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A COMPARISON PRINCIPLE FOR SINGULAR PARABOLIC EQUATIONS IN THE HEISENBERG GROUP

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Abstract. In this work, we prove a comparison principle for singular parabolic equations with boundary conditions in the context of the Heisenberg group. In particular, this result applies to interesting equations, such as the parabolic infinite Laplacian, the mean curvature flow equation and more general homogeneous diffusions.

1. Introduction

The notion of viscosity solution was firstly introduced by Crandall and Lions [8] in the context of scalar nonlinear first order equations. This concept was related to some previous work by Evans [12]. In general terms, the definition of viscosity solutions for parabolic problems may be motivated as follows: consider a $\mathcal{C}^2$-regular function $u : \Omega \times (0,T) \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an open set and $T > 0$ is given. Suppose that u solves the differential inequality

$$u_t(z) + F(z, u(z), \nabla u(z), \nabla^2 u(z)) \leq 0 \tag{1.1}$$

for all $z = (t,p) \in \Omega \times (0,T)$. Here, $F : [0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n(\mathbb{R}) \to \mathbb{R}$ is a given function, which is assumed to be degenerate elliptic:

$$F(t,p,r,\eta,\mathcal{X}) \leq F(t,p,r,\eta,\mathcal{Y}) \quad \text{whenever } \mathcal{Y} \leq \mathcal{X},$$

so that

$$u_t(z) + F(z, u(z), \nabla u(z), \nabla^2 u(z)) = 0 \tag{1.2}$$

is a parabolic equation. Suppose now that φ is a smooth function defined in $\Omega \times (0,T)$ so that the difference $u - \varphi$ attains a maximum at a point $\hat{z} \in \Omega \times (0,T)$. Then, it follows that

$$u_t(\hat{z}) = \varphi_t(\hat{z}), \quad \nabla u(\hat{z}) = \nabla \varphi(\hat{z}), \quad \nabla^2 u(\hat{z}) \leq \nabla^2 \varphi(\hat{z}).$$

From the degenerate ellipticity of F, we derive

$$\begin{aligned} &\varphi_t(\hat{z}) + F(\hat{z}, u(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})) \\ &\leq u_t(\hat{z}) + F(\hat{z}, u(\hat{z}), \nabla u(\hat{z}), \nabla^2 u(\hat{z})) \leq 0. \end{aligned} \tag{1.3}$$

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Observe that the extremes of these inequalities do not depend on the derivative of u. Hence, this suggests to define an arbitrary function u to be a generalized or weak subsolution of (1.1) in $\Omega \times (0,T)$ if for each $\hat{z} \in \Omega \times (0,T)$, a test function φ that touches u from above at $\hat{z}$ always satisfies

$$\varphi_t(\hat{z}) + F(\hat{z}, \varphi(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})) \leq 0.$$

The notion of viscosity supersolution is defined analogously. Finally, a viscosity solution of (1.2) is a subsolution and a supersolution. The primary virtues of this theory are that it allows merely continuous functions to be solutions of fully nonlinear equations of second order, that it provides very general existence and uniqueness theorems and that it yields precise formulations of general boundary conditions. Moreover, it has a great flexibility in passing to limits in various settings. For a more complete treatment of viscosity solutions in the Euclidean framework see [1, 2, 8, 9, 11] and the references therein. For extension of the definition to singular equations, see for instance the book [15].

In this work, we are concerned with the development of a comparison principle for a large class of boundary value problems, in the Heisenberg group $\mathcal{H}$, of the form

$$\begin{gathered} u_t + F(t,p,u,\nabla_{\mathcal{H}}u, (\nabla^2_{\mathcal{H}}u)^*) = 0, \quad \text{in } (0,T)\times\Omega \quad (E) \\ u(t,p) = g(t,p) \quad p \in \partial\Omega,\ t\in[0,T) \quad (BC) \\ u(0,p) = h(p) \quad p \in \overline{\Omega} \qquad (IC) \end{gathered} \tag{1.4}$$

Here $\Omega \subset \mathcal{H}$ is open and bounded, and $F = F(t,p,r,\eta,\mathcal{X})$ is assumed to be possibly singular at $\eta = 0$ (extra assumptions on F will be provided in Section 3.1). See Section 2 for definitions of the horizontal gradient $\nabla_{\mathcal{H}}u$ and the symmetrized Hessian matrix $(\nabla^2_{\mathcal{H}}u)^*$ in $\mathcal{H}$. A general comparison principle for parabolic equations in the Heisenberg group for everywhere continuous F was introduced in [3]. (See [11] for the related result in the Euclidean context). With respect to singular parabolic equations, we can quote the particular case of the horizontal mean curvature flow equation treated in [13], for which a comparison principle for axisymmetric surfaces was proven. (See also [15, 7] for the Euclidean treatment of singular parabolic equations). In this work, we prove that under some extra assumptions on F, a comparison principle for the boundary value problem (1.4) holds for solutions u which are symmetric with respect to some class of surfaces $p_3 = G(p_1,p_2)$ (see (4.1)) in the sense that

$$u(t,p_1,p_2,p_3) = u(t,\hat{p}_1,\hat{p}_2,p_3) \quad \text{whenever } G(p_1,p_2) = G(\hat{p}_1,\hat{p}_2).$$

We would like to point out that some of the arguments used in our proof of the comparison principle are similar to those from the works [13, 3] and the seminal paper [11], adapted to our framework and generality.

The organization of the paper is as follows. In Section 2, we provide a brief introduction to the Heisenberg group. In the next Section 3, we discuss the parabolic boundary problem we intend to study, the notions of viscosity solutions and we provide the main assumptions to prove the comparison principle, which is formulated and proven in Section 4. We close the paper with Section 5, where we give examples of applications.

2. Preliminaries on the Heisenberg group

In this section, we introduce the definition of the Heisenberg group $\mathcal{H}$ together with its differential and metric structures. The notion of parabolic jet on $\mathcal{H}$ and its characterization in terms of smooth functions are also explained.

2.1. **The symmetric three dimensional Heisenberg group.**

We consider the first order Heisenberg group $\mathcal{H} = (\mathbb{R}^3, \cdot)$, where $\cdot$ is the group operation defined by

$$p \cdot q = \Big(p_1 + q_1, p_2 + q_2, p_3 + q_3 + \frac{1}{2}(p_1 q_2 - p_2 q_1)\Big),$$

for all $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3) \in \mathbb{R}^3$. The group $\mathcal{H}$ is a Lie group with Lie algebra $\mathfrak{h}$ generated by the basis

$$\begin{aligned} X_1 &= \frac{\partial}{\partial p_1} - \frac{p_2}{2}\frac{\partial}{\partial p_3} \\ X_2 &= \frac{\partial}{\partial p_2} + \frac{p_1}{2}\frac{\partial}{\partial p_3} \\ X_3 &= \frac{\partial}{\partial p_3}, \end{aligned} \tag{2.1}$$

where $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. Observe that the following Heisenberg uncertainty principle holds:

$$[X_1, X_2] = X_3.$$

The exponential mapping takes the vector $p_1 X_1 + p_2 X_2 + p_3 X_3$ in the Lie algebra $\mathfrak{h}$ to the point p in the Lie group $\mathcal{H}$. This allows us to identify vectors in $\mathfrak{h}$ with points in $\mathcal{H}$.

On the Heisenberg group, an important role is played by the distribution $\mathcal{H}^h$ generated by the linearly independent vector fields X_1 and X_2, called the horizontal distribution. Thus, this space at p, denoted by $\mathcal{H}^h_p$, is a two dimensional linear space generated by the vectors $X_1(p)$ and $X_2(p)$. As $[X_1, X_2] = X_3 \notin \mathcal{H}^h$, the horizontal distribution is not involutive, and hence, by Frobenius theorem, it is not integrable; that is, there is no surface locally tangent to $\mathcal{H}^h$.

2.2. **Carnot-Carathéodory distance.**

A curve $c(s) = (c_1(s), c_2(s), c_3(s))$ is horizontal if $c'(s) \in \mathcal{H}^h_{c(s)}$. Moreover, by Chow's theorem any two points p and q in $\mathcal{H}$ can be joined by a smooth horizontal curve. Hence, the set

$$S_{p,q} = \{c : c(0) = p, c(1) = q, \ c \text{ is horizontal}\} \neq \emptyset.$$

The length of a horizontal curve c is given by

$$l(c) = \int_0^1 \sqrt{g(c'(s), c'(s))}\,ds,$$

where g is the subRiemannian metric. The Carnot-Carathéodory distance is defined as $d_C : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$,

$$d_C(p, q) = \inf\{l(c) : c \in S_{p,q}\}.$$

One may verify that d_C satisfies the distance axioms and that it is complete. This metric induces a homogeneous norm on $\mathcal{H}$, denoted $|\cdot|$, by

$$|p| = d_C(0, p),$$

and we have the estimate

$$|p| \sim \|(p_1, p_2)\|_E + |p_3|^{1/2}. \tag{2.2}$$

Here, $\|\cdot\|_E$ stands for the Euclidean norm in $\mathbb{R}^n$. This estimate leads to define the left-invariant Heisenberg gauge $|\cdot|_{\mathcal{H}}$ that is compatible to the Carnot-Carathéodory distance, and is defined as follows:

$$|p|_{\mathcal{H}} = \left[(p_1^2 + p_2^2)^2 + 16p_3^2\right]^{1/4}.$$

For the rest of this article, we shall consider all topological notions with respect to the metric space $(\mathcal{H}, d_C)$. Also, for any $p \in \mathcal{H}$ and $\delta > 0$, we write

$$B_{\mathcal{H}}(p, \delta) = \left\{q \in \mathcal{H} : |q^{-1} \cdot p| < \delta\right\},$$

to denote the ball in the Heisenberg group with center at p and radius δ.

2.3. **Analysis on $\mathcal{H}$.** The left translation $L_p : \mathcal{H} \to \mathcal{H}$ is defined by

$$L_p(q) = p \cdot q.$$

Observe that L_p is an affine map. Indeed:

$$L_p(q) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_2/2 & p_1/2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

and the determinant of the matrix on the right-hand side is 1. It follows then that the left-invariant Haar measure of $\mathcal{H}$ is the Lebesgue measure $\mathcal{L}$ of $\mathbb{R}^3$ (which is in fact also right invariant).

For a smooth function $u : \mathcal{H} \to \mathbb{R}$ the horizontal gradient $\nabla_{\mathcal{H}}$ of u at a point p is the projection of the gradient of u at p onto the horizontal space $\mathcal{H}_p^h$,

$$\nabla_{\mathcal{H}} u = (X_1 u) X_1 + (X_2 u) X_2.$$

The symmetrized horizontal second derivative matrix, denoted by $(\nabla^2_{\mathcal{H}} u)^*$ is given by

$$(\nabla^2_{\mathcal{H}} u)^* = \begin{pmatrix} X_1^2 u & \frac{1}{2}(X_1 X_2 u + X_2 X_1 u) \\ \frac{1}{2}(X_1 X_2 u + X_2 X_1 u) & X_2^2 u \end{pmatrix}.$$

With this notation and the estimate (2.2), the Taylor expansion for a smooth u around p_0 reads as

$$u(p_0) = u(p) + \langle \nabla u(p_0), p_0^{-1} \cdot p \rangle + \frac{1}{2} \langle (\nabla^2_{\mathcal{H}} u(p_0))^* p_0^{-1}\ cdotp, p_0^{-1} \cdot p \rangle + o(|p_0^{-1} \cdot p|^2$$

For more about the Heisenberg group, the interested reader is referred to [3, 6, 16, 5], and the references therein.

2.4. **Parabolic subelliptic jets.** We start by defining the parabolic superjets of a function u at a point $(t_0, p_0) \in (0, \infty) \times \mathcal{H}$, denoted by $P^{2,+}u(t_0, p_0)$, as the set of all triples $(\tau, \eta, \mathcal{X}) \in \mathbb{R} \times \mathbb{R}^3 \times S^2(\mathbb{R})$ that satisfies

$$u(t, p) \le u(t_0, p_0) + \tau(t - t_0) + \langle \eta, p_0^{-1} \cdot p \rangle + \frac{1}{2} \langle \mathcal{X} h, h \rangle + o(|t - t_0| + |p_0^{-1} \cdot p|^2_{\mathcal{H}}), \tag{2.3}$$

as $(t, p) \to (t_0, p_0)$. Here, h denotes the horizontal projection of $p_0^{-1} \cdot p$. We define the parabolic subject $P^{2,-}u(t_0, p_0)$ by

$$P^{2,-}u(t_0, p_0) = -P^{2,+}(-u)(t_0, p_0).$$

As in the subelliptic case (see [10] for the Euclidean case, [3] for the subelliptic case), it was shown in [4] that

$$P^{2,+}u(t_0,p_0) = \big\{(\varphi_t(t_0,p_0), \nabla\varphi(t_0,p_0), (\nabla^2_{\mathcal{H}}\varphi(t_0,p_0))^*) : \varphi \in \mathcal{A}u(t_0,p_0)\big\},$$

where

$$\mathcal{A}u(t_0,p_0) = \{\varphi \in \mathcal{C}^2(\mathcal{H}\times(0,T)) : u-\varphi \text{ has a strict local maximum at } (t_0,p_0)\}.$$

Similarly, one has

$$P^{2,-}u(t_0,p_0) = \big\{(\varphi_t(t_0,p_0), \nabla\varphi(t_0,p_0), (\nabla^2_{\mathcal{H}}\varphi(t_0,p_0))^*) : \varphi \in \mathcal{B}u(t_0,p_0)\big\},$$

where

$$\mathcal{B}u(t_0,p_0) = \{\varphi \in \mathcal{C}^2(\mathcal{H}\times(0,T)) : u-\varphi \text{ has a strict local minimum at } (t_0,p_0)\}.$$

We also define the closure of second order superjets and subjets.

Definition 2.1. The closure of the second order superjet of an upper-semicontinuous function u at a point (t_0,p_0), denoted by $\overline{P}^{2,+}u(t_0,p_0)$, is defined as the set of $(\tau,\eta,\mathcal{X}) \in \mathbb{R}\times\mathbb{R}^3\times S^2(\mathbb{R})$, such that there exist sequences of points (t_n,p_n) and $(\tau_n,\eta_n,\mathcal{X}_n) \in P^{2,+}u(t_n,p_n)$ such that

$$(t_n,p_n,u(t_n,p_n),\tau_n,\eta_n,\mathcal{X}_n) \to (t_0,p_0,u(t_0,p_0),\tau,\eta,\mathcal{X}), \quad \text{as } n\to\infty.$$

Similarly, the closure of the second order subjet of a lower-semicontinuous function u at a point (t_0,p_0), denoted by $\overline{P}^{2,-}u(t_0,p_0)$, is defined as the set of $(\tau,\eta,\mathcal{X}) \in \mathbb{R}\times\mathbb{R}^3\times S^2(\mathbb{R})$, such that there exist sequences of points (t_n,p_n) and $(\tau_n,\eta_n,\mathcal{X}_n) \in P^{2,-}u(t_n,p_n)$ such that

$$(t_n,p_n,u(t_n,p_n),\tau_n,\eta_n,\mathcal{X}_n) \to (t_0,p_0,u(t_0,p_0),\tau,\eta,\mathcal{X}),\ \textit{as } n\to\infty.$$

3. General setting

3.1. **The parabolic problem under study and the notions of viscosity solutions.** Let Ω be an open and bounded domain in $\mathcal{H}$. We consider the following class of problems:

$$\begin{gathered} u_t + F(t,p,u,\nabla_{\mathcal{H}}u,(\nabla^2_{\mathcal{H}}u)^*) = 0, \quad \text{in } (0,T)\times\Omega \quad (E) \\ u(t,p) = g(t,p) \quad p\in\partial\Omega,\ t\in[0,T) \quad (BC) \\ u(0,p) = h(p) \quad p\in\overline{\Omega} \quad (IC) \end{gathered} \tag{3.1}$$

Here $g \in \mathcal{C}([0,T)\times\overline{\Omega})$, $h\in\mathcal{C}(\overline{\Omega})$ and $F : [0,T]\times\overline{\Omega}\times\mathbb{R}\times(\mathbb{R}^2\setminus\{0\})\times S^2(\mathbb{R}) \to \mathbb{R}$ is assumed to satisfy the following properties:

(1) F is continuous in $[0,T]\times\overline{\Omega}\times\mathbb{R}\times(\mathbb{R}^2\setminus\{0\})\times S^2(\mathbb{R})$, and there is a modulus of continuity ω so that
$$|F(t,p,r,\eta,\mathcal{X}) - F(s,q,r,\eta,\mathcal{X})| \le \omega\big(|t-s| + d_C(p,q)\big), \tag{3.2}$$
for all $r\in\mathbb{R}$, $\eta\in\mathbb{R}^2\setminus\{0\}$ and all $\mathcal{X}\in S^2(\mathbb{R})$. (Observe that the modulus ω is the same for all r,η and $\mathcal{X}$.)

(2) F is proper, that is,
$$(r,\mathcal{X}) \to F(t,p,r,\eta,\mathcal{X})$$
is increasing in $r\in\mathbb{R}$ and decreasing in $\mathcal{X}\in S^2(\mathbb{R})$.

(3) $F_*(t,p,r,0,\mathcal{O}) = F^*(t,p,r,0,\mathcal{O}) = 0$ for all $t,p,r\in[0,T]\times\overline{\Omega}\times\mathbb{R}$. F_* and F^* are locally bounded in the set $[0,T]\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2\times S^2(\mathbb{R})$.

(4) $F(t,p,r,\eta_\epsilon,\mathcal{Y}_\epsilon) - F(t,p,r,\eta_\epsilon,\mathcal{X}_\epsilon) \le o(1)$, uniformly in t, p, r, and η_ϵ uniformly bounded, and all $\mathcal{X}_\epsilon, \mathcal{Y}_\epsilon \in S^2(\mathbb{R})$ so that

$$\mathcal{X}_\epsilon - \mathcal{Y}_\epsilon \le o(1)I,$$

where I is the identity matrix.

Remark 3.1. We may replace assumptions (1) and (4) by the following stronger assumption: there exist constants $K, L > 0$ such that

$$F(t,p,r,\eta,\mathcal{Y}) - F(s,q,r',\beta,\mathcal{X}) \le K\big(d_C(p,q) + |s-t| + |r-r'| + |\beta-\eta|\big) + L\sigma,$$

for all $p, q \in \overline{\Omega}$, $s, t \in [0,T]$, $r, r' \in \mathbb{R}$, $\eta, \beta \in \mathbb{R}^2 \setminus \{0\}$ and all $\mathcal{X}, \mathcal{Y} \in S^2(\mathbb{R})$ so that

$$\mathcal{X} \le \mathcal{Y} + \sigma I.$$

Next, we introduce the definition of viscosity solution to the singular parabolic equation (E) in the context of the Heisenberg group.

Definition 3.2. An upper (respectively, lower) semicontinuous function $u : (0,T)\times\Omega \to \mathbb{R}\cup\{\pm\infty\}$ is a viscosity subsolution (resp. supersolution) in $(0,T)\times\Omega$ to (E) if for all $(t_0,p_0) \in (0,T)\times\Omega$ and all smooth $\varphi \in \mathcal{A}u(t_0,p_0)$ (resp. $\varphi \in \mathcal{B}u(t_0,p_0)$) there holds

$$\varphi_t(t_0,p_0) + F_*(t_0,p_0,u(t_0,p_0),\nabla_{\mathcal{H}}\varphi(t_0,p_0),(\nabla^2_{\mathcal{H}}\varphi(t_0,p_0))^*) \le 0.$$
$$(\text{resp. } \varphi_t(t_0,p_0) + F^*(t_0,p_0,u(t_0,p_0),\nabla_{\mathcal{H}}\varphi(t_0,p_0),(\nabla^2_{\mathcal{H}}\varphi(t_0,p_0))^*) \ge 0.)$$

A continuous function u is a viscosity solution if it is a viscosity subsolution and a viscosity supersolution.

It is also possible to deal with the singularity of F at $\eta = 0 \in \mathbb{R}^2$ by restricting the set of test functions to the set

$$\mathcal{A}_0 = \{\varphi \in \mathcal{C}^\infty((0,\infty)\times\mathcal{H}) : \nabla_{\mathcal{H}}\varphi(t,p) = 0 \text{ implies } (\nabla^2_{\mathcal{H}}\varphi(t,p))^* = 0\}.$$

This is the content of Definition 3.3. Another way is to use parabolic jets as in Definition 3.4 below. We shall see in Lemma 3.5 that these definitions are equivalent.

Definition 3.3. An upper (respectively, lower) semicontinuous function $u : (0,T)\times\Omega \to \mathbb{R}\cup\{\pm\infty\}$ is a subsolution (resp. supersolution) to (E) if:

(i) $u < \infty$ (resp. $u > -\infty$) in $(0,T)\times\Omega$;
(ii) For any smooth function φ and $(t_0,p_0) \in (0,T)\times\Omega$ such that $\varphi \in \mathcal{A}u(t_0,p_0)$ (resp. $\varphi \in \mathcal{B}u(t_0,p_0)$), the function φ satisfies

$$\varphi_t + F(t_0,p_0,u(t_0,p_0),\nabla_{\mathcal{H}}\varphi,(\nabla^2_{\mathcal{H}}\varphi)^*) \le 0, \text{ at } (t_0,p_0),$$
$$(\text{resp. } \varphi_t + F(t_0,p_0,u(t_0,p_0),\nabla_{\mathcal{H}}\varphi,(\nabla^2_{\mathcal{H}}\varphi)^*) \ge 0, \text{ at } (t_0,p_0).)$$

if $\nabla_{\mathcal{H}}\varphi(t_0,p_0) \ne 0$, and $\varphi_t(t_0,p_0) \le 0$ (respectively $\varphi_t(t_0,p_0) \ge 0$.) when $\nabla_{\mathcal{H}}\varphi(t_0,p_0) = 0$ and $(\nabla^2_{\mathcal{H}}\varphi(t_0,p_0))^* = 0$.

Definition 3.4. An upper (respectively, lower) semicontinuous function $u : (0,T)\times\Omega \to \mathbb{R}\cup\{\pm\infty\}$ is a subsolution (resp. supersolution) to (E) if:

(i) $u < \infty$ (resp. $u > -\infty$) in $(0,T)\times\Omega$;

(ii) For any $(t_0, p_0) \in (0,T) \times \Omega$ and any $(\tau, \eta, \mathcal{X}) \in \overline{P}^{2,+} u(t_0, p_0)$ (resp. $(\tau, \eta, \mathcal{X}) \in \overline{P}^{2,-} u(t_0, p_0)$), we have

$$\tau + F_*(t_0, p_0, u(t_0, p_0), \eta, \mathcal{X}) \le 0$$
$$(\text{resp. } \tau + F^*(t_0, p_0, u(t_0, p_0), \eta, \mathcal{X}) \ge 0.$$

It is straightforward to check that Definition 3.2 and Definition 3.4 are equivalent. Moreover Definition 3.2 implies Definition 3.3. Hence, it is remains to prove the converse. This is the content of the next result, which is [13, Proposition 3.1].

Lemma 3.5. *An upper-semicontinuous function u is a subsolution to (E) in the sense of Definition 3.2 if and only if it is a subsolution in the sense of Definition 3.3. A similar statement holds for supersolutions.*

Proof. The proof is the same as that of [13, Proposition 3.1]. We just mention how to treat with the dependence of F on t, p and r, which is not the case considered in [13]. First of all, it is clear that Definition 3.2 implies Definition 3.3. To prove the converse, assume that u is a subsolution according to Definition 3.3. Let $(\hat{t}, \hat{p}) \in (0,T) \times \Omega$ and let φ a smooth function such that

$$\max_{(0,T)\times\Omega} (u - \varphi) = (u - \varphi)(\hat{t}, \hat{p}).$$

As in [13], we let

$$\Phi^\tau(t, p, q) = u(t, p) - \tau |q^{-1} \cdot p|^4 - \varphi(t, p).$$

Then $\Phi(t,p,q) = \limsup^*_{\tau\to\infty} \Phi^\tau(t,p,q) = -\infty$ when $p \ne q$, and $\Phi(t,p,p) = \limsup^*_{\tau\to\infty} \Phi^\tau(t,p,p) = u(t,p) - \varphi(t,p)$. By the convergence of maximum points [15, Lemma 2.2.5]), there exists a sequence (t^τ, p^τ, q^τ) converging to $(\hat{t}, \hat{p}, \hat{p})$ such that Φ^τ attains a maximum at (t^τ, p^τ, q^τ). Moreover

$$\lim_{\tau\to\infty} \Phi^\tau(t^\tau, p^\tau, q^\tau) = \Phi(\hat{t}, \hat{p}, \hat{p}). \tag{3.3}$$

Hence, since $\varphi(t^\tau, p^\tau) \to \varphi(\hat{t}, \hat{p})$ as $\tau \to \infty$, we derive from (3.3) and the upper semicontinuity of u that

$$\begin{aligned} u(\hat{t}, \hat{p}) &= \liminf_{\tau\to\infty} \big(u(t^\tau, p^\tau) - \tau |(q^\tau)^{-1} \cdot p^\tau|^4 \big) \\ &\le \liminf_{\tau\to\infty} u(t^\tau, p^\tau) \\ &\le \limsup_{\tau\to\infty} u(t^\tau, p^\tau) \le u(\hat{t}, \hat{p}). \end{aligned} \tag{3.4}$$

Concluding that

$$\lim_{\tau\to\infty} u(t^\tau, p^\tau) = u(\hat{t}, \hat{p}). \tag{3.5}$$

With (3.5) in mind, we may proceed with the proof as in [13]. □

Finally, we proceed as in [11, 4] and we introduce the definition of viscosity solution to the problem (3.1).

Definition 3.6. A subsolution $u(t,p)$ to problem (3.1) is a viscosity subsolution to (E), $u(t,p) \le g(t,p)$ on $\partial\Omega$, $t \in [0,T)$, and $u(0,p) \le h(p)$ in $\overline{\Omega}$. Supersolutions and solutions are defined in an analogous manner.

4. Comparison principle

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function verifying that there is no point $(p_1, p_2) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$p_1 \frac{\partial G}{\partial p_1}(p_1, p_2) + p_2 \frac{\partial G}{\partial p_2}(p_1, p_2) = 0. \tag{4.1}$$

In particular, observe that the Euclidean gradient $(\partial G/\partial p_1(p_1, p_2), \partial G/\partial p_2(p_1, p_2))$ is not zero for all $(p_1, p_2) \neq (0, 0)$. We are interested in a comparison principle for the problem (3.1) in the case of sub or supersolutions u which are symmetric with respect to the surface $p_3 = G(p_1, p_2)$:

$$u(t, p_1, p_2, p_3) = u(t, \hat{p}_1, \hat{p}_2, p_3), \quad \text{when } G(p_1, p_2) = G(\hat{p}_1, \hat{p}_2).$$

This is the content of our main result.

Theorem 4.1. *Let u and v be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution to* (3.1)*. Assume that either u or v are symmetric with respect to the surface $p_3 = G(p_1, p_2)$. Then:*

$$u \leq v \quad \text{in } [0, T) \times \overline{\Omega}.$$

Proof. Let us assume that u is symmetric with respect to the surface $p_3 = G(p_1, p_2)$. To obtain a contradiction, we assume that there exists a point $(\bar{t}, \bar{p}) \in (0, T) \times \Omega$ so that

$$(u - v)(\bar{t}, \bar{p}) > 0.$$

Then, we are able to find a positive number $\delta > 0$ satisfying

$$u(\bar{t}, \bar{p}) - v(\bar{t}, \bar{p}) - \frac{\delta}{T - \bar{t}} > 0.$$

Let $(\hat{t}, \hat{p}) \in [0, T) \times \overline{\Omega}$ so that

$$M = u(\hat{t}, \hat{p}) - v(\hat{t}, \hat{p}) - \frac{\delta}{T - \hat{t}} = \max_{[0,T)\times\overline{\Omega}} \Big(u(t, p) - v(t, p) - \frac{\delta}{T - t} \Big) > 0. \tag{4.2}$$

As usual in the proof of comparison principles, we double the variables and proceed with the penalizing process defining the function M^τ by

$$M^\tau(t, p, s, q) = u(t, p) - v(s, q) - \tau g^2(p, q) - \frac{\tau}{2}(t - s)^2 - \frac{\delta}{T - t},$$

where $g(p, q) = |p \cdot q^{-1}|^4_{\mathcal{H}}$. This is the same penalizing process as in [13]. We take maximizers $(t^\tau, p^\tau, s^\tau, q^\tau) \in ([0, T) \times \overline{\Omega})^2$ of M^τ. In view of (4.2) and the boundedness from above of the functions u and $-v$, the points t^τ lie in a compact subset of $[0, T)$ for τ large. Moreover, the inequality

$$M^\tau(t^\tau, p^\tau, s^\tau, q^\tau) \geq M^\tau(\hat{t}, \hat{p}, \hat{t}, \hat{p}) \tag{4.3}$$

implies that

$$|p^\tau \cdot (q^\tau)^{-1}|_{\mathcal{H}} \to 0 \quad \text{and} \quad |t^\tau - s^\tau| \to 0, \tag{4.4}$$

as $\tau \to \infty$. This fact, together with the compactness of the set $\overline{\Omega}$ yield the existence of a point $(t_0, p_0) \in [0, T) \times \overline{\Omega}$ such that $p^\tau, q^\tau \to p_0$ and $t^\tau, s^\tau \to t_0$. It also follows from (4.3) and the above convergences, that

$$0 \leq \limsup_{\tau\to\infty} \Big(\tau g^2(p^\tau, q^\tau) + \frac{\tau}{2}(t^\tau - s^\tau)^2 \Big) \leq 0.$$

Concluding that

$$\tau g^2(p^\tau, q^\tau) + \frac{\tau}{2}(t^\tau - s^\tau)^2 \to 0 \quad \text{as } \tau \to \infty.$$

In addition, the fact $u(0,p) \le v(0,p)$ implies that $t_0 \neq 0$. Indeed, if $t_0 = 0$, then

$$0 < M = M^\tau(\hat{t}, \hat{p}, \hat{t}, \hat{p}) \le \lim_{\tau\to\infty} M^\tau(t^\tau, p^\tau, s^\tau, q^\tau) = u(0,p_0) - v(0,p_0) - \frac{\delta}{T} \le 0$$

which is a contradiction. Hence, $t_0 \in (0,T)$. On the other hand, we also need to check that $p_0 \in \Omega$. Observe that

$$M \le \lim_{\tau\to\infty} M^\tau(t^\tau, p^\tau, s^\tau, q^\tau) = u(t_0,p_0) - v(t_0,p_0) - \frac{\delta}{T - t_0}. \tag{4.5}$$

Hence, If $p_0 \in \partial\Omega$, the property $u(t,p) \le v(t,p)$ on $[0,T) \times \partial\Omega$ says that the right-hand side above is negative, which contradicts that $M > 0$. Therefore, p_0 should be an interior point. Thus, we may apply the parabolic Euclidean Crandall-Ishii lemma [11, Theorem 8.3]) to the functions

$$(t,p) \to u(t,p) - v(s^\tau, q^\tau) - \tau g^2(p, q^\tau) - \frac{\tau}{2}(t - s^\tau)^2 - \frac{\delta}{T - t}$$

and

$$(s,q) \to u(t^\tau, p^\tau) - v(s,q) - \tau g^2(p^\tau, q) - \frac{\tau}{2}(t^\tau - s)^2 - \frac{\delta}{T - t^\tau}$$

which have, respectively, a maximum at the points (t^τ, p^τ) and (s^τ, q^τ). To do so, we need to check [11, Condition 8.5], namely, that there is an $r > 0$ such that for every $M > 0$ there is a C such that for all $(b, \beta, X) \in P^{2,+}_{Eucl} u(t,p)$: if $|u(p,t)| + |\beta| + \|X\| \le M$ and $|t - t^\tau| + \|p - p^\tau\|_E < r$ hold, then $b \le C$, with an analogous statement for $-v$. Indeed, if this is not true, for all $r > 0$, there is an $M > 0$ such that for all C there is $(b, \beta, X) \in P^{2,+}_{Eucl} u(p,t)$ so that $|u(t,p)| + |\beta| + \|X\| \le M$ and $|t - t^\tau| + \|p - p^\tau\|_E < r$ but $b > C$. This would imply that

$$\big(b, (DL_p\beta, DL_p X DL_p^T)_{2\times 2}\big) \in P^{2,+} u(t,p)$$

which contradicts the fact that u is a subsolution for large C in view of the local boundedness of the function F_*. A similar argument applies to $-v$. Hence, we may apply [11, Theorem 8.3] to obtain matrices $X^\tau, Y^\tau \in S^3(\mathbb{R})$ such that

$$\Big(\frac{\delta}{(T - t^\tau)^2} + \tau(t^\tau - s^\tau), \tau \nabla_p g^2(p^\tau, q^\tau), X^\tau\Big) \in \overline{P}^{2,+}_{Eucl} u(t^\tau, p^\tau) \tag{4.6}$$

$$\Big(\tau(t^\tau - s^\tau), -\tau \nabla_q g^2(p^\tau, q^\tau), Y^\tau\Big) \in \overline{P}^{2,-}_{Eucl} v(s^\tau, q^\tau),$$

with the property that

$$\begin{pmatrix} X^\tau & 0 \\ 0 & -Y^\tau \end{pmatrix} \le \tau\big[\nabla^2_{p,q} g^2(p^\tau, q^\tau) + (\nabla^2_{p,q} g^2(p^\tau, q^\tau))^2\big]. \tag{4.7}$$

It is also clear that

$$\Big(\frac{\delta}{(T - t^\tau)^2} + \tau(t^\tau - s^\tau), \tau DL_{p^\tau} \nabla_p g^2(p^\tau, q^\tau), (DL_{p^\tau} X^\tau DL^T_{p^\tau})_{2\times 2}\Big) \in \overline{P}^{2,+} u(t^\tau, p^\tau) \tag{4.8}$$

and

$$\Big(\tau(t^\tau - s^\tau), -\tau DL_{q^\tau} \nabla_q g^2(p^\tau, q^\tau), (DL_{q^\tau} Y^\tau DL^T_{q^\tau}\Big) \in \overline{P}^{2,-} v(s^\tau, q^\tau).$$

As in [3] and [13], let

$$\begin{aligned}w_{p^\tau} &= (DL_{p^\tau})^T(w_1,w_2,0)^T = \Big(w_1,w_2,\frac{1}{2}(p_1^\tau w_2 - p_2^\tau w_1)\Big),\\ w_{q^\tau} &= (DL_{q^\tau})^T(w_1,w_2,0)^T = \Big(w_1,w_2,\frac{1}{2}(q_1^\tau w_2 - q_2^\tau w_1)\Big).\end{aligned}$$

Then, defining the matrices $\mathcal{X}^\tau, \mathcal{Y}^\tau \in S^2(\mathbb{R})$ as

$$\begin{aligned}\mathcal{X}^\tau &= (DL_{p^\tau} X^\tau DL_{p^\tau}^T)_{2\times 2},\\ \mathcal{Y}^\tau &= (DL_{p^\tau} Y^\tau DL_{p^\tau}^T)_{2\times 2},\end{aligned}$$

we deduce from (4.7) that

$$\begin{aligned}&\langle \mathcal{X}^\tau w, w\rangle - \langle \mathcal{Y}^\tau w, w\rangle\\ &= \langle X^\tau w_{p^\tau}, w_{p^\tau}\rangle - \langle Y^\tau w_{q^\tau}, w_{q^\tau}\rangle\\ &\le \tau\Big\langle \big[\nabla^2_{p,q} g^2(p^\tau,q^\tau) + (\nabla^2_{p,q} g^2(p^\tau,q^\tau))^2\big](w_{p^\tau}\oplus w_{q^\tau}), w_{p^\tau}\oplus w_{q^\tau}\Big\rangle = o(1),\end{aligned} \tag{4.9}$$

locally uniformly in $\|w\|$.

Because of the singularity of F at $\eta = 0$, we have to consider two cases. Firstly, assume that

$$\eta^\tau = \tau\nabla^p_{\mathcal{H}} g^2(p^\tau,q^\tau) = -\tau\nabla^q_{\mathcal{H}} g^2(p^\tau,q^\tau) \ne 0$$

for all large τ. Using that u is a subsolution and v is a supersolution to equation (E), we obtain

$$\frac{\delta}{(T-t^\tau)^2} + \tau(t^\tau - s^\tau) + F(t^\tau,p^\tau,u(t^\tau,p^\tau),\eta^\tau,\mathcal{X}^\tau) \le 0, \tag{4.10}$$

$$\tau(t^\tau - s^\tau) + F(s^\tau,q^\tau,v(s^\tau,q^\tau),\eta^\tau,\mathcal{Y}^\tau) \ge 0, \tag{4.11}$$

Subtracting (4.11) from (4.10), we have

$$\frac{\delta}{(T-t^\tau)^2} + F(t^\tau,p^\tau,u(t^\tau,p^\tau),\eta^\tau,\mathcal{X}^\tau) - F(s^\tau,q^\tau,v(s^\tau,q^\tau),\eta^\tau,\mathcal{Y}^\tau) \le 0. \tag{4.12}$$

Whence

$$0 < \frac{\delta}{(T-t^\tau)^2} \le F(s^\tau,q^\tau,v(s^\tau,q^\tau),\eta^\tau,\mathcal{Y}^\tau) - F(t^\tau,p^\tau,u(t^\tau,p^\tau),\eta^\tau,\mathcal{X}^\tau).$$

We now estimate the difference on the right-hand side. From assumption (1) on F, we obtain

$$\begin{aligned}&F(s^\tau,q^\tau,v(s^\tau,q^\tau),\eta^\tau,\mathcal{Y}^\tau) - F(t^\tau,p^\tau,u(t^\tau,p^\tau),\eta^\tau,\mathcal{X}^\tau)\\ &\le \omega\big(|t^\tau - s^\tau| + d_C(p^\tau,q^\tau)\big) + F(t^\tau,p^\tau,v(s^\tau,q^\tau),\eta^\tau,\mathcal{Y}^\tau)\\ &\quad - F(t^\tau,p^\tau,u(t^\tau,p^\tau),\eta^\tau,\mathcal{X}^\tau).\end{aligned} \tag{4.13}$$

By (4.5), we have

$$u(t^\tau,p^\tau) - v(s^\tau,q^\tau) > 0,$$

for large enough τ. Hence, by assumption (2), (4.9) and assumption (4), the inequality (4.13) becomes

$$F(s^\tau,q^\tau,v(s^\tau,q^\tau),\eta^\tau,\mathcal{Y}^\tau) - F(t^\tau,p^\tau,u(t^\tau,p^\tau),\eta^\tau,\mathcal{X}^\tau) \le o(1) \text{ as } \tau\to\infty. \tag{4.14}$$

Then we obtain the contradiction

$$0 < \frac{\delta}{(T-t_0)^2} \le o(1).$$

Secondly, suppose that $\eta^{\tau_j} = 0$ for a subsequence $\tau_j \to \infty$. One has to distinguish two subcases:

Subcase 1: If $g(p^{\tau_j}, q^{\tau_j}) = 0$, then reasoning as in [13], we obtain the contradiction

$$\frac{\delta}{(T-t_0)^2} \leq 0.$$

Subcase 2: Suppose $g(p^{\tau_j}, q^{\tau_j}) \neq 0$, then it follows that $\nabla^p_{\mathcal{H}} g(p^{\tau_j}, q^{\tau_j}) = 0$. We first prove that $p_1^{\tau_j} = p_2^{\tau_j} = 0$. Assume to get a contradiction that $(p_1^{\tau_j})^2 + (p_2^{\tau_j})^2 \neq 0$. Observe that the function $p \to u(p, q^{\tau_j}, t^{\tau_j}) - \tau g^2(p, q^{\tau_j})$ attains a maximum at p^{τ_j}, which is an interior point of Ω for τ_j large enough. For all point $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3) \neq 0$ closed to p^{τ_j} such that

$$\hat{p}_3 = G(\hat{p}_1, \hat{p}_2) = G(p_1^{\tau_j}, p_2^{\tau_j}) = p_3^{\tau_j},$$

we have by the assumed symmetry of u that

$$u(\hat{p}, t^{\tau_j}) - \tau g^2(\hat{p}, q^{\tau_j}) \leq u(p^{\tau_j}, t^{\tau}) - \tau g^2(p^{\tau_j}, q^{\tau_j})$$

which yields:

$$g(\hat{p}, q^{\tau_j}) \geq g(p^{\tau_j}, q^{\tau_j}).$$

The method of Lagrange multipliers says that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\begin{gathered}\frac{\partial g}{\partial p_1}(p^{\tau_j}, q^{\tau_j}) = \lambda \frac{\partial G}{\partial p_1}(p^{\tau_j}) \\ \frac{\partial g}{\partial p_2}(p^{\tau_j}, q^{\tau_j}) = \lambda \frac{\partial G}{\partial p_2}(p^{\tau_j}).\end{gathered} \tag{4.15}$$

From the assumption $\eta^{\tau_j} = 0$ and (4.15), we obtain

$$\begin{gathered}\lambda \frac{\partial G}{\partial p_1}(p^{\tau_j}) - \frac{p_2^{\tau_j}}{2} \frac{\partial g}{\partial p_3}(p^{\tau_j}, q^{\tau_j}) = 0 \\ \lambda \frac{\partial G}{\partial p_2}(p^{\tau_j}) + \frac{p_1^{\tau_j}}{2} \frac{\partial g}{\partial p_3}(p^{\tau_j}, q^{\tau_j}) = 0.\end{gathered} \tag{4.16}$$

If $\lambda \neq 0$, then

$$\begin{gathered}p_1^{\tau_j} \frac{\partial G}{\partial p_1}(p^{\tau_j}) = \frac{p_1^{\tau_j} p_2^{\tau_j}}{2\lambda} \frac{\partial g}{\partial p_3}(p^{\tau_j}, q^{\tau_j}) \\ p_2^{\tau_j} \frac{\partial G}{\partial p_2}(p^{\tau_j}) = -\frac{p_1^{\tau_j} p_2^{\tau_j}}{2\lambda} \frac{\partial g}{\partial p_3}(p^{\tau_j}, q^{\tau_j}).\end{gathered} \tag{4.17}$$

Adding these equations gives a contradiction to (4.1). Hence, $\lambda = 0$. Thus, equations (4.15) and (4.16) take the form

$$\begin{gathered}0 = \frac{\partial g}{\partial p_1}(p^{\tau_j}, q^{\tau_j}) = 4\big[(p_1^{\tau_j} - q_1^{\tau_j})^2 + (p_2^{\tau_j} - q_2^{\tau_j})^2\big](p_1^{\tau_j} - q_1^{\tau_j}) \\ 0 = \frac{\partial g}{\partial p_2}(p^{\tau_j}, q^{\tau_j}) = 4\big[(p_1^{\tau_j} - q_1^{\tau_j})^2 + (p_2^{\tau_j} - q_2^{\tau_j})^2\big](p_2^{\tau_j} - q_2^{\tau_j}) \\ 0 = \frac{\partial g}{\partial p_3}(p^{\tau_j}, q^{\tau_j}) = 2\Big(p_3^{\tau_j} - q_3^{\tau_j} + \frac{1}{2} p_2^{\tau_j} q_1^{\tau_j} + \frac{1}{2} p_1^{\tau_j} q_2^{\tau_j}\Big),\end{gathered} \tag{4.18}$$

which yields $p^{\tau_j} = q^{\tau_j}$, contradicting the assumption $g(p^{\tau_j}, q^{\tau_j}) \neq 0$.

Therefore, the claim is proved. As in [13], it also follows that $q_1^{\tau_j} = q_2^{\tau_j}$ which implies

$$(\nabla^{p,2}_{\mathcal{H}} g^2)^*(p^{\tau_j}, q^{\tau_j}) = (\nabla^{q,2}_{\mathcal{H}} g^2)^*(p^{\tau_j}, q^{\tau_j}) = 0.$$

An application of Definition 3.3, gives the contradiction:

$$\frac{\delta}{(T-t^{\tau_j})^2}+\tau_j(t^{\tau_j}-s^{\tau_j})\leq 0$$

and

$$\tau_j(t^{\tau_j}-s^{\tau_j})\geq 0.$$

Hence, the proof is complete. □

Remark 4.2. Another way of proving Theorem 4.1 is the following: for each $\delta>0$, the function

$$\tilde{u}=u-\frac{\delta}{T-t}$$

is also a subsolution. Indeed, if φ is a smooth function touching $\tilde{u}$ from above at some point $(t_0,p_0)\in(0,T)\times\Omega$ such that

$$\max_{(0,T)\times\Omega}(\tilde{u}-\varphi)=(\tilde{u}-\varphi)(t_0,p_0)$$

then it clear that

$$\max_{(0,T)\times\Omega}(u-\tilde{\varphi})=(u-\tilde{\varphi})(t_0,p_0),$$

where

$$\tilde{\varphi}(t,p)=\varphi(t,p)+\frac{\delta}{T-t}.$$

Observe that $\nabla_{\mathcal{H}}\tilde{\varphi}=\nabla_{\mathcal{H}}\varphi$ and $(\nabla_{\mathcal{H}}\tilde{\varphi})^*=(\nabla_{\mathcal{H}}\varphi)^*$. Hence, if $\nabla_{\mathcal{H}}\varphi\neq 0$, we have by Definition 3.3 and assumption (2) on F that

$$\begin{aligned}\varphi_t+F(t,p,\tilde{u},\nabla_{\mathcal{H}}\varphi,(\nabla^2_{\mathcal{H}}\varphi)^*)&\leq\tilde{\varphi}_t+F(t,p,u,\nabla_{\mathcal{H}}\tilde{\varphi},(\nabla^2_{\mathcal{H}}\tilde{\varphi})^*)-\frac{\delta}{(T-t)^2}\\&\leq-\frac{\delta}{(T-t)^2}<0.\end{aligned}\tag{4.19}$$

If $\nabla_{\mathcal{H}}\varphi=0$ and $(\nabla^2_{\mathcal{H}}\varphi)^*=0$, then by Definition 3.3,

$$\varphi_t=\tilde{\varphi}_t-\frac{\delta}{(T-t)^2}\leq-\frac{\delta}{(T-t)^2}<0.\tag{4.20}$$

Therefore, without loss of generality, we may assume that the subsolution u satisfies

$$u_t+F(t,p,u,\nabla_{\mathcal{H}}u,(\nabla^2_{\mathcal{H}}u)^*)\leq-\frac{\delta}{T^2}<0,$$

in the sense of viscosity subsolution; that is, u is a subsolution of

$$u_t+\overline{F}(t,p,u,\nabla_{\mathcal{H}}u,(\nabla^2_{\mathcal{H}}u)^*)\leq 0$$

where

$$\overline{F}(t,p,r,\eta,\mathcal{X})=F(t,p,r,\eta,\mathcal{X})+\frac{\delta}{T^2}.$$

Also we may assume that the subsolution u satisfies

$$\lim_{t\to T}u(t,p)=-\infty,\ \text{uniformly in }\overline{\Omega}.\tag{4.21}$$

Then we take the limit $\delta\to 0$ to obtain the desired result for any subsolution u. Proceeding as above, we assume $u(\bar{t},\bar{p})-v(\bar{t},\bar{p})>0$ for some point $(\bar{t},\bar{p})\in\Omega\times[0,T)$. The function M^τ is redefined as follows

$$M^\tau(t,p,s,q)=u(t,p)-v(s,q)-\tau g^2(p,q)-\frac{\tau}{2}(t-s)^2.$$

As above, we derive

$$p^\tau, q^\tau \to p_0, \quad t^\tau, s^\tau \to t_0, \quad \text{as } \tau \to \infty.$$

Observe that by assumption (4.21) on u, the points t^τ lie in a compact subset of $[0,T)$. Moreover, $(p_0,t_0) \in \Omega \times (0,T)$, and we may apply the parabolic Crandall-Ishii lemma as before. Recall that the term $\delta/(T-t^\tau)^2$ does not appear now in the expressions (4.6) and (4.8). The proof proceed as above, getting the contradictions:

$$0 < \frac{\delta}{T^2} \le F(s^\tau, q^\tau, v(s^\tau,q^\tau), \eta^\tau, \mathcal{Y}^\tau) - F(t^\tau, p^\tau, u(t^\tau,p^\tau), \eta^\tau, \mathcal{X}^\tau) \le o(1),$$

when $\eta^\tau \neq 0$ for all large τ, and

$$\frac{\delta}{T^2} \le 0$$

in the two subcases of the case $\eta^{\tau_j} \neq 0$ for a subsequence $\tau_j \to \infty$.

5. Examples

5.1. **Parabolic infinite Laplacian.** We consider the following parabolic equation in the Heisenberg group $\mathcal{H}$:

$$u_t - \Delta^N_{\infty,\mathcal{H}} u = 0, \quad \text{in } \mathcal{H} \times (0,T). \tag{5.1}$$

Here, the operator $-\Delta^N_{\infty,\mathcal{H}}$ denotes the normalized ∞-Laplacian in the Heisenberg group, and it is defined, for all u such that $\nabla_{\mathcal{H}} u \neq 0$, as follows:

$$\begin{aligned} -\Delta^N_{\infty,\mathcal{H}} u &= -\frac{1}{|\nabla_{\mathcal{H}} u|^2} \big\langle (\nabla^2_{\mathcal{H}} u)^* \nabla_{\mathcal{H}} u; \nabla_{\mathcal{H}} u \big\rangle \\ &= -\frac{1}{|\nabla_{\mathcal{H}} u|^2} \sum_{i,j=1}^{2} X_i u X_j u X_i X_j u. \end{aligned} \tag{5.2}$$

For a vector $\eta \in \mathbb{R}^2 \setminus \{0\}$, and $\mathcal{X} \in S^2(\mathbb{R})$, we introduce the function

$$F_\infty(\eta, \mathcal{X}) = -\sum_{i,j=1}^{2} \frac{\eta_i \eta_j}{|\eta|^2} \mathcal{X}_{ij}. \tag{5.3}$$

Hence, equation (5.1) can be written whenever $\nabla_{\mathcal{H}} u \neq 0$ as

$$u_t + F_\infty(\Delta_{\mathcal{H}} u, (\Delta^2_{\mathcal{H}} u)^*) = 0, \quad \text{in } \mathcal{H} \times (0,T). \tag{5.4}$$

Moreover, observe that

$$F^*_\infty(0,0) = F_{\infty,*}(0,0) = 0,$$

and that, for all pairs $(\eta, \mathcal{X}) \in (\mathbb{R}^2 \setminus \{0\}) \times S^2(\mathbb{R})$,

$$F^*_\infty(\eta, \mathcal{X}) = F_{\infty,*}(\eta, \mathcal{X}) = F_\infty(\eta, \mathcal{X}).$$

Finally, it is clear that F also satisfies assumption (4) in Section 3.1. Therefore, Theorem 4.1 tells us that there exists a unique symmetric viscosity solution to the problem

$$\begin{gathered} u_t - \Delta^N_{\infty,\mathcal{H}} u = 0, \quad \text{in } \Omega \times (0,T) \\ u(t,p) = g(t,p) \quad p \in \partial\Omega,\ t \in [0,T) \\ u(0,p) = h(0,p) \quad p \in \overline{\Omega} \end{gathered} \tag{5.5}$$

for $T > 0$, $g \in \mathcal{C}([0,T) \times \overline{\Omega})$ and $h \in \mathcal{C}(\overline{\Omega})$ given.

5.2. **Mean curvature flow equation.** We consider now the following problem involving the mean curvature flow equation:

$$\begin{gathered} u_t - \operatorname{tr}\Big[\Big(I - \frac{\nabla_{\mathcal{H}} u \otimes \nabla_{\mathcal{H}} u}{|\nabla_{\mathcal{H}} u|^2}\Big)(\nabla^2_{\mathcal{H}} u)^*\Big)\Big] = 0, \quad \text{in } \Omega \times (0,T) \\ u(t,p) = u_0(t,p) \quad p \in \partial\Omega,\ t \in [0,T) \\ u(0,p) = u_0(0,p) \quad p \in \overline{\Omega} \end{gathered} \tag{5.6}$$

A derivation and interpretation of the problem (5.6) in the Euclidean setting may be seen in [15] and [7], and in [6] and [14] for the analogue in the Heisenberg group.

Let $F : \mathbb{R}^2 \setminus \{0\} \times S^2(\mathbb{R}) \to \mathbb{R}$ be given by

$$F(\eta, \mathcal{X}) = -\operatorname{tr}\Big[\Big(I - \frac{\eta \otimes \eta}{|\eta|^2}\Big)\mathcal{X}\Big].$$

Observe that F satisfies all the assumptions (1)–(4) from Section 3.1. By Theorem 4.1, the boundary value problem (5.6) admits a unique viscosity solution which is symmetric in the sense specified in Theorem 4.1.

5.3. **Homogeneous diffusions in $\mathcal{H}$.** Consider the following one parameter family of Cauchy problems in the Heisenberg group:

$$\begin{gathered} u_t + C_p \Delta^1_{p,\mathcal{H}} u = 0, \quad \text{in } \Omega \times (0,T) \\ u(t,p) = u_0(t,p) \quad p \in \partial\Omega,\ t \in [0,T) \\ u(0,p) = u_0(0,p) \quad p \in \overline{\Omega} \end{gathered} \tag{5.7}$$

where

$$C_p = \frac{p}{p+1},$$

and the 1-homogeneous p-Laplacian Δ^1_p is defined, for $1 \le p \le \infty$, by

$$\Delta^1_{p,\mathcal{H}} u = \begin{cases} \big(1 - \frac{1}{p}\big) F_1((\nabla^2_{\mathcal{H}} u)^*) + \big(\frac{2}{p} - 1\big) F\big(\nabla_{\mathcal{H}} u, (\nabla^2_{\mathcal{H}} u)^*\big), & \text{if } 1 \le p \le 2 \\ \frac{1}{p} F_1((\nabla^2_{\mathcal{H}} u)^*) + \big(1 - \frac{2}{p}\big) F_\infty(\nabla_{\mathcal{H}} u, (\nabla^2_{\mathcal{H}} u)^*), & \text{if } p > 2. \end{cases}$$

Here $F_1 : S^2(\mathbb{R}) \to \mathbb{R}$ is given by

$$F_1(\mathcal{X}) = -\operatorname{tr} \mathcal{X}.$$

Our result Theorem 4.1 indicates that the problem (5.7) has a unique symmetric (with respect to a surface $p_3 = G(p_1, p_2)$) viscosity solution.

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