Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 128, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OSCILLATION CRITERIA FOR EVEN-ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLES

SHUHUI WU, PARGAT SINGH CALAY, ZHANYUAN HOU

Abstract. In this article, we study the oscillatory behavior of solutions to even-order nonlinear neutral difference equations of the form

$$\Delta_\tau^m(x(t)-px(t-r))+f(t,x(g(t)))=0.$$

Using an integral transformation, the Riccati transformation, and iteration, we obtain sufficient conditions for all solutions to be oscillatory. Examples are also given to illustrate the obtained criteria.

1. Introduction

Difference equations have attracted a great deal of attention of researchers in mathematical, biological, physical sciences, and economy. This is specially due to the applications in various problems of biology, physics, economy, and so on. The topics studied for oscillation of the solutions have been investigated intensively and the references [1]–[17] are just a few examples.

In this article, we study even-order nonlinear neutral difference equations with continuous variable of the form

$$\Delta_\tau^m(x(t)-px(t-r))+f(t,x(g(t)))=0, \tag{1.1}$$

where m is an even integer $m \geq 4$, $p \geq 0$, τ and r are positive constants, $\Delta_\tau x(t) = x(t+\tau)-x(t)$, $0 < g(t) < t$, $g \in C^1([t_0,\infty),\mathbb{R}^+)$, $g'(t) > 0$, and $f \in C([t_0,\infty) \times \mathbb{R},\mathbb{R})$. Throughout this article we assume that

$$g(t+\tau) \geq g(t)+\tau \quad \text{for } t \geq t_0, \tag{1.2}$$

$$f(t,\ u)/u \geq q(t) > 0 \quad \text{for } u \neq 0 \text{ and } q \in C(\mathbb{R},\mathbb{R}^+). \tag{1.3}$$

Let $t_0^* = \min\{g(t_0),\ t_0-r\}$ and $I_0 = [t_0^*,\ t_0]$. A function x is called the *solution* of (1.3) with $x(t) = \varphi(t)$ for $t \in I_0$ and $\varphi \in C(I_0,\mathbb{R})$ if it satisfies (1.3) for $t \geq t_0$.

A solution x is called oscillatory if it has arbitrarily large zeros; otherwise it is non-oscillatory. x is eventually positive if there exists $t_1 \geq t_0$, such that $x(t) > 0$ for all $t \geq t_1$. Eventually negative definite is defined similarly.

This article is organized as follows. The main results is stated in section 2 and leave the proofs for section 5. Examples will be presented in section 3 to

---

2010 *Mathematics Subject Classification.* 39A10, 39A21.
Key words and phrases. Neutral difference equations; oscillation; continuous variable.

Submitted November 2, 2014. Published May 7, 2015.

demonstrate the application of the obtained results. In section 4, some lemmas will be given to be used in the proofs of the main results.

2. Statement of main results

The assumptions on g guarantee the existence and differentiability of its inverse g^{-1}. Let

$$\bar{q}(t) = \alpha \min_{t \le s \le t+m\tau} \{q(s)\} \Big(\min_{g(t) \le s \le g(t)+m\tau} \{(g^{-1}(s))'\} \Big)^m, \tag{2.1}$$

where $0 < \alpha < 1$. The function $\bar{q}$ will play an important role in the oscillatory criteria for (1.3). Throughout this article, we use the symbol $\lceil a \rceil$ to denote the smallest integer not less than a.

Theorem 2.1. *Assume that for some $t_1 \ge t_0$,*

$$\sum_{i=0}^{\infty} \bar{q}(t_1 + i\tau) = \infty \,. \tag{2.2}$$

Then for every solution $x(t)$ of (1.1), is either oscillatory, or for any $T \ge t_0$ there exists a $t_2 > T$ such that $|x(t_2)| \le p|x(t_2 - r)|$.

Theorem 2.2. *In addition to (2.2), we assume that $0 < p < 1$ and there is a positive integer k_0 and a $t_1 \ge t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \le n$ for all large enough n. Moreover, assume that there is a sequence $\{n_k\} \to \infty$ such that*

$$\sum_{i=m_1(n_k)}^{n_k} \bar{q}(t_1 + i\tau) \ge \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{2.3}$$

for all k large enough. Then for every solution $x(t)$ of (1.1), either $x(t)$ or $x(t) - px(t-r)$ or both are oscillatory.

Corollary 2.3. *In addition to (2.2), we assume that $0 < p < 1$ and there is a positive integer k_0 and a $t_1 \ge t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \le n$ for all sufficiently large n. Moreover, we assume that there is a sequence $\{n_k\} \to \infty$ such that*

$$\sum_{i=m_1(n_k)}^{n_k} (i - m_1 + 1)\bar{q}(t_1 + i\tau) \ge \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{2.4}$$

for all k large enough. Then for every solution $x(t)$ of (1.1), either $x(t)$ or $x(t) - px(t-r)$ or both are oscillatory.

Remark 2.4. Note that the requirement (2.4) for $\bar{q}(t)$ is weaker than (2.3) since $(i - m_1 + 1) \ge 1$ holds in (2.4).

Corollary 2.5. *In addition to (2.2), we assume that $0 < p < 1$ and that there is a positive integer k_0 and a $t_1 \ge t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \le n$ for all sufficiently large n. Moreover, assume that there is a sequence $\{n_k\} \to \infty$, and an integer l, $1 \le l < m$ such that*

$$\frac{1}{l!} \sum_{i=m_1(n_k)}^{n_k} (i - m_1 + 1)(i - m_1 + 2) \dots (i - m_1 + l)\bar{q}(t_1 + i\tau) \ge \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{2.5}$$

for all k large enough. Then for every solution $x(t)$ of (1.1), either $x(t)$ or $x(t) - px(t-r)$ or both are oscillatory.

Remark 2.6. Note that (2.5) coincides with (2.4) for $l = 1$. For $l > 1$,
$$\frac{(i-m_1+1)(i-m_1+2)\dots(i-m_1+l)}{l!} \geq i - m_1 + 1 \geq 1\,.$$
Thus, (2.5) is weaker than (2.4) and (2.3).

Theorem 2.7. *In addition to (2.2), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1+n\tau)-t_1+k_0r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover assume that there is a sequence $\{n_k\} \to \infty$ such that*
$$\sum_{i=m_1(n_k)}^{n(k)} \bar{q}(t_1+i\tau) \geq \frac{1}{k_0} \tag{2.6}$$
for k large enough. Then for every solution $x(t)$ of (1.1), either $x(t)$ or $x(t)-x(t-r)$ or both are oscillatory.

Corollary 2.8. *In addition to (2.2), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n_k) = \lceil (g(t_1+n\tau)-t_1+k_0r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover assume that there is a sequence $\{n_k\} \to \infty$ such that*
$$\sum_{i=m_1(n_k)}^{n_k} (i-m_1+1)\bar{q}(t_1+i\tau) \geq \frac{1}{k_0} \tag{2.7}$$
for k large enough. Then for every solution $x(t)$ of (1.1), either $x(t)$ or $x(t)-x(t-r)$ or both are oscillatory.

Corollary 2.9. *In addition to (2.2), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n_k) = \lceil (g(t_1+n\tau)-t_1+k_0r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover we assume that there is a sequence $\{n_k\} \to \infty$ and an integer l, $1 \leq l \leq m-1$ such that*
$$\frac{1}{l!}\sum_{i=m_1(n_k)}^{n_k} (i-m_1+1)(i-m_1+2)\dots(i-m_1+l)\bar{q}(t_1+i\tau) \geq \frac{1}{k_0} \tag{2.8}$$
for k large enough. Then, for every solution $x(t)$ of (1.1), either $x(t)$ or $x(t) - x(t-r)$ or both are oscillatory.

Theorem 2.10. *In addition to (2.2), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n_k) = \lceil (g(t_1+n\tau)-t_1+k_0r)/\tau \rceil \leq n$ for all large enough n. Moreover assume that there is a sequence $\{n_k\} \to \infty$ such that*
$$\sum_{i=m_1(n_k)}^{n_k} \bar{q}(t_1+i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{2.9}$$
holds for all k large enough. Then, for every bounded solution $x(t)$ of (1.1), either $x(t)$ or $x(t)-px(t-r)$ or both are oscillatory.

Corollary 2.11. *In addition to (2.2), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1+n\tau)-t_1+k_0r)/\tau \rceil \leq n$ for all large enough n. Moreover assume that there is a sequence $\{n_k\} \to \infty$ such that*
$$\sum_{i=m_1(n_k)}^{n_k} (i-m_1+1)\bar{q}(t_1+i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{2.10}$$

holds for all k large enough. Then for every bounded solution $x(t)$ of (1.1)*, either $x(t)$ or $x(t)-px(t-r)$ or both are oscillatory.*

Corollary 2.12. *In addition to* (2.2)*, we assume that $p>1$ and that there is a positive integer k_0 and a $t_1\geq t_0$ satisfying $m_1(n)=\lceil(g(t_1+n\tau)-t_1+k_0r)/\tau\rceil\leq n$ for all sufficiently large n. Moreover assume that there is a sequence $\{n_k\}\to\infty$ and an integer l, $1\leq l<m$, such that*

$$\frac{1}{l!}\sum_{i=m_1(n_k)}^{n_k}(i-m_1+1)(i-m_1+2)\dots(i-m_1+l)\,\bar{q}(t_1+i\tau)\geq\frac{p^{k_0}(1-p)}{1-p^{k_0}}\tag{2.11}$$

holds for all k large enough. Then, for every bounded solution $x(t)$ of equation (1.1)*, either $x(t)$ or $x(t)-px(t-r)$ or both are oscillatory.*

Remark 2.13. Note that

$$\frac{(i-m_1+1)(i-m_1+2)\dots(i-m_1+l-1)}{(l-1)!}\geq i-m_1+1\geq 1$$

holds. Thus, (2.11) is weaker that (2.10) and (2.9).

Corollary 2.14. *In addition to* (2.2)*, we assume that $p>1$ and that there is a positive integer k_0 and a $t_1\geq t_0$ satisfying $m_1(n)=\lceil(g(t_1+n\tau)-t_1+k_0r)/\tau\rceil\leq n$ for all sufficiently large n. Moreover assume that there is a sequence $\{n_k\}\to\infty$ such that*

$$\frac{1}{(n-1)!}\sum_{i=m_1(n_k)}^{n_k}\frac{(i-m_1+n-1)!}{(i-m_1)!}\,\bar{q}(t_1+i\tau)\geq\frac{p^{k_0}(1-p)}{1-p^{k_0}}\tag{2.12}$$

holds for all k large enough. Then, for every bounded solution $x(t)$ of (1.1)*, either $x(t)$ or $x(t)-px(t-r)$ or both are oscillatory.*

3. Examples

Three illustrating examples are given here to demonstrate the applications of the obtained oscillatory criteria.

Example 3.1. Consider the linear difference equation

$$\Delta_\tau^{2n}(x(t)-px(t-r))+\frac{1}{t}x(t-\frac{\sigma}{1+\beta t})=0\tag{3.1}$$

for $t>0$, where n is a positive integer, $p\geq 0$, $\beta\geq 0$, the constants r, τ and σ are positive. Viewing (3.1) as (1.1), we have $q(t)=1/t$ and $g(t)=t-\sigma/(1+\beta t)$. Then, according to (2.1), $\bar{q}(t)=\alpha/(t+2n\tau)$ for $\beta=0$ and

$$\bar{q}(t)=\frac{\alpha}{t+2n\tau}\Big(1-\frac{\sigma\beta}{(1+\beta t)^2+\sigma\beta}\Big)^{2n}$$

for $\beta>0$. Since $\bar{q}_{2n}(t)\geq\alpha'/(t+2n\tau)$ for some $\alpha'>0$ and all $t\geq 0$, $\bar{q}_{2n}$ satisfies (2.2) with $t_1=0$. By Theorem 2.1, for every solution $x(t)$ of (3.1), either $x(t)$ is oscillatory or for any $T\geq t_0$ there exists a $t_2>T$ such that $|x(t_2)|<p|x(t_2-r)|$. In particular, when $p=0$, every solution of (3.1) is oscillatory.

Example 3.2. *Consider the difference equation*

$$\Delta_\pi^{2n}(x(t)-px(t-\pi))+8x(t-\pi)+\frac{8\sigma}{1+t^2}x^3(t-\pi)=0,\tag{3.2}$$

where $\sigma \geq 0$ is a constant. Regarding (3.2) as (1.1), we have $\tau = \pi$, $r = \pi$, $g(t) = t - \pi$ and $q(t) = 8$. Then, for some $\alpha \in (0,1)$, $\bar{q}_{2n} = 8\alpha$ by (2.1) so (2.2) is satisfied. For $p = 1$, $k_0 = 1$ and $t_1 = t$, we have $m_l = l$ and

$$\sum_{s=m_l}^{l} (s + 1 - m_l)\bar{q}_{2n}(t_1 + s\tau) = 8\alpha > 1 = \frac{1}{k_0}$$

if $\alpha > 1/8$. Also we have

$$\sum_{s=m_l}^{l} (s + 1 - m_l)\bar{q}_{2n}(t_1 + s\tau) = 8\alpha > p = \frac{(1-p)p^{k_0}}{1 - p^{k_0}}$$

if $p \in (0,1) \cup (1,8)$ and $\alpha > p/8$. According to Theorems 2.2 and 2.7, for every solution $x(t)$ of (3.2), either $x(t)$ or $x(t) - px(t - r)$ is oscillatory if $0 < p \leq 1$. Furthermore, by Theorem 2.10, for every bounded solution $x(t)$ of (3.2), either $x(t)$ or $x(t) - px(t - r)$ is oscillatory if $1 < p < 8$.

Example 3.3. Consider the difference equation

$$\Delta_\tau^{2n}(x(t) - x(t - r)) + 2^{2n+1}x(t - 3) = 0, \tag{3.3}$$

where τ and r are positive odd integers. Viewing (3.3) as (1.1), we have $g(t) = t - 3$ and $q(t) = 2^{2n+1}$. Then, for some $\alpha \in (0,1)$, $\bar{q}_{2n} = \alpha 2^{2n+1}$ by (2.1) so (2.2) is satisfied. For $p = 1$, $k_0 = 1$ and $t_1 = t$, we have $m_l = l$ and

$$\sum_{s=m_l}^{l} (s + 1 - m_l)\bar{q}_{2n}(t_1 + s\tau) = \alpha 2^{2n+1} > 1 = \frac{1}{k_0}$$

if $\alpha > 2^{-(2n+1)}$. According to Theorem 2.7, for every solution $x(t)$ of (3.3), either $x(t)$ or $x(t) - x(t - r)$ is oscillatory.

4. Related lemmas

In this section, we present the lemmas which will be needed in the proofs of the main results. The following lemma can be found in [1, page 31].

Lemma 4.1. *Let $u(k)$ be defined on $N(a)$, where $a \in N$, and $u(k) > 0$ with $\Delta^m u(k)$ of constant sign on $N(a)$ for any positive integer m and not identically zero. Then, there exists an integer h, $0 \leq h \leq m$, with $m + h$ odd for $\Delta^m u(k) \leq 0$ or $m + h$ even for $\Delta^m u(k) \geq 0$ such that*

(i) *$h \leq m - 1$ implies $(-1)^{h+i}\Delta^i u(k) > 0$ for all $k \in N(a)$, $h \leq i \leq m - 1$,*
(ii) *$h \geq 1$ implies $\Delta^i u(k) > 0$ for all $k \in N(a)$, $1 \leq i \leq h - 1$.*

By applying the above result to the difference with continuous variables, we have the following lemma.

Lemma 4.2. *Let $y(t)$ be defined on $[t_0, +\infty)$ where $t_0 \in \mathbb{R}$, and $y(t) > 0$ with $\Delta_\tau^m y(t)$ of constant sign on $[t_0, +\infty)$ for any positive integer m and not identically zero. Then, there exists an integer h, $0 \leq h \leq m$, with $m + h$ odd for $\Delta_\tau^m y(t) \leq 0$ or $m + h$ even for $\Delta_\tau^m y(t) \geq 0$ such that*

(i) *$h \leq m - 1$ implies $(-1)^{h+i}\Delta_\tau^i y(t) > 0$ for all $t \in [t_0, \infty)$, $h \leq i \leq m - 1$,*
(ii) *$h \geq 1$ implies $\Delta_\tau^i y(t) > 0$ for all $t \in [t_0, +\infty)$, $1 \leq i \leq h - 1$.*

Proof. Let t_1 be any constant real number in $[t_0, +\infty)$. For this fixed t_1, by the assumption, we have $y(t_1 + k\tau)$ defined for any $k \in \{0, 1, \dots\}$, and $y(t_1 + k\tau) > 0$ with $\Delta_\tau^m y(t_1 + k\tau)$ of constant sign for any $k \in \{0, 1, \dots\}$ and for any positive integer m and not identically zero. Thus, by Lemma 4.1, the conclusion holds with the replacement of t by $t_1 + k\tau$ for all $k \in N$. Since $t_1 \in [t_0, \infty)$ is arbitrary, we can see that the conclusion holds for $t \in [t_0, +\infty)$. $\square$

Lemma 4.3 ([1, page 289])**.** *Let $y(t)$ be an m times differentiable function on $\mathbb{R}_+$ of constant sign satisfying $y^{(m)}(t) \not\equiv 0$ and $y^{(m)}(t)y(t) \le 0$ on $[t_1, \infty)$. Then the following statements hold.*

- (i) *There exists a $t_2 \ge t_1$ such that the functions $y^{(j)}(t)$, $j = 1, 2, \dots, m-1$, are of constant sign on $[t_2, \infty)$.*
- (ii) *There exists an integer $k < m$ which is odd (even) when m is even (odd), such that*

$$
\begin{gathered}
y(t)y^{(j)}(t) > 0 \quad \text{for } j = 0, 1, \dots, k,\ t \ge t_2, \\
(-1)^{m+j+1} y(t) y^{(j)}(t) > 0 \quad \text{for } j = k+1, \dots, m,\ t \ge t_2.
\end{gathered}
$$

Lemma 4.4 ([5, page 289])**.** *Assume that $y(t)$, $y'(t), \dots, y^{(m-1)}(t)$ are absolutely continuous and of constant sign on the interval (t_0, ∞), and assume $y^{(m)}(t)y(t) \ge 0$. Then either $y^{(k)}(t)y(t) \ge 0$, $k = 0, 1, \dots, m$ or there exists an integer $l, 0 \le l \le m-2$, which is even (odd) when m is even (odd), such that*

$$
\begin{gathered}
y^{(k)}(t)y(t) \ge 0, \quad \text{for } k = 0, 1, \dots, l, \\
(-1)^{m+k} y^{(k)}(t) y(t) \ge 0, \quad \text{for } k = l+1, \dots, m.
\end{gathered}
$$

Lemma 4.5. *Assume that $x(t)$ is an eventually positive (negative) solution of* (1.1) *such that $y(t) = x(t) - px(t-r) > 0\,(< 0)$ eventually. Then $\Delta_\tau y(t) > 0\,(< 0)$ and $\Delta_\tau^{m-1} y(t) > 0\,(< 0)$ hold eventually.*

Proof. Suppose $x(t) > 0$ and $y(t) > 0$ hold eventually. Due to $g(t) < t$, $g'(t) > 0$ and (1.2), there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \ge t_1$. Further, (1.1) becomes

$$
\Delta_\tau^m y(t) + f(t, x(g(t))) = 0.
$$

According to (1.3), $f(t, x(g(t))) \ge q(t)x(g(t)) > 0$ for $t \ge t_1$ hold. Therefore,

$$
\Delta_\tau^m y(t) \le -q(t)x(g(t)) < 0 \tag{4.1}
$$

for all large enough t, namely, $\Delta_\tau^m y(t) < 0$ eventually. By Lemma 4.2, h could be odd with $1 \le h \le m-1$. For all cases, we could obtain $\Delta_\tau y(t) > 0$ and $\Delta_\tau^{m-1} y(t) > 0$ eventually. If $x(t) < 0$ and $y(t) < 0$ hold eventually, then (4.1) becomes $\Delta_\tau^m y(t) \ge -q(t)x(g(t)) > 0$. Applying Lemma 4.2 to $-y(t)$, we obtain $\Delta_\tau y(t) < 0$ and $\Delta_\tau^{m-1} y(t) < 0$. $\square$

Lemma 4.6. *Let the hypothesis of Lemma 4.5 be satisfied. Moreover, let $\bar{q}(t)$ be defined by* (2.1)*. Set*

$$
u(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \dots \int_{t_{m-2}}^{t_{m-2}+\tau} dt_{m-1} \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta)\, d\theta.
$$

Then u satisfies $u^{(m)}(t) = \Delta_\tau^m y(t) < 0$ (> 0), $u(t) > 0$ (< 0), $u'(t) > 0$ (< 0), $u^{(m-1)}(t) > 0$ (< 0), $\Delta_\tau^{m-1} u(t) > 0$ (< 0), and

$$\Delta_\tau^m u(t) + \bar{q}(t)\, u(g(t) - kr) \sum_{i=0}^{k} p^i \le 0\, (\ge 0)$$

for each fixed number k and for all large enough t.

Proof. Suppose $x(t) > 0$ and $y(t) > 0$ hold eventually. According to the definition of $u(t)$ and (4.1), we can see that $u(t) > 0$, $u^{(m)}(t) = \Delta_\tau^m y(t) < 0$ and

$$\Delta_\tau^m y(t) + q(t)x(g(t)) \le 0 \tag{4.2}$$

for sufficiently large t. Taking into account the definition of $y(t)$, we have

$$\Delta_\tau^m y(t) + q(t)(y(g(t)) + px(g(t) - r)) \le 0.$$

By repeating the above process k times, we deduce

$$\Delta_\tau^m y(t) + q(t) \sum_{i=0}^{k} p^i y(g(t) - ir) + q(t)p^{k+1} x(g(t) - (k+1)r) \le 0.$$

Therefore, since $q(t)p^{k+1}x(g(t) - (k+1)r) \ge 0$, it follows that

$$\Delta_\tau^m y(t) + q(t) \sum_{i=0}^{k} p^i\, y(g(t) - ir) \le 0.$$

Furthermore,

$$u^{(m)}(t) + q(t) \sum_{i=0}^{k} p^i\, y(g(t) - ir) \le 0. \tag{4.3}$$

Then, for large enough t, the assumptions on g and q give

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_{m-2} \dots \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{s_{m-1}}^{s_{m-1}+\tau} y(g(\theta) - ir) q(\theta)\, d\theta \\
&\ge \min_{t \le l \le t+m\tau} \{q(l)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_{m-2} \dots \\
&\quad \times \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{s_{m-1}}^{s_{m-1}+\tau} y(g(\theta) - ir)\, d\theta \\
&\ge \min_{t \le l \le t+m\tau} \{q(l)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))' ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))'\, ds_2 \dots \\
&\quad \times \int_{s_{m-2}}^{g(g^{-1}(s_{m-2})+\tau)} (g^{-1}(s_{m-1}))' ds_{m-1} \int_{s_{m-1}}^{g(g^{-1}(s_{m-1})+\tau)} y(\theta - ir)(g^{-1}(\theta))' d\theta \\
&\ge \min_{t \le l \le t+m\tau} \{q(l)\} \Big(\min_{g(t) \le s \le g(t)+m\tau} (g^{-1}(s))' \Big)^m \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \dots \\
&\quad \times \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{s_{m-1}}^{s_{m-1}+\tau} y(\theta - ir) d\theta \\
&\ge \min_{t \le l \le t+m\tau} \{q(l)\} \Big(\min_{g(t) \le s \le g(t)+m\tau} (g^{-1}(s))' \min)^m u(g(t) - ir) \\
&\ge \bar{q}(t) u(g(t) - ir).
\end{aligned}$$

Thus, integration on both sides of (4.3) gives

$$\Delta_\tau^m u(t) + \bar{q}(t)\sum_{i=0}^{k} p^i u(g(t)-ir) \le 0. \tag{4.4}$$

According to the definition of $u(t)$, the equality

$$u'(t) = \int_t^{t+\tau} dt_2 \int_{t_2}^{t_2+\tau} dt_3 \dots \int_{t_{m-2}}^{t_{m-2}+\tau} dt_{m-1} \int_{t_{m-1}}^{t_{m-1}+\tau} \Delta_\tau y(\theta)\, d\theta$$

holds. Then it follows from Lemma 4.5 that $u'(t) > 0$. Similarly, we have

$$u^{(m-1)}(t) = \int_t^{t+\tau} \Delta_\tau^{m-1} y(\theta) d\theta$$

so $u^{(m-1)}(t) > 0$ from Lemma 4.5. Hence,

$$\Delta_\tau^{m-1} u(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \dots \int_{t_{m-2}}^{t_{m-2}+\tau} u^{(m-1)}(\theta) d\theta > 0.$$

Further, (4.4) implies

$$\Delta_\tau^m u(t) + \bar{q}(t) u(g(t)-kr) \sum_{i=0}^{k} p^i \le 0$$

for each fixed natural number k and for all large enough t. If $x(t) < 0$ and $y(t) < 0$ hold eventually, then $u(t) < 0$, $u^{(m)}(t) = \Delta_\tau^m y(t) > 0$ and $\Delta_\tau^m y(t) + q(t)x(g(t)) \ge 0$ for large enough t. Moreover, (4.3) becomes

$$u^{(m)}(t) + q(t)\sum_{i=0}^{k} p^i y(g(t)-ir) \ge 0$$

and (4.4) becomes

$$\Delta_\tau^m u(t) + \bar{q}(t)\sum_{i=0}^{k} p^i u(g(t)-ir) \ge 0.$$

That $u'(t) < 0$ and $u^{(m-1)}(t) < 0$ follow from $\Delta_\tau y(t) < 0$ and $\Delta_\tau^{m-1} y(t) < 0$. Then $\Delta_\tau^{m-1} u(t) < 0$ follows from the integration of $u^{(m-1)}(t)$. Since $u(t)$ is decreasing, each $u(g(t)-ir)$ can be replaced by $u(g(t)-kr)$ in the above inequality. □

5. Proofs of the main results

Proof of Theorem 2.1. Let $x(t)$ be a solution of (1.1) satisfying $x(t) > 0$ and $x(t) - px(t-r) > 0$ for all large t. Let $y(t)$ be as in Lemma 4.5 and $u(t)$ be as in Lemma 4.6. Furthermore, for any positive integer k, we have

$$\Delta_\tau^m u(t) + \bar{q}(t) u(g(t)-kr) \sum_{i=0}^{k} p^i \le 0,$$

where $u(g(t)-kr) > 0$. Define the Riccati transformation by

$$v(t) = \frac{\Delta_\tau^{m-1} u(t)}{u(g(t)-kr)}.$$

Notice that $v(t) > 0$. Moreover we deduce

$$\Delta_\tau v(t) = v(t+\tau) - v(t)$$

$$
\begin{aligned}
&= \frac{\Delta_\tau^{m-1}u(t+\tau)}{u(g(t+\tau)-kr)} - \frac{\Delta_\tau^{m-1}u(t)}{u(g(t)-kr)}\\
&= \frac{u(g(t)-kr)\Delta_\tau^{m-1}u(t+\tau) - u(g(t+\tau)-kr)\Delta_\tau^{m-1}u(t)}{u(g(t+\tau)-kr)u(g(t)-kr)}\\
&= \frac{u(g(t)-kr)\Delta_\tau^{m-1}u(t+\tau) + u(g(t+\tau)-kr)(\Delta_\tau^{m}u(t) - \Delta_\tau^{m-1}u(t+\tau))}{u(g(t+\tau)-kr)u(g(t)-kr)}\\
&\le \frac{\Delta_\tau^{m}u(t)}{u(g(t)-kr)} - \frac{\Delta_\tau^{m-1}u(t+\tau)\Delta_\tau u(g(t)-kr)}{u(g(t)-kr)u(g(t+\tau)-kr)}\\
&\le -\bar{q}(t)\sum_{i=0}^{k} p^i - v(t+\tau)\frac{\Delta_\tau u(g(t)-kr)}{u(g(t)-kr)}\\
&\le -\bar{q}(t)\sum_{i=0}^{k} p^i.
\end{aligned}
$$

Therefore, there exists a $t_1 > t_0$ such that

$$
\Delta_\tau v(t_1+j\tau) + \bar{q}(t_1+j\tau)\sum_{i=0}^{k} p^i \le 0. \tag{5.1}
$$

Summing both sides of (5.1) from 0 to n, we have

$$
v(t_1+(n+1)\tau) - v(t_1) + \sum_{i=0}^{k} p^i \sum_{j=0}^{n} \bar{q}(t_1+j\tau) \le 0.
$$

Thus

$$
\sum_{i=0}^{k} p^i \sum_{j=0}^{n} \bar{q}(t_1+j\tau) < v(t_1) < \infty,
$$

which leads to a contradiction to (2.2). If $x(t)$ is a solution of (1.1) satisfying $x(t) < 0$ and $y(t) < 0$ eventually, from Lemmas 4.5 and 4.6, the above argument about $v(t)$ is still valid and also leads to a contradiction. Therefore, the conclusion of the theorem holds. □

Proof of Theorem 2.2. According to Theorem 2.1, if (2.2) holds, we have that every solution $x(t)$ of (1.1) is either oscillatory or for any $T \ge t_0$, there exists one $t_2 > T$ such that $|x(t_2)| \le p|x(t_2 - r)|$.

Assume that (1.1) has an eventually positive solution $x(t)$ such that $y(t) = x(t) - px(t-\tau)$ is not oscillatory. Then from Theorem 2.1, we deduce that $y(t) < 0$ for all large enough t. Let $z(t) = -y(t)$. Therefore, $z(t) > 0$ and

$$
\Delta_\tau^m z(t) - f(t, x(g(t))) = 0.
$$

Moreover,

$$
\Delta_\tau^m z(t) \ge q(t)x(g(t)) > 0
$$

so

$$
\Delta_\tau^m z(t) - q(t)x(g(t)) \ge 0. \tag{5.2}
$$

For $z(t)$, according to Lemma 4.2, h is even. So $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \le i \le m-2$, and $|\Delta_\tau^j z(t)| > 0$ for all odd number j with $1 \le j \le m-1$.

We show that $\Delta_\tau z(t) < 0$. Indeed, if $\Delta_\tau z(t) > 0$, then, since $\Delta_\tau^2 z(t) > 0$, we may assume $\Delta_\tau z(t_1 + k\tau) > l > 0$ for a large enough t_1 and all $k \in N$. Then

$$\sum_{i=0}^{d} \Delta_\tau z(t_1 + i\tau) = z(t_1 + (d+1)\tau) - z(t_1) \geq (d+1)l.$$

Let $d \to \infty$, then $z(t_1 + (d+1)\tau) \to +\infty$. We have $\lim_{t\to\infty} x(t) = 0$ by repeating $x(t) < px(t-r)$ for $0 < p < 1$. Thus, by the definition of $z(t)$, we have $\lim_{t\to\infty} z(t) = 0$ which contradicts $z(t_1 + d\tau) \to +\infty$ as $d \to \infty$. Thus, $\Delta_\tau z(t) < 0$.

So, according to Lemma 4.2 again, $h = 0$. Thus, $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \leq i \leq m-2$, and $\Delta_\tau^j z(t) < 0$ for all odd number j with $1 \leq j \leq m-1$.

Notice $x(t) = (x(t+r) + z(t+r))/p$. Hence, from (5.2), it follows that

$$\Delta_\tau^m z(t) - \frac{q(t)}{p} z(g(t)+r) - \frac{q(t)}{p} x(g(t)+r) \geq 0,$$

and further

$$\Delta_\tau^m z(t) - q(t)\sum_{i=1}^{k} \frac{1}{p^i}\, z(g(t)+ir) - \frac{q(t)}{p^k} x(g(t)+kr) \geq 0.$$

So,

$$\Delta_\tau^m z(t) - q(t)\sum_{i=1}^{k} \frac{1}{p^i}\, z(g(t)+ir) > 0 \tag{5.3}$$

since $x(g(t)+kr) > 0$. Let

$$u(t) = \int_0^\tau ds_1 \int_{s_1}^{s_1+\tau} ds_2 \dots \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{t+s_{m-1}}^{t+s_{m-1}+\tau} z(\theta)d\theta.$$

Then we have $u^{(m)}(t) > 0$ and $u(t) > 0$. Since

$$u'(t) = \int_0^\tau ds_1 \int_{s_1}^{s_1+\tau} ds_2 \dots \int_{s_{m-2}}^{s_{m-2}+\tau} \Delta_\tau z(t+s_{m-1})ds_{m-1},$$

then $\Delta_\tau z(t) < 0$ implies $u'(t) < 0$. Moreover, $u^{(i)}(t) > 0$ for all even number i with $2 \leq i \leq m-2$, and $u^{(j)}(t) < 0$ for all odd number j with $1 \leq j \leq m-1$.

Integrating (5.3) and from the proof of Lemma 4.6 replacing $y(t)$ by $z(t)$, we have

$$\Delta_\tau^m u(t) - \bar{q}(t)\sum_{i=1}^{k} \frac{1}{p^i} u(g(t)+ir) > 0,$$

which leads to

$$\Delta_\tau^m u(t) - \bar{q}(t)u(g(t)+kr)\sum_{i=1}^{k} \frac{1}{p^i} > 0.$$

Due to $\sum_{i=1}^{k} 1/p^i = (1-p^k)/(p^k(1-p))$, we deduce that

$$\Delta_\tau^m u(t) \geq \frac{1-p^k}{p^k(1-p)}\bar{q}(t)\, u(g(t)+kr) > 0.$$

Replacing k by k_0 and t by $t_1 + i\tau$ in the above inequalities yield

$$\Delta_\tau^m u(t_1+i\tau) \geq \frac{1-p^{k_0}}{p^{k_0}(1-p)}\, \bar{q}(t_1+i\tau)u(g(t_1+i\tau)+k_0 r).$$

Summing up both sides of the above inequality for i from s to n and since $u'(t) < 0$, we have

$$\Delta_\tau^{m-1} u(t_1+(n+1)\tau) - \Delta_\tau^{m-1} u(t_1+s\tau) \geq \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{i=s}^{n} \bar{q}(t_1+i\tau),$$

which implies

$$-\Delta_\tau^{m-1} u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{i=s}^{n} \bar{q}(t_1+i\tau) \tag{5.4}$$

due to $\Delta_\tau^{m-1} u(t) < 0$. For the above inequality, we will reduce the order of $\Delta_\tau^{j} u(t_1+s\tau)$ by rewriting it as $\Delta_\tau^{j-1} u(t+(s+1)\tau) - \Delta_\tau^{j-1} u(t+s\tau)$ for $j = 1, 2, \dots, n-1$. Taking into account the fact that all even terms are positive and all odd terms are negative, we will write off all the negative terms from the left hand side of this inequality. It yields

$$u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{i=s}^{n} \bar{q}(t_1+i\tau).$$

Since $g(t_1+n\tau)+k_0 r \leq t_1+m_1\tau$ and u is decreasing, by taking $s = m_1$, we obtain

$$u(t_1+m_1\tau) > u(t_1+m_1\tau) \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{i=m_1}^{n} \bar{q}(t_1+i\tau),$$

i.e.,

$$\sum_{i=m_1}^{n} \bar{q}(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (2.3). If $x(t)$ is an eventually negative solution such that $y(t)$ is not oscillatory, then $y(t) > 0$ holds eventually. The above reasoning with an obvious minor modification also leads to a contradiction. Therefore, for every solution $x(t)$, either $x(t)$ or $y(t)$ is oscillatory. $\square$

Proof of Corollary 2.3. Without loss of generality, we suppose (1.1) has an eventually positive solution $x(t)$ such that $y(t) = x(t) - px(t-r)$ is not oscillatory. The proof is the same as that of Theorem 2.2 up to (5.4). By the same technique we reduce the order of the difference on the left hand side of this inequality down to the second order and it yields

$$\Delta_\tau^{2} u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{i=s}^{n} \bar{q}(t_1+i\tau).$$

Summing the above inequality for s from m_1 to n, we have

$$\Delta_\tau u(t_1+(n+1)\tau) - \Delta_\tau u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}(t_1+i\tau).$$

Due to $\Delta_\tau u(t) < 0$, it follows from the above inequality that

$$-\Delta_\tau u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}(t_1+i\tau),$$

so

$$u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0r) \sum_{s=m_1}^{n}\sum_{i=s}^{n}\bar{q}(t_1+i\tau).$$

According to $g(t_1+n\tau)+k_0r \le t_1+m_1\tau$ and u is decreasing, it follows that

$$u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(t_1+m_1\tau) \sum_{i=m_1}^{n}(i-m_1+1)\bar{q}(t_1+i\tau),$$

i.e.,

$$\sum_{i=m_1}^{n}(i-m_1+1)\bar{q}(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (2.4). Thus conclusion holds. □

Proof of Corollary 2.5. The proof is the same as that of Theorem 2.2 up to (5.4). We reduce the order of the difference at the left hand of this inequality down to the lth order as we did in the proof of Theorem 2.2. Since $1 \le l < m$, l is odd, we obtain

$$-\Delta_\tau^l u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0r) \sum_{i=s}^{n}\bar{q}(t_1+i\tau),$$

and if l is even,

$$\Delta_\tau^l u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0r) \sum_{i=s}^{n}\bar{q}(t_1+i\tau).$$

We can reach the same conclusion for the above two cases. Thus, we only give the details of the proof when l is odd. Summing up the above inequality for s from m_l to n, we have

$$\begin{aligned}
&-\Delta_\tau^{l-1}u(t_1+(n+1)\tau)+\Delta_\tau^{l-1}u(t_1+m_l\tau)\\
&> \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0r) \sum_{s=m_l}^{n}\sum_{i=s}^{n}\bar{q}(t_1+i\tau).
\end{aligned}$$

Since $\Delta_\tau^{l-1}u(t) > 0$, the above inequality implies

$$\Delta_\tau^{l-1}u(t_1+m_l\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0r) \sum_{s=m_l}^{n}\sum_{i=s}^{n}\bar{q}(t_1+i\tau).$$

By repeating the above procedure, we obtain

$$u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0r) \sum_{m_2=m_1}^{n}\sum_{m_3=m_2}^{n}\cdots\sum_{s=m_l}^{n}\sum_{i=s}^{n}\bar{q}(t_1+i\tau).$$

Because $g(t_1+n\tau)+k_0r \le t_1+m_1\tau$ and u is decreasing, we have

$$\begin{aligned}
1 &> \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{m_2=m_1}^{n}\sum_{m_3=m_2}^{n}\cdots\sum_{s=m_l}^{n}\sum_{i=s}^{n}\bar{q}(t_1+i\tau)\\
&= \frac{1-p^{k_0}}{p^{k_0}(1-p)}\Big(\sum_{m_2=m_1}^{n}\sum_{m_3=m_2}^{n}\cdots
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i=m_{l-1}}^{n} \frac{1}{2!}(i-m_{l-1}+1)(i-m_{l-1}+2)\bar{q}(t_1+i\tau)\Big) \\
&\dots \\
&= \frac{1-p^{k_0}}{p^{k_0}(1-p)}\Big(\sum_{i=m_1}^{n} \bar{q}(t_1+i\tau) \sum_{m_2=m_1}^{i} \frac{1}{(l-1)!}(i-m_2+1)(i-m_2+2) \\
&\quad \times \dots (i-m_2+(l-1))\Big) \\
&= \frac{1-p^{k_0}}{p^{k_0}(1-p)}\Big(\sum_{i=m_1}^{n} \frac{1}{l!}(i-m_1+1)(i-m_1+2)\times \dots (i-m_1+l)\bar{q}(t_1+i\tau)\Big),
\end{aligned}
$$

i.e.,

$$
\frac{1}{l!}\sum_{i=m_1}^{n}(i-m_1+1)\dots(i-m_1+l)\bar{q}(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.
$$

This inequality contradicts (2.5). Thus the conclusion holds. □

Proof of Theorem 2.7. The proof is similar to that of Theorem 2.2. However, the proof of the feature of $z(t)$ is different from that of Theorem 2.2 due to $p = 1$. We, hence, just give the proof about the feature of $z(t)$. For $z(t)$, by Lemma 4.2, we notice h could be even with $2 \le h \le m-2$. So $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \le i \le m-2$, and $|\Delta_\tau^j z(t)| > 0$ for all odd number j with $1 \le j \le m-1$.

If $\Delta_\tau z(t) > 0$, from the proof of Theorem 2.2 we have $z(t_1+d\tau) \to +\infty$ as $d \to \infty$ for some $t_1 \ge t_0$. Since $p = 1$, from $0 < x(t) < x(t-r)$, we know that $x(t)$ is bounded on $[t_0, \infty)$. Thus, $z(t)$ is bounded on $[t_0, \infty)$. This contradicts $z(t_1+d\tau) \to +\infty$ as $d \to \infty$. Thus, $\Delta_\tau z(t) < 0$. So, according to Lemma 4.2 again, $h = 0$. Thus, $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \le i \le m-2$ and $\Delta_\tau^j z(t) < 0$ for all odd number j with $1 \le j \le m-1$.

The rest of the proof is as in Theorem 2.2, replacing $p^{k_0}(1-p)/(1-p^{k_0})$ by $1/k_0$. □

The proofs of the following corollaries are very similar to those of Corollaries 2.3-2.5 except minor changes. Thus, we omit them.

Proof of Theorem 2.10. Suppose that $x(t)$ is a bounded eventually positive solution of (2.2). The proof of Theorem 2.2 is then still valid for Theorem 2.10 subject to a few obvious minor changes. Therefore, we omit the proof of the results following equation (1.1) with $p > 1$. □

Acknowledgements. S. Wu acknowledges support from the NNSF of China via Grant 11171306 and the interdisciplinary research funding from Zhejiang University of Science and Technology via Grant 2012JC09Y.

References

[1] R. Agarwal; *Difference equations and inequalities*, New York: Marcel Dekker, 2 (2000).

[2] R. Agarwal, M. Bohner, T. Li, C. Zhang; *Oscillation of third-order nonlinear delay differential equations*, Taiwanese J. Math. 17 (2013), 545-558.

[3] Yaşr Bolat, Öer Akin, H. Yildirim; *Oscillation criteria for a certain even order neutral difference equation with an oscillating coefficient*, Appl. Math. Lett. 22 (2009) 590-594.

[4] H. El-Morshedy; *New oscillation criteria for second order linear difference equations with positive and negative coefficients*, Comput. Math. Appl. 58 (2009) 1988-1997.

[5] L. Erbe, Q. Kong, B. Zhang; *Oscillation Theory for Functional Differential Equations*, Now York: Marcel Dekker, (1995).
[6] B. Karpuz, Ö. Öalan, M. Yildiz; *Oscillation of a class of difference equations of second order*, Math. Comput. Modelling 49 (2009), 912-917.
[7] H. Li, C. Yeh; *Oscillation criteria for second order neutral delay difference equations*, Comput. Math. Appl. 36(10-12) (1998), 123-132.
[8] Q. Meng, J. Yan; *Bounded oscillation for second-order nonlinear neutral difference equations in critical and non-critical states*, J. Comput. Appl. Math. 211 (2008) 156-172.
[9] N. Parhi, A. Panda; *Nonoscillation and oscillation of solutions of a class of third order difference equations*, J. Math. Anal. Appl. 336 (2007), 213-223.
[10] H. Qin, N. Shang, Y. Lu; *A note on oscillation criteria of second order nonlinear neutral delay differential equations*, Comput. Math. Appl. 56 (2008) 2987-2992.
[11] I. Stavroulakis; *Oscillation of delay difference equations*, Comput. Math. Appl. 29(7) (1995), 83-88.
[12] S. Wu, Z. Hou; *Oscillation criteria for a class of neutral difference equations with continuous variable*, J. Math. Anal. Appl. 290 (2004) 316-323.
[13] S. Wu, P. S. Calay, Z. Hou; *Oscillation criteria for a class of third order neutral difference equations with continuous variable*, preprint.
[14] B. G. Zhang, J. Yan, S. K. Choi; *Oscillation for difference equations with continuous variable*, Comput. Math. Appl. 36(9) (1998) 11-18.
[15] Z. Zhang, P. Bi, J. Chen; *Oscillation of second order nonlinear difference equation with continuous variable*, J. Math. Anal. Appl. 255(1) (2001), 349-357.
[16] X. Zhou; *Oscillatory and asymptotic properties of higher order nonlinear neutral difference equations with oscillating coefficients*, Appl. Math. Lett. 21 (2008) 1142-1148.
[17] X. Zhou, W. Zhang; *Oscillatory and asymptotic properties of higher order nonlinear neutral difference equations*, Appl. Math. Comput. 203 (2008) 679-689.

Shuhui Wu (corresponding author)
School of Science, Zhejiang University of Science and Technology, 318 Liuhe Road, Hangzhou 310023, China
E-mail address: s.wu@zust.edu.cn, shuhui75@hotmail.com

Pargat Singh Calay
Faculty of Computing, London Metropolitan University, 166–220 Holloway Road, London N7 8DB, UK
E-mail address: p.calay@londment.ac.uk

Zhanyuan Hou
Faculty of Computing, London Metropolitan University, 166–220 Holloway Road, London N7 8DB, UK
E-mail address: z.how@londonmet.ac.uk