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# LIMIT BEHAVIOR OF MONOTONE AND CONCAVE SKEW-PRODUCT SEMIFLOWS WITH APPLICATIONS

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Abstract. In this article, we study the long-time behavior of monotone and concave skew-product semiflows. We show that if there are two strongly ordered omega limit sets, then one of them is a copy of the base. Thus, we obtain a global attractor result. As an application, we consider a delay differential equation.

## 1. Introduction

Recently, monotone skew-product semiflows generated by nonautonomous systems, in particular almost periodic systems, have extensively investigated, see [3, 5, 6, 7, 8, 9, 10]. Hetzer and Shen [3] considered the convergence of positive solutions of almost periodic competitive diffusion systems. Jiang and Zhao [5] established the 1-covering property of the omega limit set for monotone and uniformly stable skew-product semiflows with the componentwise separating property of bounded and ordered full orbits, which is an important property for considering the long-time behavior of skew-product semiflows. Novo et al [6, 7, 8] considered the skew-product semiflow generated by almost periodic systems. Under the assumption that there existed two strongly ordered minimal subsets or completely strongly ordered minimal subsets, a complete description of the long-time behavior of the trajectories was given and a global picture of the dynamics was provided for a class of monotone and convex skew-product semiflows. Zhao [10] proved a global attractivity theory for a class of skew-product semiflows.

In conclusion, the properties of the omega limit set of skew-product semiflows, especially its structure, play an important role in considering the convergent behavior of the orbit. Shen and Yi [9] told us if the omega limit set $\mathcal{O}$ is linearly stable, then there exists an integral number $N$ such that $\mathcal{O}$ is the $(N-1)$-almost periodic extension; i.e., there exists a subset $Y_0 \subset Y$ (the definition of $Y$ see Section 2) such that for any $g_0 \in Y_0$, $\mathrm{card}(\mathcal{O} \cap \pi^{-1}(g_0)) = N$ ($\pi$ is the natural projector). If it is uniformly stable, then it is the extension of $Y$; i.e., $\mathrm{card}(\mathcal{O} \cap \pi^{-1}(g)) = N$ for any $g \in Y$. This is not enough to understand the structure of the omega limit set thoroughly. If we can obtain the conclusion that $\mathcal{O}$ is the copy of the base $Y$; i.e., $\mathrm{card}(\mathcal{O} \cap \pi^{-1}(g)) = 1$ for any $g \in Y$, it would give a complete description

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for the long-time behavior of the orbit. For this purpose, under the assumption of the existence of two completely strongly ordered omega limit sets and motivated by [6, 7], we deduce that one of them is an equilibrium point set if monotonicity and concavity are satisfied. Naturally, it is a copy of the base. Furthermore, we establish the convergent results for skew-product semiflows.

This article is organized as follows. In Section 2, we present some definitions and notation of skew-product semiflows. In Section 3, we establish global attractor results and consider an almost periodic delay differential equation.

## 2. Preliminaries

Let $(Y, d)$ be a compact metric space. A continuous flow $(Y, \sigma, \mathbb{R})$ is defined by a continuous mapping $\sigma : Y \times \mathbb{R} \to Y$, $(g, t) \mapsto \sigma(g, t)$, which satisfies (i) $\sigma_0 = id$, (ii) $\sigma_t \cdot \sigma_s = \sigma_{t+s}$, for all $t, s \in \mathbb{R}$, where $\sigma_t(g) := \sigma(g, t) = g \cdot t$ for $g \in Y$ and $t \in \mathbb{R}$ with $g \cdot 0 = g$ and $g \cdot (s + t) = (g \cdot s) \cdot t$. A continuous flow $(Y, \sigma, \mathbb{R})$ is **distal** if for any two distinct points $g_1$ and $g_2$ in $Y$, $\inf_{t\in\mathbb{R}} d(\sigma(g_1, t), \sigma(g_2, t)) > 0$.

A semiflow $(X, \Phi, \mathbb{R}^+)$ on Banach space $X$ is a continuous map $\Phi : X \times \mathbb{R}^+ \to X$, $(x, t) \mapsto \Phi(x, t)$, which satisfies (i) $\Phi_0 = id$, (ii) $\Phi_t \cdot \Phi_s = \Phi_{t+s}$, where $\Phi_t(x) := \Phi(x, t)$ for $x \in X$ and $t \geq 0$.

A compact, positively invariant subset $S$ of a semiflow $(X, \Phi, \mathbb{R}^+)$ is **minimal** if it contains no nonempty, closed and proper positively invariant subset. If $X$ itself is minimal, then $(X, \Phi, \mathbb{R}^+)$ is called minimal semiflow.

In this article, we assume that $(X, X^+)$ is an ordered Banach space with $\operatorname{int} X^+ \neq \emptyset$, where $\operatorname{int} X^+$ denotes the interior of the cone $X^+$. For $x, y \in X$, we write $x \leq y$ if $y - x \in X^+$; $x < y$ if $y - x \in X^+ \backslash \{0\}$; $x \ll y$ if $y - x \in \operatorname{int} X^+$. In addition, the norm of Banach space $X$ is **monotone**, namely, if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$ (see [7]).

The ordering on $X$ induces the ordering on $Y \times X$ in the following way:

$$\begin{gathered}
(g, x) \leq (g, y) \Leftrightarrow y - x \in X^+, \quad \forall g \in Y, \\
(g, x) < (g, y) \Leftrightarrow y - x \in X^+,\ x \neq y, \quad \forall g \in Y, \\
(g, x) \ll (g, y) \Leftrightarrow y - x \in \operatorname{int} X^+, \quad \forall g \in Y.
\end{gathered}$$

Consider a skew-product semiflow: $\Pi : \mathbb{R}^+ \times Y \times X \to Y \times X$,

$$(t, g, x) \mapsto (g \cdot t, u(t, g, x)). \tag{2.1}$$

We assume that $(Y, \sigma, \mathbb{R})$ is a minimal flow defined by $\sigma : Y \times \mathbb{R} \to Y$, $(g, t) \mapsto g \cdot t$ and $u$ is locally $C^1$ in $x \in X$; that is, $u$ is $C^1$ in $x$, and $u_x$ is continuous in $g \in Y$, $t > 0$ in a neighborhood of each compact subset of $Y \times X$. Moreover, for any $v \in X$, $\lim_{t\to 0^+} u_x(t, g, x)v = v$ uniformly in every compact subset of $Y \times X$. Sometimes, we also use the notation $\Pi_t(g, x) \equiv \Pi(t, g, x)$. We denote $\pi : Y \times X \to Y$ as the natural projection.

The forward orbit of $(g_0, x_0)$ is written as

$$O(g_0, x_0) = \{\Pi(t, g_0, x_0) : t \geq 0\}.$$

If $u(t, g_0, x_0)$ is convergent as $t \to \infty$, we can define the omega limit set of $(g_0, x_0)$ as

$$\mathcal{O}(g_0, x_0) = \{(g, x) \in Y \times X : \exists t_n \to \infty \text{ such that } g_0 \cdot t_n \to g,\ u(t_n, g_0, x_0) \to x\}.$$

Given a subset $K \subset Y \times X$, let us introduce the projection set of $K$ into the fiber space

$$K_Y := \{g \in Y : \text{ there exists } x \in X \text{ such that } (g, x) \in K\} \subset Y.$$

An **equilibrium** is a map $a : Y \to X$ such that $a(g \cdot t) = u(t, g, a(g))$, for all $g \in Y$, $t \geq 0$. A set $E \subset Y \times X$ is called an **equilibrium point set** if there exists a map $a$ such that $a(g) = x$, for all $(g, x) \in E$ and $a(g \cdot t) = u(t, g, a(g))$, for all $g \in E_Y$, $t \geq 0$.

We say that the skew-product semiflow (2.1) is **monotone** if

$$u(t, g, y) \geq u(t, g, x), \quad \forall y \geq x, \ t \geq 0, \tag{2.2}$$

and **strongly monotone** if

$$u(t, g, y) \gg u(t, g, x), \quad \forall y \gg x, \ t \geq 0.$$

The skew-product semiflow (2.1) is said to be **eventually strongly monotone** if there exists $t_0 > 0$ such that

$$u(t, g, y) \gg u(t, g, x), \quad \forall y > x, \ t > t_0 \tag{2.3}$$

and it preserves the ordering; i.e.,

$$u(t, g, y) >_r u(t, g, x), \quad \forall y >_r x, \ t > 0,$$

where $>_r$ denotes the relations $\geq$, $>$ or $\gg$.

The skew-product semiflow (2.1) is called **concave**, if, whenever $x \leq y$,

$$u(t, g, \lambda y + (1 - \lambda)x) \geq \lambda u(t, g, y) + (1 - \lambda)u(t, g, x) \tag{2.4}$$

for $g \in Y$, $\lambda \in [0, 1]$ and $t \in \mathbb{R}^+$; **strongly concave**, if, whenever $x \ll y$,

$$u(t, g, \lambda y + (1 - \lambda)x) \gg \lambda u(t, g, y) + (1 - \lambda)u(t, g, x) \tag{2.5}$$

for $g \in Y$, $\lambda \in (0, 1)$ and $t \in \mathbb{R}^+$.

From the continuous hypothesis for $u$, (2.4) is equivalent to, whenever $y \geq x$,

$$u_x(t, g, x)(y - x) \geq u_x(t, g, y)(y - x)$$

for $g \in Y$ and $t \in \mathbb{R}^+$. Similarly, (2.5) is equivalent to, whenever $y \gg x$,

$$u_x(t, g, x)(y - x) \gg u_x(t, g, y)(y - x)$$

for $g \in Y$ and $t \in \mathbb{R}^+$. Since $x \leq \lambda y + (1 - \lambda)x$ and $\lambda y + (1 - \lambda)x \leq y$, we have

$$u_x(t, g, y)(y - x) \leq u(t, g, y) - u(t, g, x) \leq u_x(t, g, x)(y - x) \tag{2.6}$$

for $g \in Y$ and $t \in \mathbb{R}^+$.

Let $y \geq x$, we have

$$u(t, g, y) - u(t, g, x) = \int_0^1 u_x(t, g, \lambda y + (1 - \lambda)x)(y - x) d\lambda.$$

A forward orbit $\{\Pi(t, g_0, x_0) | t \geq 0\}$ of the skew-product semiflow (2.1) is said to be **uniformly stable** if for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$, such that if $s > 0$ and $\|u(s, g_0, x_0) - u(s, g_0, x)\| \leq \delta(\epsilon)$, we have

$$\|u(t + s, g_0, x_0) - u(t + s, g_0, x)\| \leq \epsilon, \ \forall t \geq 0.$$

A forward orbit $\{\Pi(t, g_0, x_0) | t \geq 0\}$ of the skew-product semiflow (2.1) is said to be **uniformly asymptotically stable** if it is uniformly stable and there is $\delta_0 > 0$

with the following property: for each $\epsilon > 0$ there exists a $t_0(\epsilon) > 0$ such that if $s \geq 0$ and $\|u(s, g_0, x_0) - u(s, g_0, x)\| \leq \delta_0$, we get

$$\|u(t+s, g_0, x_0) - u(t+s, g_0, x)\| \leq \epsilon, \ \forall t \geq t_0(\epsilon).$$

## 3. Global attractor result

In this section, we assume that the skew-product semiflow (2.1) satisfies eventually strong monotonicity and (strong) concavity. Based on this, we establish the global attractor results.

**Definition 3.1.** Two subsets $S_1$, $S_2$ of $Y \times X$ are ordered $S_1 \leq S_2$ if for each $(g, x_1) \in S_1$, there exists $(g, x_2) \in S_2$ such that $x_1 \leq x_2$. We say $S_1 < S_2$ if $S_1 \leq S_2$ and they are different.

**Definition 3.2.** We say the subset $S_1$, $S_2$ of $Y \times X$ to be ordered $S_1 \ll S_2$ if for each $(g, x_1) \in S_1$, there exists $(g, x_2) \in S_2$ such that $x_1 \ll x_2$.

**Definition 3.3.** Two subsets $S_1$, $S_2$ are said to be completely strongly ordered $S_1 \ll_C S_2$ if $x_1 \ll x_2$ holds for all $(g, x_1) \in S_1$ and $(g, x_2) \in S_2$.

**Definition 3.4.** Let $M \subset Y \times X$ be a compact, positively invariant subset of the skew-product semiflow (2.1). For $(g, x) \in M$, we define the **Lyapunov exponent** $\lambda(g, x)$ as

$$\lambda(g, x) = \limsup_{t\to\infty} \frac{\ln \|u_x(t, g, x)\|}{t}.$$

The number $\lambda_M = \sup_{(g,x)\in M} \lambda(g, x)$ is called the **upper Lyapunov exponent** on $M$. If $\lambda_M \leq 0$, then $M$ is said to be **linearly stable**.

In addition, the following assumptions are necessary.

- (A1) Every bounded forward orbit $\{\Pi(t, g, x) : t \geq 0\}$ is precompact.
- (A2) $u(t, g, 0) = 0$, for all $g \in Y$, $t \in \mathbb{R}^+$.

**Theorem 3.5.** *Assume that* (A2) *holds and* $\mathcal{O} \subset Y \times \operatorname{int} X^+$ *with* $\lambda_{\mathcal{O}} < 0$. *Then* $\mathcal{O}$ *is uniformly asymptotically stable, that is, for each* $g \in Y$, *the forward orbit* $\{\Pi(t, g, a(g)|t \geq 0\}$ *is uniformly asymptotically stable. Moreover,* $\mathcal{O}$ *is the copy of the base* $Y$, *i.e.,* $\operatorname{card}(\mathcal{O} \cap \pi^{-1}(g)) = 1$, *for all* $g \in Y$.

*Proof.* The proof of the uniformly asymptotical stability is completely similar to [6, Theorem 8.1], we omit the details here.

In view of the theory of [9] about the structure of omega limit sets, we deduce that $\mathcal{O}$ is an $(N-1)$-extension of $Y$ as $\lambda_{\mathcal{O}} < 0$, that is, $\operatorname{card}(\mathcal{O} \cap \pi^{-1}(g)) = N$ for any $g \in Y$, where $N$ is an integral number, and hence, we denote $\mathcal{O} \cap \pi^{-1}(g) = \{x_1(g), \dots, x_N(g)\}$. Since $X^+$ is a normal cone and $\operatorname{int} X^+ \neq \emptyset$, it is easy to deduce that, for each $g \in Y$, the finite set $\{x_1(g), \dots, x_N(g)\}$ is bounded with respect to the ordering induced by $X^+$. Thus, there exists the supremum

$$b(g) = \sup\{x_1(g), \dots, x_N(g)\},$$

which is a continuous map on $Y$. The positive invariance and monotonicity of the semiflow imply that

$$b(g \cdot t) \leq u(t, g, b(g)), \quad \forall g \in Y, \ t \geq 0. \tag{3.1}$$

Furthermore, we claim that $b$ is invariant under the flow $\sigma$, that is, $b(g \cdot t) = u(t, g, b(g))$ for each $g \in Y$ and $t \geq 0$.

On the contrary, we assume that there exist $g \in Y$ and $s > 0$ such that

$$b(g \cdot s) < u(s, g, b(g)). \tag{3.2}$$

Our assumption implies that $x_i \gg 0$, $i = 1, \dots, N$, from which we deduce that $b(g) \gg 0$. For $e \gg 0$ we define $e$-norm by

$$\|x\|_e =: \inf\{\gamma > 0 : -\gamma e \leq_K x \leq_K \gamma e\}. \tag{3.3}$$

Let $e = b(g) \gg 0$ and

$$\alpha = \inf\{\|b(g) - x_i(g)\|_e : i = 1, \dots, N\}. \tag{3.4}$$

Obviously, $\alpha < 1$ and there exists $j \in \{1, \dots, N\}$ such that $\alpha = \|b(g) - x_j(g)\|_e$. Hence, $b(g) - x_j(g) \leq \alpha b(g)$, which is equivalent to

$$x_j(g) \geq (1 - \alpha) b(g).$$

The monotonicity and concavity of the skew-product semiflow and (A2) imply that

$$u(s, g, x_j(g)) \geq (1 - \alpha) u(s, g, b(g)) > (1 - \alpha) b(g \cdot s).$$

If $\alpha = 0$, then we obtain $b(g \cdot s) \geq x_j(g \cdot s) = u(s, g, x_j(g)) \geq u(s, g, b(g))$, which contradicts to (3.2), and hence, $\alpha$ is strictly positive. Moreover, the eventually strong monotonicity and strong concavity of the semiflow show that

$$u(s + t_0, g, x_j(g)) \gg (1 - \alpha) u(t_0, g \cdot s, b(g \cdot s)).$$

The property of cones implies that we can find $0 < \alpha_0 < \alpha$ such that

$$u(s + t_0, g, x_j(g)) \gg (1 - \alpha_0) u(t_0, g \cdot s, b(g \cdot s)).$$

Using the eventually strong monotonicity and strong concavity of the semiflow again, it then follows from (3.1) that

$$u(t, g, x_j(g)) \gg (1 - \alpha_0) b(g \cdot t), \quad \forall t \geq s + t_0.$$

Since the flow is minimal, there exists a sequence $t_n \to \infty$ such that

$$\lim_{n \to \infty} (g \cdot t_n, u(t_n, g, x_j(g)) = (g, x_k(g))$$

for some $k \in \{1, \dots, N\}$. Thus, we have

$$x_k(g) \geq (1 - \alpha_0) b(g);$$

i.e., $b(g) - x_k(g) \leq \alpha_0 b(g) = \alpha_0 e$, which contradicts to (3.4). Hence, $b$ is invariant under the flow $\sigma$.

Define

$$\mathcal{O}_b = \{(g, b(g)) : g \in Y\}.$$

Finally, we verify that $\mathcal{O}_b = \mathcal{O}$. On the contrary, assume that there exist $g \in Y$ and $j \in \{1, \dots, N\}$ such that $b(g) > x_j(g)$. The eventually strong monotonicity of the semiflow implies that $b(g) \gg x_j(g)$, for all $g \in Y$, $j \in \{1, \dots, N\}$, which contradicts that $b$ is the supremum. Hence, we get $\mathcal{O}_b = \mathcal{O}$. Furthermore, the conclusion that $\mathcal{O}$ is a copy of the base $Y$ can be obtained straight. □

**Corollary 3.6.** *Let the assumptions of Theorem 3.5 hold. Then $\mathcal{O}$ is an equilibrium point set.*

*Proof.* By Theorem 3.5, we have

$$\mathcal{O} = \{(g, b(g)) : g \in Y\},$$

and the map $g \mapsto b(g)$ is a bijection with $b(g \cdot t) = u(t, g, b(g))$, $\forall g \in Y$, $t \geq 0$. Hence, $\mathcal{O}$ is the equilibrium point set. □

**Lemma 3.7.** *Assume that two omega limit sets satisfy $\mathcal{O}_1 \ll_C \mathcal{O}_2$. Then there exists a positive constant $c_1$ such that*

$$\|u_x(t, g, x_2)\| \leq c_1, \quad \forall (g, x_2) \in \mathcal{O}_2,\ t \geq 0.$$

*Proof.* In view of the proof of [6, Lemma 5.6], we know that, for $e \gg 0$ there exists a constant $\bar{c}$ (depending on $e$) such that

$$\|u_x(t, g, x)\| \leq \bar{c}\|u_x(t, g, x)e\|, \quad \forall (g, x) \in Y \times X,\ t \geq 0. \tag{3.5}$$

The conclusion of [6, Lemma 5.3] implies that there exists a positive constant $\beta > 0$ such that $x_2 - x_1 \geq \beta e$, for all $(g, x_1) \in \mathcal{O}_1$, $(g, x_2) \in \mathcal{O}_2$. The positiveness of the linear operator $u_x(t, g, x_2)$ shows that

$$u_x(t, g, x_2)(x_2 - x_1) \geq \beta u_x(t, g, x_2)e.$$

The monotonicity and concavity of the semiflow and (2.6) show that

$$\|u_x(t, g, x_2)\| \leq \frac{\bar{c}}{\beta}\|u_x(t, g, x_2) - u_x(t, g, x_1)\|, \quad \forall t \geq 0.$$

From the above and the compact positive invariance of $\mathcal{O}_1$ and $\mathcal{O}_2$ we can conclude that there exists a positive constant $c_1$ such that

$$\|u_x(t, g, x_2)\| \leq c_1, \quad \forall (g, x_2) \in \mathcal{O}_2,\ t \geq 0.$$

The proof is complete. □

**Proposition 3.8.** *If $\mathcal{O}_1 \ll_C \mathcal{O}_2$ holds, then $\mathcal{O}_2$ is a linearly stable set, i.e., $\lambda_{\mathcal{O}_2} \leq 0$.*

*Proof.* By Definition 3.4 and Lemma 3.7, the conclusion can be obtained immediately. □

**Proposition 3.9.** *There exists the function $g \mapsto a(g)$ such that the set*

$$Y_0 = \{g \in Y : (g, a(g)) \in \mathcal{O}\}$$

*is the continuous point set of the mapping $g \mapsto a(g)$.*

*Proof.* It is sufficient to prove that for any $g_k \to g$ there exists $g \mapsto a(g)$ such that $a(g_k) \to a(g)$. Because of the minimality of the flow, we only to prove $a(g \cdot t_k) \to a(g \cdot t_0)$ for any $t_k \to t_0$. Let $(g, x) \in \mathcal{O}$, from the definition of the omega limit set, there exists a sequence $t_n \to \infty$ such that $g_0 \cdot t_n \to g$, $u(t_n, g_0, x_0) \to x$. Let

$$a(g) := \lim_{n\to\infty} u(t_n, g_0, x_0) = x.$$

Then

$$\begin{aligned}
a(g \cdot t_0) &= \lim_{n\to\infty} u(t_n, g_0 \cdot t_0, u(t_0, g_0, x_0)) \\
&= \lim_{n\to\infty} u(t_n + t_0, g_0, x_0) \\
&= \lim_{n\to\infty} u(t_0, g_0 \cdot t_n, u(t_n, g_0, x_0)) \\
&= u(t_0, g, x),
\end{aligned}$$

and for any $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{k\to\infty} a(g\cdot t_k) &= \lim_{k\to\infty}\lim_{n\to\infty} u(t_n, g_0\cdot t_k, u(t_k, g_0, x_0))\\ &= \lim_{k\to\infty}\lim_{n\to\infty} u(t_k, g_0\cdot t_n, u(t_n, g_0, x_0))\\ &= \lim_{k\to\infty} u(t_k, g, x)\\ &= u(t_0, g, x) = a(g\cdot t_0).\end{aligned}$$

The proof is complete. □

From [6, Proposition 6.1], we have the following result .

**Proposition 3.10.** *Suppose that $\mathcal{O}_1 \ll_C \mathcal{O}_2$. If $\lambda_{\mathcal{O}_2} = 0$, there exist positive constant $\hat{c}$ and $c$ such that*

$$\hat{c} \le \|u_x(t, g, x_2)\| \le c, \quad \forall (g, x_2) \in \mathcal{O}_2,\ t \ge 0. \tag{3.6}$$

**Proposition 3.11.** *Assume that $\mathcal{O}_1 \ll_C \mathcal{O}_2$ holds and $\lambda_{\mathcal{O}_2} = 0$. Then there exists a minimal subset $\mathcal{O}^*$ of $Y \times X$ such that $\mathcal{O}_1 \ll \mathcal{O}^* < \mathcal{O}_2$.*

*Proof.* As in Proposition 3.9, define $Y_0 = \{g \in Y : (g, a(g)) \in \mathcal{O}_2\}$. Let $g_0 \in Y_0$, from the definition of $Y_0$, we have $(g_0, a(g_0)) \in \mathcal{O}_2$. Since $\mathcal{O}_1 \ll_C \mathcal{O}_2$, for each $(g_0, x_1) \in \mathcal{O}_1$, we have $x_1 \ll a(g_0)$. Fixed $0 < \alpha < 1$, define

$$y_\alpha = \alpha x_1 + (1-\alpha)a(g_0).$$

Obviously, $x_1 \ll y_\alpha < a(g_0)$. The precompactness of the forward orbit $\{\pi(t, g_0, y_\alpha) : t \ge \delta,\ \delta > 0\}$ implies that its closure contains a minimal subset, denoted by $\mathcal{O}_\alpha$, i.e.,

$$\mathcal{O}_\alpha \subset \operatorname{cls}\{(g_0\cdot t, u(t, g_0, y_\alpha)) : t \ge \delta\}.$$

The monotonicity of the skew-product semiflow implies $\mathcal{O}_1 \le \mathcal{O}_\alpha \le \mathcal{O}_2$. In the following, we prove that $\mathcal{O}_\alpha$ is required.

First we check $\mathcal{O}_1 \ll \mathcal{O}_\alpha$. For $(g, z) \in \mathcal{O}_\alpha$, there exist a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \Pi(t_n, g_0, y_\alpha) = (g, z).$$

The concavity implies that

$$u(t_n, g_0, y_\alpha) \ge \alpha u(t_n, g_0, x_1) + (1-\alpha)u(t_n, g_0, a(g_0)).$$

In addition, there exists a subsequence (assume the whole sequence), $(g, z_1) \in \mathcal{O}_1$ and $(g, z_2) \in \mathcal{O}_2$ such that

$$\lim_{n\to\infty} \Pi(t_n, g_0, x_1) = (g, z_1), \quad \lim_{n\to\infty} \Pi(t_n, g_0, a(g_0)) = (g, z_2).$$

Hence, we have

$$z \ge \alpha z_1 + (1-\alpha)z_2.$$

Since $\mathcal{O}_1 \ll_C \mathcal{O}_2$, $z_1 \ll z_2$ holds, from which we have $z \gg z_1$, Definition 3.2 tells us $\mathcal{O}_1 \ll \mathcal{O}_\alpha$.

In the following we prove $\mathcal{O}_2 \ne \mathcal{O}_\alpha$. On the contrary, we assume that $\mathcal{O}_2 = \mathcal{O}_\alpha$ with $(g_0, a(g_0)) \in \mathcal{O}_2 \cap \mathcal{O}_\alpha$. Thus, there exists a sequence $t_k \to \infty$ such that $\lim_{n\to\infty} \Pi(t_k, g_0, y_\alpha) = (g_0, a(g_0))$. Proposition 3.10 implies that there exist a positive constant $\hat{c} > 0$ such that $\hat{c} \le \|u_x(t, g_0, a(g_0))\|$, $\forall t \ge 0$. From the inequality (2.6) we deduce that for all $k \in \mathbb{N}$,

$$u(t_k, g_0, a(g_0)) - u(t_k, g_0, y_\alpha) \ge u_x(t_k, g_0, a(g_0))(a(g_0) - y_\alpha)$$

$$= \alpha u_x(t_k, g_0, a(g_0))(a(g_0) - x_1).$$

It then follows from (3.5) and the monotonicity of the skew-product semiflow that for $e = (a(g_0) - x_1)$, we can find $l$ (which only depends on $a(g_0)$ and $x_1$) such that

$$\|a(g_0 \cdot t_k) - u(t_k, g_0, y_\alpha)\| \geq l > 0, \quad \forall k \in \mathbb{N}.$$

This contradicts that $g_0$ is a point of continuity of $a(g_0)$, which implies $\lim_{n\to\infty}(g_0 \cdot t_k, u(t_k, g_0, y_\alpha)) = (g_0, a(g_0))$. The proof is complete. □

**Theorem 3.12.** *If $\mathcal{O}_1 \ll_C \mathcal{O}_2$, then $\lambda_{\mathcal{O}_2} < 0$.*

*Proof.* Proposition 3.8 implies that $\lambda_{\mathcal{O}_2} \leq 0$, hence, it is sufficient to prove $\lambda_{\mathcal{O}_2} \neq 0$. On the contrary, we assume that $\lambda_{\mathcal{O}_2} = 0$. It follows from Proposition 3.11 that there exists the subset $\mathcal{O}^*$ of $Y \times X$ such that $\mathcal{O}_1 \ll \mathcal{O}^* < \mathcal{O}_2$. Let $g_0 \in Y_0$, then $(g_0, a(g_0)) \in \mathcal{O}_2$ and there exist $(g_0, z) \in \mathcal{O}^*$ and $(g_0, x_1) \in \mathcal{O}_1$ such that

$$x_1 \ll z < a(g_0).$$

Let $e = a(g_0) - x_1 \gg 0$ in (3.3) and define

$$\gamma = \inf\{\|a(g_0) - x\|_e : (g_0, x) \in \mathcal{O}^*\}.$$

It is easy to see that there exists $(g_0, x) \in \mathcal{O}^*$ such that $\gamma = \|a(g_0) - x\|_e$ with $0 < \gamma < 1$, which implies that $a(g_0) - x \leq \gamma(a(g_0) - x_1)$; i.e.,

$$x \geq (1-\gamma)a(g_0) + \gamma x_1.$$

Since $a(g_0) \gg x_1$, the monotonicity and strong concavity of the skew-product semiflow implies that

$$u(t, g_0, x) \gg (1-\gamma)u(t, g_0, a(g_0)) + \gamma u(t, g_0, x_1). \tag{3.7}$$

In view of the property of the cone, there exists $\gamma_0$ with $0 < \gamma_0 < \gamma$ such that

$$u(t, g_0, x) \gg (1-\gamma_0)a(g_0 \cdot t) + \gamma_0 u(t, g_0, x_1),$$

Hence, there exists $(g_0, y) \in \mathcal{O}^*$ such that

$$y \geq (1-\gamma_0)a(g_0) + \gamma_0 x_1;$$

i.e., $a(g_0) - y \leq \gamma_0(a(g_0) - x_1) = \gamma_0 e$, which implies that $\|a(g_0) - y\|_e \leq \gamma_0 < \gamma$. This contradicts the definition of $\gamma$. □

**Theorem 3.13.** *If $\mathcal{O}_1 \ll_C \mathcal{O}_2$, then $\mathcal{O}_2$ is the copy of the base $Y$, i.e., for each $g \in Y$, $\operatorname{card}(\mathcal{O}_2 \cap \pi^{-1}(g)) = 1$.*

*Proof.* Since $\mathcal{O}_1 \ll_C \mathcal{O}_2$, Theorem 3.12 tells us $\lambda_{\mathcal{O}_2} < 0$, the remaining is concluded by Theorem 3.5. □

Next, we introduce the main result of this article.

**Theorem 3.14.** *If* (A1) *and* (A2) *hold, then for any $(g, x) \in Y \times X^+ \setminus \{0\}$ either*

- (i) $\lim_{t\to\infty} \|u(t, g, x)\| = +\infty$, *or*
- (ii) *there exists an equilibrium point set $\mathcal{O}^* \subset Y \times \operatorname{int} X^+$ such that $\mathcal{O}(g, x) = \mathcal{O}^*$ and $\lim_{t\to\infty} \|u(t, g, x) - u(t, g, x^*)\| = 0$, where $(g, x^*) = \mathcal{O}^* \cap \pi^{-1}(g)$.*

*Proof.* On the contrary, we assume that (i) does not hold; i.e., the forward orbit of the skew-product semiflow is bounded, From (A1) we know $\{\Pi(t,g,x)|t \geq 0\}$ is precompact. The eventually strong monotonicity implies that if $(g,x) \in Y \times (X^+ \setminus \{0\})$, then $\mathcal{O}(g,x) =: \mathcal{O}^* \subset Y \times \operatorname{int} X^+$. It then follows from (A2) that $\mathcal{O}(g,0) =: \mathcal{O}^0 \subset Y \times \{0\}$. Hence, $\mathcal{O}^0 \ll_C \mathcal{O}^*$. Thus, Theorem 3.12 implies that $\lambda_{\mathcal{O}^*} < 0$. Furthermore, Theorem 3.13 and Corollary 3.6 show that $\mathcal{O}^*$ is a copy of the base $Y$ and an equilibrium set, i.e., $\operatorname{card}(\mathcal{O}^* \cap \pi^{-1}(g)) = 1$, for all $g \in Y$.

Next we prove that $\lim_{t\to\infty} \|u(t,g,x) - u(t,g,x^*)\| = 0$. On the contrary, we assume there exists a sequence $t_n \to \infty$ and a positive constant $\epsilon > 0$ such that $\|u(t_n,g,x) - u(t_n,g,x^*)\| > \epsilon$ for all $n \geq 1$. Denote $\lim_{n\to\infty} \Pi(t_n,g,x) = (\bar{g},\bar{x}_1)$ and $\lim_{n\to\infty} \Pi(t_n,g,x^*) = (\bar{g},\bar{x}_2)$, where $(g,x^*) = \mathcal{O}^* \cap \pi^{-1}(g)$. Since $\operatorname{card}(\mathcal{O}^* \cap \pi^{-1}(\bar{g})) = 1$, we have $\bar{x}_1 = \bar{x}_2$. Thus, $0 = \|\bar{x}_1 - \bar{x}_2\| = \lim_{n\to\infty} \|u(t_n,g,x) - u(t_n,g,x^*)\| \geq \epsilon$, a contradiction holds. Hence, $\lim_{t\to\infty} \|u(t,g,x) - u(t,g,x^*)\| = 0$. □

Consider the almost periodic delay differential equation

$$\begin{aligned} y'(t) &= f(t,y(t),y(t-1)), \quad \forall t \in \mathbb{R}^+, \\ y(s) &= \phi(s), \quad \forall s \in [-1,0], \end{aligned} \tag{3.8}$$

where $\phi \in C^+ := C([-1,0],\mathbb{R}^n_+)$, the function $f = (f_1,f_2,\dots,f_n) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$ is almost periodic ( Let $(X,d)$ be metric space, a function $f \in C(\mathbb{R},X)$ is said to be **almost periodic** if for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that every interval of $\mathbb{R}$ of length $l$ contains at least one point of the set $T(\epsilon) = \{\tau \in \mathbb{R} : d(f(t+\tau),f(t)) < \epsilon, \forall t \in \mathbb{R}\}$). In addition, we propose the following properties:

(i) for each $y,z \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $i \neq j$, $\frac{\partial f_i}{\partial y_j}(t,y,z) \geq 0$; If $\tilde{I}$ and $\tilde{J}$ form a partition of $N = \{1,2,\dots,n\}$, then there exist $\delta > 0$, $i \in \tilde{I}$ and $j \in \tilde{J}$, such that
$$\Big|\frac{\partial f_i}{\partial y_j}(t,y,z)\Big| \geq \delta, \quad \forall y,z \in \mathbb{R}^n, t \in \mathbb{R};$$

(ii) for $y,z \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $i,j \in \{1,2,\dots,n\}$, $\frac{\partial f_i}{\partial z_j}(t,y,z) \geq 0$. Furthermore, There exists $\delta > 0$ such that
$$\Big|\frac{\partial f_i}{\partial z_j}(t,y,z)\Big| \geq \delta;$$

(iii) there exists $g_0 \in Y$ such that $f$
  (a) is concave with respect to $(y,z)$, i.e., whenever $y^1 \leq y^2$, $z^1 \leq z^2$,
  $$f(t,\lambda(y^1,z^1) + (1-\lambda)(y^2,z^2)) \geq \lambda f(t,(y^1,z^1)) + (1-\lambda) f(t,(y^2,z^2))$$
  for $\lambda \in [0,1]$ and $t \in \mathbb{R}^+$;
  (b) is strongly concave with respect to $(y,z)$; i.e., whenever $y^1 \ll y^2$, $z^1 \ll z^2$,
  $$f(t,\lambda(y^1,z^1) + (1-\lambda)(y^2,z^2)) \gg \lambda f(t,(y^1,z^1)) + (1-\lambda) f(t,(y^2,z^2));$$
  for $\lambda \in (0,1)$ and $t \in [0,1]$;

(iv) $f(\cdot,0,0) \equiv 0$.

We embed (3.8) into the skew-product semiflow $\Pi : \mathbb{R}^+ \times Y \times C^+ \to Y \times C^+$

$$\Pi(t,g,\phi) \mapsto (\sigma_t(g), u(t,g,\phi)), \tag{3.9}$$

where for $\theta \in [-1,0]$, $u(t,g,\phi)(\theta) = y(t+\theta,g,\phi)$, and $\sigma_t(g(s,\cdot,\cdot)) = g(s,\cdot,\cdot)\cdot t = g(t+s,\cdot,\cdot)$. $y(t,g,\phi)$ is the solution of the equation

$$y'(t) = g(t, y(t), y(t-1)), \tag{3.10}$$

and for $\theta \in [-1,0]$ and $g = (g_1, g_2, \dots, g_n) \in Y$, $y(\theta,g,\phi) = \phi(\theta)$, where

$$Y := \mathrm{cls}\{f_t | t \geq 0, \quad f_t(s,\cdot,\cdot) = f(t+s,\cdot,\cdot)\},$$

the closure is defined in the topology of uniform convergence on compact set. From the above we deduce that $Y$ is compact metric space and $(Y, \sigma, \mathbb{R}^+)$ is minimal. By the standard theory of delay differential equations (refer to [2, 4]), we know that for all $g \in Y$ and initial value $\phi \in C$, (3.8) admit a unique solution $y(t,g,\phi)$, i.e., for $\theta \in [-1,0]$, $y(\theta,g,\phi) = \phi(\theta)$. If $y(t,g,\phi)$ is the unique solution of (3.8) in the existence interval of $t$, then $u(t,g,\phi)$ exists for all $t > 0$, and the forward orbit $\{u(t,g,\phi) | t \geq 1+\delta\}$ is precompact for $\delta > 0$.

**Theorem 3.15.** *The skew-product semiflow* (3.9) *is eventually strongly monotone and satisfies concavity and strongly concavity, respectively; i.e., there exists $g_0 \in Y$ such that*

$$\lambda u(t,g,v) + (1-\lambda)u(t,g,w) \leq u(t,g,\lambda v + (1-\lambda)w)$$

*whenever $w \geq v$, $t \geq 0$, $\lambda \in [0,1]$ and $g \in Y$, and*

$$\lambda u(t,g_0,v) + (1-\lambda)u(t,g_0,w) \ll u(t,g_0,\lambda v + (1-\lambda)w)$$

*whenever $w \gg v$, $t \geq 1$ and $\lambda \in (0,1)$.*

*Proof.* The eventually strong monotonicity can be obtained from [6, 7]. Let $\lambda \in (0,1)$ and $Z_g(t) = \lambda y(t,g,v) + (1-\lambda)y(t,g,w)$, so

$$Z_g' = \lambda g(t, y(t,g,v), v(t-1)) + (1-\lambda) g(t, y(t,g,w), w(t-1)), \ \forall t \in [0,1].$$

By the monotonicity of the skew-product semiflow, if $v \leq w$, then $y(t,g,v) \leq y(t,g,w)$. It then follows from (iii)(a) that

$$Z_g'(t) \leq g(t, Z_g(t), \lambda v(t-1) + (1-\lambda)w(t-1)), \quad \forall t \in [0,1].$$

From (i), (ii) and comparison theorems for this kind of ordinary differential equation (see [1]), we have

$$\lambda y(t,g,v) + (1-\lambda)y(t,g,w) \leq y(t,g,\lambda v + (1-\lambda)w), \quad \forall t \in [0,1]$$

An inductive argument shows that for each $n \in \mathbb{N}$,

$$\lambda y(t,g,v) + (1-\lambda)y(t,g,w) \leq y(t,g,\lambda v + (1-\lambda)w), \quad \forall t \in [n,n+1].$$

Hence,

$$\lambda u(t,g,v) + (1-\lambda)u(t,g,w)) \leq u(t,g,\lambda v + (1-\lambda)w), \quad \forall t \geq 0.$$

If $v \ll w$, the strong monotonicity implies $y(t,g_0,v) \ll y(t,g_0,w)$. From (iii)(b), for each $t \in [1,2]$,

$$z_{g_0}'(t) \ll g_0(t, z_{g_0}(t), \lambda v(t-1) + (1-\lambda)w(t-1)).$$

Using a same process, comparison theorems provide $Z_{g_0}(t) \ll y(t,g_0,\lambda v + (1-\lambda)w)$. Hence,

$$\lambda y(t,g_0,v) + (1-\lambda)y(t,g_0,w)) \ll y(t,g_0,\lambda v + (1-\lambda)w), \quad \forall t > 0.$$

That is,

$$\lambda u(t,g_0,v) + (1-\lambda)u(t,g_0,w)) \ll u(t,g_0,\lambda v + (1-\lambda)w), \quad \forall t > 1.$$

The proof is complete. □

**Theorem 3.16.** *If* (3.8) *admits a bounded solution* $y(t,\phi)$*, then there exists an almost periodic solution* $y^*(t)$*,* $\lim_{t\to\infty}\|y(t,\phi)-y^*(t)\|=0$ *for* $\phi\in C^+$ *with* $\phi(0)>0$*.*

*Proof.* Theorem 3.15 tells us that the skew-product semiflow (3.9) is eventually strongly monotone and (strongly) concave. For any $(g,\phi)\in Y\times C^+$ with $\phi(0)>0$, we conclude $\mathcal{O}^*:=\mathcal{O}(g,\phi)\subset Y\times \operatorname{int}C^+$. It then follows from Theorem 3.14 that $\lim_{t\to\infty}\|y(t,\phi)-y^*(t)\|=0$, where $(g,y^*(t))=\mathcal{O}^*\cap\pi^{-1}(g)$. □

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