\delta \in (0,\delta_0]$, $\delta_0 \le \alpha/2$, there exist two positive numbers $\epsilon = \epsilon(\delta)$ and $C = C(\delta)$ such that for every $\xi^\pm \in \mathbb{R}$, the functions $v^\pm(x,t)$ defined by*

$$v^+(x,t) := 1+\delta - [1-(\alpha-2\delta)e^{-\epsilon t}]\zeta(-\epsilon(x-\xi^+ + Ct)),$$
$$v^-(x,t) := -\delta + [1-(1-\alpha-2\delta)e^{-\epsilon t}]\zeta(\epsilon(x-\xi^- - Ct))$$

are a supersolution and a subsolution of (1.1) *on* $[0,+\infty)$*, respectively.*

Proof. Since $du - b(u) > 0$ for $u \in (0,\alpha) \cup (1, 1+\delta)$, we obtain

$$M_1 = M_1(\delta) = \min\{du - bu : u \in [\delta, \alpha-\delta/2]\} > 0.$$

By (H5), we choose $\mu \in [1/2, 1)$ and $\gamma > 0$ satisfying

$$d\mu > b'(1) + \gamma \tag{5.1}$$

and

$$0 \le b'(\eta) < b'(1) + \gamma \quad \text{for } \eta \in [1, 1+\delta].$$

Let $\epsilon^* = \epsilon^*(\delta) > 0$ be such that $\epsilon^* \le \min\{\delta/2, 2(1-\mu)\delta\}$, and let $k = k(\delta) \in (0,1)$ be such that

$$0 \le \zeta(s) < \epsilon^*/2, \quad \text{if } s < -2+k, \tag{5.2}$$

$$1 \ge \zeta(s) > 1-\epsilon^*/2, \quad \text{if } s > 2-k. \tag{5.3}$$

Take $\nu = \nu(\delta) < 0$ small enough such that $(1+\nu)(2-k/2) > 2-k$. Take $\epsilon = \epsilon(\delta) > 0$ sufficiently small and $M_0 = M_0(\delta) > 0$ sufficiently large such that $\alpha e^{\epsilon\tau} < 1$, $\epsilon M_0 \le -\nu(2-k)$,

$$-\epsilon\alpha - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy + d(1+\mu\delta) - \delta(b'(1)+\gamma) > 0 \tag{5.4}$$

and

$$\begin{aligned} &-M_1 + \epsilon\alpha + 2\epsilon\int_{\mathbb{R}} |y|J(y)dy + \epsilon\tau b'_{\max}e^{\epsilon\tau} \\ &+ b'_{\max}\epsilon^* + 2b'_{\max}\Big[\int_{M_0}^{+\infty} + \int_{-\infty}^{-M_0} K(y)dy\Big] < 0. \end{aligned} \tag{5.5}$$

Define

$$\sigma := \min\{\zeta'(s) : -2+k/2 \le s \le 2-k/2\} > 0.$$

Then take $C = C(\delta) > 0$ large enough so that

$$\begin{aligned} C\epsilon(1-\alpha)\sigma &> \epsilon\alpha + 2\epsilon\int_{\mathbb{R}} |y|J(y)dy + \epsilon\tau b'_{\max}e^{\epsilon\tau} \\ &\quad + \max\{|du - b(u)| : u \in [\delta, 1+\delta]\} + 2b'_{\max}. \end{aligned} \tag{5.6}$$

It is easy to see that for $t \ge -\tau$ and $x \in \mathbb{R}$,

$$\delta \le v^+(x,t) \le 1+\delta, \quad -\delta \le v^-(x,t) \le 1-\delta.$$

Set $\xi = x - \xi^+ + Ct$. Then

$$\begin{aligned} S(v^+)(x,t) :&= \frac{\partial v^+}{\partial t} - J*v^+ + v^+ + dv^+ - \int_{\mathbb{R}} K(y)b(v^+(x-y,t-\tau))dy \\ &= -\epsilon(\alpha-2\delta)e^{-\epsilon t}\zeta(-\epsilon\xi) + \epsilon C[1-(\alpha-2\delta)e^{-\epsilon t}]\zeta'(-\epsilon\xi) \\ &\quad + [1-(\alpha-2\delta)e^{-\epsilon t}]\Big[\int_{\mathbb{R}} J(y)\zeta(-\epsilon(\xi-y))dy - \zeta(-\epsilon\xi)\Big] \\ &\quad + dv^+ - \int_{\mathbb{R}} K(y)b(v^+(x-y,t-\tau))dy \\ &\ge -\epsilon\alpha + \epsilon C(1-\alpha)\zeta'(-\epsilon\xi) + [1-(\alpha-2\delta)e^{-\epsilon t}] \\ &\quad \times \int_{\mathbb{R}} J(y)|\zeta(-\epsilon(\xi-y)) - \zeta(-\epsilon\xi)|dy \\ &\quad + dv^+ - \int_{\mathbb{R}} K(y)b(v^+(x-y,t-\tau))dy \\ &\ge -\epsilon\alpha + \epsilon C(1-\alpha)\zeta'(-\epsilon\xi) - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy \\ &\quad + dv^+ - \int_{\mathbb{R}} K(y)b(v^+(x-y,t-\tau))dy. \end{aligned} \tag{5.7}$$

By a direct computation, it follows that for all $t \ge -\tau$,

$$\frac{\partial}{\partial t}v^+(x,t) = -\epsilon(\alpha-2\delta)e^{-\epsilon t} + \epsilon C[1-(\alpha-2\delta)e^{-\epsilon t}]\zeta'(-\epsilon\xi)$$

$$\geq -\epsilon(\alpha - 2\delta)e^{-\epsilon t} \geq -\epsilon e^{\epsilon\tau},$$

and hence, for all $t \geq 0$,

$$\begin{aligned} &b(v^+(x,t-\tau)) - b(v^+(x,t)) = b'(\eta_1)[v^+(x,t-\tau) - v^+(x,t)] \\ &= -\tau b'(\eta_1)\frac{\partial}{\partial t}v^+(x,t^*) \leq \epsilon\tau b'(\eta_1)e^{\epsilon\tau} \leq \epsilon\tau b'_{\max}e^{\epsilon\tau}, \end{aligned}$$

where $t^* \in [t-\tau, t]$ and $\eta_1 = \theta v^+(x,t) + (1-\theta)v^+(x,t-\tau)$.

On the other hand, for $t \geq 0$, one has

$$\begin{aligned} &|v^+(x-y,t-\tau) - v^+(x,t-\tau)| \\ &= (1-(\alpha-2\delta)e^{-\epsilon(t-\tau)})|\zeta(-\epsilon(\xi-y-C\tau)) - \zeta(-\epsilon(\xi-C\tau))| \\ &\leq |\zeta(-\epsilon(\xi-y-C\tau)) - \zeta(-\epsilon(\xi-C\tau))|. \end{aligned}$$

It then follows that

$$\begin{aligned} S(v^+)(x,t) &\geq -\epsilon\alpha + \epsilon C(1-\alpha)\zeta'(-\epsilon\xi) - 2\epsilon\int_{\mathbb{R}}|y|J(y)dy \\ &\quad + dv^+(x,t) - b(v^+(x,t)) - (b(v^+(x,t-\tau)) - b(v^+(x,t))) \\ &\quad - \int_{\mathbb{R}} K(y)(b(v^+(x-y,t-\tau)) - b(v^+(x,t-\tau)))dy \\ &\geq -\epsilon\alpha + \epsilon C(1-\alpha)\zeta'(-\epsilon\xi) - 2\epsilon\int_{\mathbb{R}}|y|J(y)dy \\ &\quad + dv^+(x,t) - b(v^+(x,t)) - (b(v^+(x,t-\tau)) - b(v^+(x,t))) \\ &\quad - \int_{\mathbb{R}} K(y)b'(\eta_2)|v^+(x-y,t-\tau) - v^+(x,t-\tau)|dy \\ &\geq -\epsilon\alpha + \epsilon C(1-\alpha)\zeta'(-\epsilon\xi) - 2\epsilon\int_{\mathbb{R}}|y|J(y)dy \\ &\quad + dv^+(x,t) - b(v^+(x,t)) - \epsilon\tau b'_{\max}e^{\epsilon\tau} \\ &\quad - b'_{\max}\int_{\mathbb{R}} K(y)|\zeta(-\epsilon(\xi-y-C\tau)) - \zeta(-\epsilon(\xi-C\tau))|dy, \end{aligned} \tag{5.8}$$

where $\eta_2 = \theta v^+(x,t-\tau) + (1-\theta)v^+(x-y,t-\tau) \in [\delta, 1+\delta]$.

We distinguish three cases:

Case (i): $-\epsilon\xi \leq -2+k/2$. In this case, $-\epsilon\xi \leq -2+k$. By (5.2), $0 \leq \zeta(-\epsilon\xi) < \epsilon^*/2$. Recall the $\epsilon^* < 2(1-\mu)\delta$, we then have

$$\begin{aligned} 1+\delta \geq v^+(x,t) &\geq 1+\delta - [1-(\alpha-2\delta)e^{-\epsilon t}]\epsilon^*/2 \\ &\geq 1+\delta - \epsilon^*/2 \geq 1+\delta-(1-\mu)\delta = 1+\mu\delta \geq 1+\delta/2 \end{aligned}$$

for all $t \geq 0$.

It then follows from (5.7) that

$$\begin{aligned} S(v^+)(x,t) &\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}}|y|J(y)dy + dv^+(x,t) - d \\ &\quad - \int_{\mathbb{R}} K(y)[b(v^+(x-y,t-\tau)) - b(1)]dy \\ &\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}}|y|J(y)dy + d\mu\delta \end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}} K(y)b'(\eta^*)(v^+(x-y,t-\tau)-1)dy \\
&\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy + d\mu\delta - \delta\int_{\mathbb{R}} K(y)b'(\eta^*)dy,
\end{aligned}
$$

where $\eta^* = \theta v^+(x-y,t-\tau) + (1-\theta) \in [1, 1+\delta]$.

Therefore, by (5.1) and (5.4), we get

$$
\begin{aligned}
S(v^+)(x,t) &\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy + d\mu\delta - \delta\int_{\mathbb{R}} K(y)b'(\eta^*)dy \\
&\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy + d\mu\delta - \delta(b'(1)+\gamma) > 0.
\end{aligned}
$$

Case (ii): $-\epsilon\xi \geq 2-k/2$. In this case, $-\epsilon\xi \geq 2-k$. By (5.3), $1-\epsilon^*/2 \leq \zeta(-\epsilon\xi) \leq 1$. It then follows that

$$
\begin{aligned}
\delta \leq v^+(x,t) &\leq (1+\delta) - [1-(\alpha-2\delta)e^{-\epsilon t}](1-\frac{\epsilon^*}{2}) \\
&\leq (1+\delta) - [1-\alpha+2\delta](1-\frac{\epsilon^*}{2}) \\
&\leq 1+\delta-1+\alpha-2\delta+\epsilon^* \\
&\leq \alpha-\delta+\epsilon^* \leq \alpha - \frac{\delta}{2}.
\end{aligned}
$$

Thus, we can see that

$$
dv^+(x,t) - b(v^+(x,t)) \geq \min\{du - b(u) : u \in [\delta, \alpha-\delta/2]\} = M_1.
$$

By the choice of ϵ and $\nu < 0$, we have

$$
\nu\xi \geq -\nu\frac{2-k/2}{\epsilon} \geq \frac{-\nu(2-k)}{\epsilon} \geq M_0,
$$

and for any $y \in [-\nu\xi, \nu\xi]$,

$$
\begin{gathered}
-\epsilon(\xi - y - C\tau) \geq -\epsilon\xi(1+\nu) + \epsilon C\tau \geq (1+\nu)(2-k/2) > 2-k, \\
-\epsilon(\xi - C\tau) \geq 2-k/2+\epsilon C\tau > 2-k.
\end{gathered}
$$

Then, we obtain

$$
\int_{-\nu\xi}^{\nu\xi} |\zeta(-\epsilon(\xi-y-C\tau)) - \zeta(-\epsilon(\xi-C\tau))|K(y)dy \leq \epsilon^*/2\int_{-\nu\xi}^{\nu\xi} K(y)dy \leq \epsilon^*.
$$

Therefore, by (5.8) and (5.5), we get

$$
\begin{aligned}
S(v^+)(x,t) &\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy - \epsilon\tau b'_{\max}e^{\epsilon\tau} + M_1 \\
&\quad - b'_{\max}\int_{-\nu\xi}^{\nu\xi} |\zeta(-\epsilon(\xi-y-C\tau)) - \zeta(-\epsilon(\xi-C\tau))|K(y)dy \\
&\quad - 2b'_{\max}\Big[\int_{-\infty}^{-\nu\xi} + \int_{\nu\xi}^{+\infty} K(y)dy\Big] \\
&\geq -\epsilon\alpha - 2\epsilon\int_{\mathbb{R}} |y|J(y)dy - \epsilon\tau b'_{\max}e^{\epsilon\tau} + M_1 - b'_{\max}\epsilon^* \\
&\quad - 2b'_{\max}\Big[\int_{-\infty}^{-\nu\xi} + \int_{\nu\xi}^{+\infty} K(y)dy\Big] > 0.
\end{aligned}
$$

Case (iii): $-2+k/2 \le -\epsilon\xi \le 2-k/2$. In this case, by (5.6), one has

$$\begin{aligned} S(v^+)(x,t) &\ge C\epsilon(1-\alpha)\sigma - \epsilon\alpha - 2\epsilon \int_{\mathbb{R}} |y|J(y)dy - \epsilon\tau b'_{\max} e^{\epsilon\tau} \\ &\quad - \max\{|du - b(u)| : u \in [\delta, 1+\delta]\} - 2b'_{\max} > 0. \end{aligned}$$

Combining Cases (i)-(iii), we obtain

$$\frac{\partial v^+}{\partial t} \ge J * v^+ - v^+ - dv^+ + \int_{\mathbb{R}} K(y) b(v^+(x-y, t-\tau))dy, \quad x \in \mathbb{R}, t \ge 0.$$

Thus, $v^+(x,t)$ is a supersolution of (1.1) on $[0,+\infty)$. Similarly, we can prove that $v^-(x,t)$ is a subsolution of (1.1) on $[0,+\infty)$. The proof is complete. □

Remark 5.2. Clearly, the functions v^+ and v^- have the following properties:

$$\begin{gathered} v^+(x,s) = 1+\delta, \quad \text{if } s \in [-\tau,0],\ x \ge \xi^+ - Cs + 2\epsilon^{-1}, \\ v^+(x,s) \ge \alpha - \delta, \quad \text{for all } s \in [-\tau,0] \text{ and } x \in \mathbb{R}, \\ v^+(x,t) = \delta + (\alpha - 2\delta)e^{-\epsilon t}, \quad \text{for all } t \ge -\tau \text{ and } x \le \xi^+ - Ct - 2\epsilon^{-1}, \\ v^-(x,s) = -\delta, \quad \text{if } s \in [-\tau,0],\ x \le \xi^+ + Cs - 2\epsilon^{-1}, \\ v^-(x,s) \le \alpha + \delta, \quad \text{for all } s \in [-\tau,0] \text{ and } x \in \mathbb{R}, \\ v^-(x,t) = 1 - \delta - (1-\alpha-2\delta)e^{-\epsilon t}, \quad \text{for all } t \ge -\tau \text{ and } x \ge \xi^- + Ct + 2\epsilon^{-1}. \end{gathered}$$

Let $U(x+ct)$ be a non-decreasing traveling wavefronts of (1.1). We define the following two functions:

$$w^{\pm}(x,t,\eta,\delta) := U(x+ct+\eta \pm \sigma_0\delta(e^{\beta_0\tau} - e^{-\beta_0 t})) \pm \delta e^{-\beta_0 t},$$

where σ_0 and β_0 are as in Lemma 4.2.

Lemma 5.3. *Let $U(x+ct)$ be a non-decreasing traveling wavefront of (1.1). Then there exists a positive number ϵ^* such that if $u(x,t)$ is a solution of (1.1) on $[0,+\infty)$ with initial data $0 \le u(x,s) \le 1$ for all $x \in \mathbb{R}$ and $s \in [-\tau,0]$, and for some $\xi \in \mathbb{R}$, $h>0$, $\delta>0$ and $T \ge 0$, there holds*

$$w_0^-(x,-cT+\xi,\delta)(s) \le u_T(x)(s) \le w_0^+(x,-cT+\xi+h,\delta)(s)$$

for all $x \in \mathbb{R}$ and $s \in [-\tau,0]$, then for any $t \ge T+\tau+1$, there exist $\hat{\xi}(t)$, $\hat{\delta}(t)$ and $\hat{h}(t)$ satisfying

$$\begin{gathered} \xi - \sigma_0(2\delta + \epsilon^* \min\{1,h\})e^{\beta_0\tau} \le \hat{\xi}(t) \le \xi + h + \sigma_0(2\delta + \epsilon^* \min\{1,h\})e^{\beta_0\tau}, \\ \hat{\delta}(t) = (\epsilon^* \min\{1,h\} + \delta)e^{-\beta_0[t-(T+1+\tau)]}, \\ \hat{h}(t) = h - 2\sigma_0\epsilon^* \min\{1,h\} + \sigma_0(3\delta + \epsilon^* \min\{1,h\})e^{\beta_0\tau} > 0, \end{gathered}$$

such that

$$w_0^-(x,-ct+\hat{\xi}(t),\hat{\delta}(t))(s) \le u_t(x)(s) \le w_0^+(x,-ct+\hat{\xi}(t)+\hat{h}(t),\hat{\delta}(t))(s)$$

for all $x \in \mathbb{R}$ and $s \in [-\tau,0]$.

Proof. By Lemma 4.2, $w^+(x,t,cT+\xi+h,\delta)$ and $w^-(x,t,cT+\xi,\delta)$ are a supersolution and a subsolution of (1.1), respectively. It is easy to see that $v(x,t) :=$

$u(x,T+t), t \geq 0$, is also a solution to (1.1) with $v_0(x)(s) = u_T(x)(s)$. Then it follows from the comparison principle that

$$w^-(x,t,cT+\xi,\delta) \leq u(x,t+T) \leq w^+(x,t,cT+\xi+h,\delta)$$

for $x \in \mathbb{R}, t \geq 0$. That is,

$$\begin{aligned} &U(x+c(t+T)+\xi-\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0 t}))-\delta e^{-\beta_0 t} \leq u(x,t+T) \\ &\leq U(x+c(t+T)+\xi+h+\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0 t}))+\delta e^{-\beta_0 t} \end{aligned}$$

for $x \in \mathbb{R}$, $t \geq 0$.

In view of Lemma 3.4, for $x \in \mathbb{R}, t > 0$, we have

$$\begin{aligned} &u(x,T+t)-w^-(x,t,cT+\xi,\delta) \\ &\geq \Theta(|x|,t)\int_0^1 [u(y,T)-w^-(y,0,cT+\xi,\delta)]dy \\ &\geq \Theta(|x|,t)\int_0^1 [u(y,T)-U(y+cT+\xi-\sigma_0\delta(e^{\beta_0\tau}-1))+\delta]dy \\ &\geq \Theta(|x|,t)\int_0^1 [u(y,T)-U(y+cT+\xi)+\delta]dy. \end{aligned} \tag{5.9}$$

By Lemma 4.1, $\lim_{|r|\to+\infty} U'(r) = 0$. Then we can fix a positive number $M > 0$ such that

$$U'(r) \leq \frac{1}{2\sigma_0} \quad \text{for all } |r| \geq M.$$

Let

$$\epsilon_1 = \frac{1}{2}\min\{U'(\eta) : |\eta| \leq 2\} > 0 \quad \text{and} \quad \overline{h} = \min\{1,h\}.$$

By the mean value theorem, we obtain

$$\int_0^1 [U(y+cT+\xi+\overline{h})-U(y+cT+\xi)]dy \geq 2\epsilon_1\overline{h}.$$

Then at least one of the following two statements is true:

(i) $\int_0^1 [u(y,T)-U(y+cT+\xi)]dy \geq \epsilon_1\overline{h}$.
(ii) $\int_0^1 [U(y+cT+\xi+\overline{h})-u(y,T)]dy \geq \epsilon_1\overline{h}$.

Subsequently, we consider only the case (i). The case (ii) is similar and thus omitted.

Let $J_1 = M_3+|c|(1+\tau)+2$, $z_0 = -cT-\xi$ and $J_2 = J_1+c+3$. For $|x-z_0| \leq J_1$, further letting $t = 1+\tau+s$ in (5.9), and $\overline{\Theta} := \Theta(J_1, 1+\tau+s)$, then we get

$$\begin{aligned} &u(x,T+1+\tau+s) \\ &\geq U(x+c^*(T+1+\tau+s)+\xi-\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0(1+\tau+s)})) \\ &-\delta e^{-\beta_0(1+\tau+s)}+\Theta(|x|,1+\tau+s))\int_0^1 [u(y,T)-U(y+cT+\xi)]dy \\ &\geq U(x-z_0+c(1+\tau+s)-\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0(1+\tau+s)})) \\ &-\delta e^{-\beta_0(1+\tau+s)}+\overline{\Theta}\epsilon_1\overline{h}. \end{aligned} \tag{5.10}$$

In addition, we have

$$\begin{aligned}
&U(x - z_0 + c(1+\tau+s) + 2\sigma_0\epsilon^*\overline{h} - \sigma_0\delta(e^{\beta_0\tau} - e^{-\beta_0(1+\tau+s)})) \\
&- U(x - z_0 + c(1+\tau+s) - \sigma_0\delta(e^{\beta_0\tau} - e^{-\beta_0(1+\tau+s)})) \\
&= U'(\eta_1)2\sigma_0\epsilon^*\overline{h} \le \overline{\Theta}\epsilon_1\overline{h},
\end{aligned} \tag{5.11}$$

where

$$\epsilon^* = \min\Big\{\frac{1}{3\sigma_0}, \quad \min_{|\eta|\le J_2} \frac{\overline{\Theta}\epsilon_1}{2\sigma_0\phi'(\eta)}\Big\},$$
$$\eta_1 = x - z_0 + c(1+\tau+s) - \sigma_0\delta(1 - e^{-\beta_0}) + \theta\cdot 2\sigma_0\epsilon^*\overline{h}, \quad \theta\in(0,1).$$

It follows that

$$|\eta_1| = |x - z_0| + c(1+\tau) + \sigma_0\delta + 2\sigma_0\epsilon^* \le J_1 + c + \sigma_0\delta + 2\sigma_0\epsilon^* \le J_2.$$

Hence, by the monotonicity of $U(\cdot)$, (5.10) and (5.11), we have

$$\begin{aligned}
&u(x, T+1+\tau+s) \\
&\ge U(x + c(T+1+\tau+s) + \xi + 2\sigma_0\epsilon^*\overline{h} - \sigma_0\delta(e^{\beta_0\tau} - e^{-\beta_0(1+\tau+s)})) \\
&\quad - \delta e^{-\beta_0(1+\tau+s)}.
\end{aligned}$$

The remainder of proof is similar to that of [30, Lemma 3.1], so is omitted. The proof is complete. □

By Lemmas 4.2, 5.1, 5.3, we can obtain the following Lemma 5.4 and Theorem 1.3. Their proofs are similar to the proofs of [37], so we omit them here.

Lemma 5.4. *Let $U(x+ct)$ be a non-decreasing traveling wavefront of* (1.1)*, and $\varphi\in[0,1]_C$ be such that*

$$\limsup_{x\to-\infty}\max_{s\in[-\tau,0]}\varphi(x,s) < \alpha < \liminf_{x\to+\infty}\min_{s\in[-\tau,0]}\varphi(x,s).$$

Then, for any $\delta>0$, there exist $T=T(\varphi,\delta)>0$, $\xi=\xi(\varphi,\delta)\in\mathbb{R}$, and $h=h(\varphi,\delta)>0$ such that

$$w_0^-(x, -cT+\xi, \delta)(s) \le u_T(x,\varphi)(s) \le w_0^+(x, cT+\xi+h, \delta)(s)$$

for all $x\in\mathbb{R}$ and $s\in[-\tau,0]$.

Acknowledgments. The first author was supported by CPSF (2013M530435), SRFDP (20126203120006), NSF of China (11401478) and Gansu Provincial Natural Science Foundation (145RJZA220). The second author was supported by NSF of China (11061030), SRFDP (20126203110004) and Gansu Provincial National Science Foundation of China (1208RJZA258).

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