STRUCTURE OF THE SOLUTION SET TO DIFFERENTIAL INCLUSIONS WITH IMPULSES AT VARIABLE TIMES

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Abstract. A topological structure of the solution set to differential inclusions with impulses at variable times is investigated. In order to do that an appropriate Banach space is defined. It is shown that the solution set is an R_δ-set. Results are new also in the case of differential equations with impulses at variable times.

1. Introduction

Impulsive differential equations and inclusions have a lot of applications in diverse fields. The moments of impulses can be chosen in various ways: randomly, fixed beforehand, determined by the state of a system. The problems with fixed time of impulses were recently investigated [6,7,9,10,12,17] and the results considering the structure of solution sets were summarised in [11].

The problems with impulses at variable times bring much more difficulties and up to now there were only existence theorems [3,4,5]. Our results develop this research area. We need to use sophisticated assumptions that would guarantee fixed amount of impulses and then we need to deal with the estimations more carefully than in case with fixed times of jumps. In the result we show that the solution set is an R_δ-set. It is worth mentioning that this result is new also in the case of differential equation with impulses at variable times that has no uniqueness of solutions and therefor has a nontrivial solution set.

There are many motivations to study the structure of solution sets of differential equations and inclusions. One of them is considering the Poincaré translation operator and discussing the problem of the existence of periodic solutions [16,8,14].

We have to have the space of functions which contains solutions of given problem to study the structure of solution set. Obviously standard Banach space with the supremum norm is insufficient to impulsive problem with the times of jumps that depend on the state. B-topology on spaces of solutions of impulsive differential inclusions is introduced in [1], however, it is only Hausdorff topology. We use this concept to create Banach space that have the same topology on common functions, and is sufficient to the considered problem.

In Section 2 we describe the problem, introduce the suitable Banach space $CJ_m([0, a])$ and recall useful theorems. In Section 3 we present the main results of the paper. The main idea is included in Theorem 3.1 in which we show that the solution set for the problem with exactly one jump is an R_δ-set. In the Theorem 3.2 we are using the result from previous Theorem proving by induction analogical statement for any fixed number of jumps. We also provide the reader with a transparent example.

2. Preliminaries

We study the problem

$$\dot{y}(t) \in F(t, y(t)), \quad \text{for } t \in [0, a], \ t \neq \tau_j(y(t)), \ j = 1, \ldots, m,$$
$$y(0) = y_0, \tag{2.1}$$
$$y(t^+) = y(t) + I_j(y(t)), \quad \text{for } t = \tau_j(y(t)), \ j = 1, \ldots, m,$$

where $F : [0, a] \times \mathbb{R}^N \multimap \mathbb{R}^N$, $I_j : \mathbb{R}^N \to \mathbb{R}^N$, $j = 1, \ldots, m$, are given impulse functions, $\tau_j \in C^1(\mathbb{R}^N, \mathbb{R})$ with $0 < \tau_j(y) < a$, and $t_y = \{t | t = \tau_k(y(t))\}$. The hypersurface $t - \tau_j(y) = 0$ is called the j-th pulse hypersurface and we denote it by Σ_j. If for each $j = 1, \ldots, m$, τ_j is a different constant function, then impulses are in the fixed times.

Our goal is to find the structure of the solution set of the previous problem, but to do that we need a space of functions with m jumps. We introduce considering space as $CJ_m([0, a]) := C([0, a]) \times (\mathbb{R} \times \mathbb{R}^N)^m$ with following interpretation: the element $(\varphi, (l_j, v_j)_{j=1}^m)$, where $l_j \in [0, a]$ we will interpret as the function with m jumps in the times j_k defined as follows:

$$\hat{\varphi}(t) := \begin{cases} \varphi(t), & 0 \leq t \leq l_{\sigma(1)}, \\ \varphi(t) + \sum_{i=1}^{j} v_{\sigma(i)}, & l_{\sigma(j)} < t \leq l_{\sigma(j+1)}, \\ \varphi(t) + \sum_{i=1}^{m} v_{\sigma(i)}, & l_{\sigma(m)} < t \leq a, \end{cases}$$

where σ is a permutation of $\{1, 2, \ldots, m\}$ such that $l_{\sigma(i)} \leq l_{\sigma(i+1)}$.

There is a mutual correspondence between the functions on interval $[0, a]$ with m jumps and the sets $\{(\varphi, (l_j, v_j)_{j=1}^m) \in CJ_m([0, a]) : l_j < l_{j+1}\}$, with $\zeta \mapsto (\check{\zeta}, (l_j, I_j(\check{\zeta}(l_j)))_{j=1}^m)$, where the function $\check{\zeta}$ is ζ with reduced jumps, l_j is j-th time of jump and the function I_j is an impulse functions.

The space $CJ_m([0, a])$ with the norm

$$\|(\varphi, (l_j, v_j)_{j=1}^m)\| := \sup_{t \in [0, a]} \|\varphi(t)\| + \sum_{j=1}^{m} (|l_j| + \|v_j\|)$$

is a Banach space.

We will find out that the wanted structure is R_δ-type. To show that we will use the following well-known theorems:

Theorem 2.1 ([15]). *Let X be an absolute neighbourhood retract and $A \subset X$ be a compact nonempty subset. Then the following statements are equivalent:*

  (a) *A is an R_δ-set,*
  (b) *for every $\epsilon > 0$ the set A is contractible in $O_\epsilon(A) = \{x \in X | dist(x, A) < \epsilon\}$,*

(c) *A is an intersection of a decreasing sequence $\{A_n\}$ of compact contractible spaces,*

(d) *A is an intersection of a decreasing sequence $\{A_n\}$ of closed contractible spaces, such that $\beta(A_n) \to 0$, where β is the Hausdorff measure of noncompactness.*

A multimap $F: X \multimap E$, where E is real Banach space, is called upper hemicontinuous (uhc) if for every functional $p \in E^*$ the function $X \ni x \mapsto \sigma_{F(x)}(p) := \sup_{y \in F(x)} \langle p, y \rangle \in \mathbb{R} \cup \{+\infty\}$ is upper semicontinuous (usc).

Theorem 2.2 (Convergence theorem [2]). *Let E and E' be Banach spaces, let the space (T, Ω, μ) be a measurable space, and the multivalued map $F : T \times E \multimap E'$ has closed and convex values and for a.e. $t \in T$ the map $F(t, \cdot) : E \multimap E'$ is uhc. Let $(u_n : T \to E)$ be a sequence of functions such that $u_n \to u$ in $L^p(T, E)$ and let sequence $(w_n) \subset L^p(T, E')$, $1 \le p < \infty$ be such that $w_n \rightharpoonup w$ in $L^p(T, E')$. If for a.e. $t \in T$ and for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$w_n(t) \in \operatorname{cl} \operatorname{conv} B(F(t, B(u_n(t), \varepsilon)), \varepsilon)$$

for $n > N$, then $w(t) \in F(t, u(t))$ for a.e. $t \in T$.

We recall Arzela- Ascoli Theorem:

Theorem 2.3. *If the family $\mathcal{F} \subset C([a, b], \mathbb{R}^N)$ of continuous functions is equicontinuous and uniformly bounded, then there exists a subsequence that converges uniformly.*

A piecewise absolutely continuous function $y : [0, a] \to \mathbb{R}^N$ is a solution of the problem with impulses (2.1) if:

(a) $y(0) = y_0$,

(b) there exists a function $f \in L^1([0, a], \mathbb{R}^N)$ such that $f(t) \in F(t, y(t))$ for a.e. $t \in [0, a]$ and $y(t) = y_0 + \sum_{j=1}^m I_j(y(t_j)) + \int_0^t f(s)ds$, where $t_j = \tau_j(y(t_j))$,

(c) the function y is left continuous at $t = \tau_j(y(t)) \in [0, a]$ and the limit $y(t^+)$ exists and $y(t^+) = y(t) + I_j(y(t))$ for $t = \tau_j(y(t))$, $j = 1, \dots, m$.

3. Structure of the solution set

We assume the following conditions on the multivalued perturbation $F : [0, a] \times \mathbb{R}^N \multimap \mathbb{R}^N$:

(F0) F has compact and convex values,

(F1) $F(\cdot, y) : [0, a] \multimap \mathbb{R}^N$ has a measurable selection for every $y \in \mathbb{R}^N$,

(F2) F is almost uniformly with respect to t H-usc, i.e. for every $y \in \mathbb{R}^N$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for a.e. $t \in [0, a]$ and for all $x \in \mathbb{R}^N$ if $\|y - x\| < \delta$, then $\sup_{\varphi \in F(t, x)} d(\varphi, F(t, y)) < \varepsilon$,

(F3) F has a sublinear growth, i.e., there exists $\alpha \in L^1([0, a])$ such that

$$\sup_{\varphi \in F(t, y)} \|\varphi\| \le \alpha(t)(1 + \|y\|) \quad \text{for a.e. } t \in [0, a] \text{ and } y \in \mathbb{R}^N.$$

Moreover, we assume the following hypotheses about impulse functions:

(H1) $I_j \in C(\mathbb{R}^N, \mathbb{R}^N)$, $j = 1, \dots, m$,

(H2) $\tau_j \in C^1(\mathbb{R}^N, \mathbb{R})$, $j = 1, \ldots, m$,
for $j = 1, \ldots, m-1$ we have:

$$0 < \tau_j(y) < \tau_{j+1}(y) < a,$$
$$\tau_j(y + I_j(y)) \le \tau_j(y) < \tau_{j+1}(y + I_j(y)),$$
$$\tau_m(y + I_m(y)) \le \tau_m(y),$$

and there exists a constant $M \ge 0$ such that $\|\tau_j{}'(y)\| \le M$ for all $y \in \mathbb{R}^N$, $j = 1, \ldots, m$,

(H3) there exists a constant $p > 0$ such that for a.e. $t \in [0, a]$

$$\sup_{\varphi \in F(t,y)} \tau_j'(y) \cdot \varphi - 1 \le -p < 0 \quad \text{for all } y \in \mathbb{R}^N, j = 1, \ldots, m.$$

Note that if $\tau_j'(y) = 0$ for $j = 1, \ldots, m$, then the problem is reduced to a problem with a fixed impulse time. Assumption (F2) and compact values of the multivalued map F implies that $F(t, \cdot)$ is usc.

Theorem 3.1. *Let the assumptions* (F0)–(F3) *hold, and* (H1)-(H3) *hold for* $m = 1$. *Then every solution of the problem* (2.1), *where* $m = 1$, *meets* Σ_1 *exactly once and the solution set* S *of this problem is an* R_δ-*set in the space* $CJ_1([0, a])$.

Proof. To simplify notation we write I and τ instead of I_1 and τ_1. We will proceed in several steps.

Step 1. A Lipschitz selection and the uniqueness of jump. For each n let $\{B(y, r_n(y))\}_{y \in \mathbb{R}^N}$ be an open covering (open balls, such that $r_n(y) \le \frac{1}{n} \to 0$) of the space $\mathbb{R}^N$, such that for every $x \in B(y, r_n(y))$ we have

$$\sup_{\varphi \in F(t,x)} d(\varphi, F(t,y)) < \frac{1}{n}. \tag{3.1}$$

There exists locally finite open point-star refinement $\mathcal{U}_n = \{U_{n,s}\}_{s \in S}$ of the cover $\{B(y, r_n(y))\}_{y \in \mathbb{R}^N}$, i.e. for every $y \in \mathbb{R}^N$ there exists $x_{y,n} \in \mathbb{R}^N$ such that $\text{st}(y, \mathcal{U}_n) \subset B(x_{y,n}, r_n(x_{y,n}))$.

We can choose it in a way that $\mathcal{U}_{n+1}$ is a refinement of the cover $\mathcal{U}_n$. Let $\{\lambda_s\}_{s \in S}$ be a locally Lipschitz partition of unity subordinated to the cover $\mathcal{U}_n$ i.e. for every $s \in S$ the function $\lambda_s : [0, a] \to \mathbb{R}^N$ satisfies the locally Lipschitz condition. For every $y_s \in \mathbb{R}^N$, $s \in S$ let a function q_s be a measurable selection of $F(\cdot, y_s)$. We define the function $g_n : [0, a] \times \mathbb{R}^N \to \mathbb{R}^N$ in the following way

$$g_n(t, y) := \sum_{s \in S} \lambda_s(y) \cdot q_s(t).$$

The set $S(y) := \{s \in S | \lambda_s(y) \ne 0\}$ is finite. If $\lambda_s(y) > 0$, so $s \in S(y)$, then $y \in \text{supp} \, \lambda_s \subset U_{n,s} \subset \text{st}(y, \mathcal{U}_n)$. There exists $x_{y,n} \in \mathbb{R}^N$ such that $\text{st}(y, \mathcal{U}_n) \subset B(x_{y,n}, r_n(x_{y,n}))$. We know that $y_s \in \text{st}(y, \mathcal{U}_n)$. We obtain

$$g_n(t, y) = \sum_{s \in S} \lambda_s(y) \cdot q_s(t) \in \text{cnv} \, F(t, \text{st}(y, \mathcal{U}_n))$$
$$\subset G_n(t, y) := \text{cl conv} \, F(t, \text{st}(y, \mathcal{U}_n)).$$

Moreover,

$$G_n(t, y) \subset \text{cl conv} \, F(t, B(x_{y,n}, r_n(x_{y,n}))),$$

so from the inequality (3.1) we have

$$G_n(t,y) \subset \mathrm{cl}\, O_{1/n}(F(t,x_{y,n})). \tag{3.2}$$

We have

$$F(t,y) \subset \cap_{n \geq 1} G_n(t,y).$$

From usc (the map F has compact values) we obtain that for every $y \in \mathbb{R}^N$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(t,B(y,\delta)) \subset O_\varepsilon(F(t,y)).$$

We have

$$\cap_{n \geq 1} G_n(t,y) \subset \cap_{n \geq 1} \mathrm{cl}\, O_{1/n}(F(t,x_{y,n}))$$
$$\subset \cap_{n \geq 1} \mathrm{cl}\, O_{1/n}\left(F\left(t, B(y, 2r_n(x_{y,n}))\right)\right)$$
$$\subset \cap_{n \geq 1} \mathrm{cl}\, O_{\frac{1}{n}+\varepsilon}\left(F(t,y)\right) = F(t,y).$$

We obtain

$$F(t,y) = \cap_{n \geq 1} G_n(t,y).$$

We have

$$G_{n+1}(t,y) \subset G_n(t,y).$$

Let the Nemitskiĭ (substitution) operator $P_{G_n} : CJ_1([0,a]) \multimap L^1(J,\mathbb{R}^N)$ be defined by

$$P_{G_n}(y) := \{\phi \in L^1([0,a],\mathbb{R}^N) | \phi(t) \in G_n(t,y(t)) \quad \text{for a.e. } t \in [0,a]\}.$$

Let S_n denote the set of solutions of the problem

$$\dot{y}(t) \in G_n(t,y(t)), \quad \text{for } t \in [0,a],\, t \neq \tau(y(t)),$$
$$y(0) = y_0, \tag{3.3}$$
$$y(t^+) = y(t) + I(y(t)), \quad \text{for } t = \tau(y(t)).$$

It is obvious that the sets S_n are nonempty, because the problem

$$\dot{y}(t) = g_n(t,y(t)), \quad \text{for } t \in [0,a],\, t \neq \tau(y(t)),$$
$$y(0) = y_0, \tag{3.4}$$
$$y(t^+) = y(t) + I(y(t)), \quad \text{for } t = \tau(y(t)),$$

for every $n \in \mathbb{N}$ has exactly one solution.

Step 1a. We denote by $t_{y_n}^j$ the time of j-th jump for the function y_n a nd if the function y_n has less that j jumps we take $t_{y_n}^j = a$. Let y_n be an arbitrary solution of the system (3.3) for $0 \leq t \leq t_{y_n}^2$. For $t \leq t_{y_n}^1$ we obtain the following form of the solution

$$y_n(t) = y_0 + \int_0^t \phi_n(s)ds,$$

where $\phi_n \in P_{G_n}(y_n)$.

There exists selection f_n (not necessarily measurable) of the multivalued map $F(\cdot,x_{y,n}(\cdot))$ such that for a.e. t we obtain $\|\phi_n(t) - f_n(t,x_{y,n}(t))\| \leq \frac{1}{n}$. We have

$$\|\phi_n(t)\| \leq \frac{1}{n} + \|f_n(t,x_{y,n}(t))\|.$$

From the assumption $(F3)$ we obtain

$$\|f_n(t,x_{y,n}(t))\| \leq \alpha(t)\big(1 + \|x_{y,n}(t)\|\big) \leq \alpha(t)\Big(1 + \|y(t)\| + \frac{1}{n}\Big).$$

So

$$\|y_n(t)\| \le \|y_0\| + \int_0^t \Big(\alpha(s)\Big(1 + \|y_n(s)\| + \frac{1}{n}\Big) + \frac{1}{n}\Big)ds.$$

From Gronwall inequality, we have

$$\sup_{t\in[0,t_{y_n}^1]} \|y_n(t)\| \le \Big(\|y_0\| + \int_0^a 2\alpha(s)ds + \frac{a}{n}\Big)e^{\int_0^a \alpha(s)ds} := K. \tag{3.5}$$

By the continuity of I there exists a constant $c > 0$ such that $\|I(y_n(t_{y_n}^1))\| \le c$ for all n. Next for $t_{y_n}^1 < t \le t_{y_n}^2$ we obtain

$$\|y_n(t)\| \le \|y_0\| + \|I(y_n(t_{y_n}^1))\| + \int_0^t \|\phi_n(s)\|ds$$

$$\le \|y_0\| + c + \int_0^t \Big(\alpha(s)\Big(1 + \frac{1}{n} + \|y_n(s)\|\Big) + \frac{1}{n}\Big)ds.$$

Again, from Gronwall inequality we obtain

$$\sup_{t\in[0,t_{y_n}^2]} \|y_n(t)\| \le Ce^{\int_0^t \alpha(s)ds} < Ce^{\int_0^a \alpha(s)ds} =: \bar{K}, \tag{3.6}$$

where $C := \|y_0\| + c + \int_0^a 2\alpha(s)ds + \frac{a}{n}$. If the solution y_n does not have jumps, then of course $\sup_{t\in[0,a]} \|y_n(t)\| \le K$.

Step 1b. Let $\bar{y}$ be a fixed function with values in $\operatorname{cl} B(0, \bar{K})$. Let $\phi_n \in G_n(t, \bar{y}(t))$, where t is such that (H3) is satisfied. We denote $\bar{y}(t) =: y$. From the assumptions (H2) and (H3) for some $v \in \operatorname{cl} B(0, 1)$ we obtain

$$\tau'(y) \cdot \phi_n - 1 = \Big(\tau'(x_{y,n}) + \tau'(y) - \tau'(x_{y,n})\Big) \cdot \phi_n - 1$$

$$= \tau'(x_{y,n}) \cdot \Big(\bar{\varphi} + \frac{1}{n}v\Big) - 1 + (\tau'(y) - \tau'(x_{y,n})) \cdot \phi_n$$

$$= \tau'(x_{y,n}) \cdot \bar{\varphi} - 1 + \frac{1}{n}\tau'(x_{y,n}) \cdot v + (\tau'(y) - \tau'(x_{y,n})) - \phi_n,$$

where $\bar{\varphi} \in F(t, x_{y,n})$. Hence

$$\tau'(y) \cdot \phi_n - 1 \le -p + \frac{\|\tau'(x_{y,n})\|}{n} + (\tau'(y) - \tau'(x_{y,n})) \cdot \phi_n$$

$$\le -p + \frac{M}{n} + \|\tau'(y) - \tau'(x_{y,n})\|\|\phi_n\|.$$

The function τ' is continuous on the compact set $\operatorname{cl} B(0, \bar{K} + 1)$, therefore it is uniformly continuous. Hence, for $y \in \operatorname{cl} B(0, \bar{K})$, $\tau'(y) - \tau'(x_{y,n}) \to 0$ (we have $\|x_{y,n}\| \le \bar{K} + \frac{1}{n}$). Moreover, the set $\{\phi_n\}$ is bounded (from the sublinear growth of F).

We have $\|\tau'(y) - \tau'(x_{y,n}))\| \|\phi_n\| \to 0$. We take $N_0 \in \mathbb{N}$ such that for every $n \ge N_0$ we have $-p + \frac{M}{n} + \|\tau'(y) - \tau'(x_{y,n})\| \|\phi_n\| < \frac{-p}{2}$. There exists a constant $p' > 0$ such that

$$\tau'(y) \cdot \phi_n - 1 < -p'.$$

Step 1c. Let us fix y_n, where $n > N_0$, the solution of the problem (3.3). We define the function $w_n : [0, a] \to \mathbb{R}$ by

$$w_n(t) := \tau(y_n(t)) - t.$$

The function w_n has value 0 in any time, in which the function y_n has a jump. By the condition (H2) with $m = 1$, we obtain $w_n(0) = \tau(y_0) > 0$ and $w_n(a) = \tau(y_n(a)) - a < a - a = 0$. If $w_n(t) \neq 0$ on $[0, a]$, then there would not be any impulse effect, therefore there would not be any jump time, so w_n would be continuous, which would contradict with the earlier inequalities. Hence every solution of the problem (3.3) has at least one jump.

Suppose that $0 < t^1_{y_n} < a$ is the first time in which the solution y_n hits the hypersurface Σ_1. Then

$$w_n(t^1_{y_n}) = 0 \quad \text{and} \quad w_n(t) > 0, \quad \text{for } t \in [0, t^1_{y_n}).$$

By assumption (H2) with $m = 1$ we obtain that

$$w_n(t^1_{y_n}{}^+) = \tau(y_n(t^1_{y_n}{}^+)) - t^1_{y_n} = \tau\left(y_n(t^1_{y_n}) + I(y_n(t^1_{y_n}))\right) - t^1_{y_n} \leq 0.$$

For a.e. $t \geq t^1_{y_n}$ we have

$$w_n'(t) = \tau'(y_n(t)) \cdot y_n'(t) - 1 = \tau'(y_n(t)) \cdot \phi_n(t) - 1 < -p' < 0,$$

where $\phi_n \in P_{G_n}(y_n)$. The function w_n in $[t^1_{y_n}, a]$ is decreasing, hence y_n hits the hypersurface Σ_1 exactly once and the time of this jump we denote t_{y_n}.

Step 2. Now we show that each sequence (y_n), where $y_n \in S_n$, has a convergent subsequence to the solution $\tilde{y}$ of the problem (2.1).

There exists exactly one jump, so from the previous estimations we have

$$\|y_n(t)\| \leq \bar{K}.$$

Consequently, the values of solutions of the problem (2.1) are contained in a ball $\mathrm{cl}\, B(0, \bar{K})$, which is convex, so in particular we know that function $g_n|_{[0,a] \times \mathrm{cl}\, B(0,\bar{K})}$ has integrable Lipschitz constant Λ.

For $t < t' \leq t_{y_n}$ we have

$$\|y_n(t) - y_n(t')\| = \|\int_t^{t'} \phi_n(s) ds\|$$

$$\leq \frac{1}{n}|t - t'| + \int_t^{t'} \alpha(s)\left(1 + \frac{1}{n} + \|y_n(s)\|\right) ds \qquad (3.7)$$

$$\leq |t - t'| + (2 + K) \int_t^{t'} \alpha(s) ds,$$

and for $t_{y_n} < t < t'$ we have

$$\|y_n(t) - y_n(t')\| = \|\int_t^{t'} \phi_n(s) ds\|$$

$$\leq \frac{1}{n}|t - t'| + \int_t^{t'} \alpha(s)\left(1 + \frac{1}{n} + \|y_n(s)\|\right) ds \qquad (3.8)$$

$$\leq |t - t'| + (2 + \bar{K}) \int_t^{t'} \alpha(s) ds.$$

Step 2.a Let us consider convergence to the time t_*, where t_* the limit of a convergent subsequence $(t_{y_{n_k}})$ of the sequence (t_{y_n}) (with $y_n \in S_n$), which exists due

to the compactness of $[0, a]$. For every $\varepsilon > 0$ there exists N_0 such that for $n_k > N_0$ we have $t_* - \varepsilon < t_{y_{n_k}}$. Note that

$$\sup_{t \in [0, t_* - \varepsilon - \delta]} \int_t^{t+\delta} \alpha(s) ds \to 0$$

with $\delta \to 0$.

From inequality (3.7) we obtain that for all $\xi > 0$ and $0 < t < t_* - \varepsilon$ there exists $\delta > 0$ such that for all $n_k > N_0$ and $t < t' < t+\delta \le t_* - \varepsilon$ we have $\|y_{n_k}(t) - y_{n_k}(t')\| \le |t' - t| + (2 + K) \int_t^{t'} \alpha(s) ds < \xi$. Therefore, the family $\{y_{n_k}\}_{n_k > N_0}$ is equicontinuous and by the inequality (3.5) uniformly bounded. By Arzela-Ascoli Theorem 2.3 (possibly going to the subsequences) we can assume that $y_{n_k} \to y_\varepsilon$ on $[0, t_* - \varepsilon]$, where y_ε is continuous function. This can be done in such a way that for any $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 > \varepsilon_2$ functions $y_{\varepsilon_1}, y_{\varepsilon_2}$ fulfil condition $y_{\varepsilon_2}|_{[0, t_* - \varepsilon_1]} = y_{\varepsilon_1}$. For $\varepsilon \searrow 0$ we obtain an extension of the function y_ε, i.e. the function $y_* : [0, t_*) \to \mathbb{R}^N$, where y_{n_k} converges pointwise to y_*. Moreover

$$\|\phi_{n_k}(t)\| \le \alpha(t) \Big(1 + \|x_{y_{n_k}, n_k}(t)\|\Big) + \frac{1}{n_k} \le \alpha(t)(2 + \bar{K}) + 1. \qquad (3.9)$$

We know that:

- $\phi_{n_k}(t) \in \operatorname{cl\,conv} F\left(t, \operatorname{st}(y_{n_k}(t), \mathcal{U}_{n_k})\right) \subset \operatorname{clO}_{1/n_k}\Big(F\big(t, B(y_{n_k}(t), \frac{1}{n_k})\big)\Big)$, by inclusion (3.2),
- $y_{n_k}(t) \to y_\varepsilon(t)$ a.e. on $[0, t_* - \varepsilon]$,
- $\phi_{n_k} \in L^1([0, t_* - \varepsilon], \mathbb{R}^N)$,
- by estimation (3.9) and weak compactness of the closed ball we obtain $\phi_{n_{k_l}} \rightharpoonup \phi$ on $[0, t_* - \varepsilon]$.

Thus by Theorem 2.2 we obtain $\phi(t) \in F(t, y_\varepsilon(t))$ for a.e. $t \in [0, t_* - \varepsilon]$. By analogy, we conclude that $\phi(t) \in F(t, y_*(t))$ a.e. on $[0, t_*)$. By weak convergence $\phi_{n_{k_l}} \rightharpoonup \phi$ on $[0, t_* - \varepsilon]$, for $\Psi(\phi_n) := \int_0^t \phi_n(s) ds$ we have

$$\int_0^t \phi(s) ds = \Psi(\phi) = \lim_{k \to \infty} \Psi(\phi_{n_k}) = \lim_{k \to \infty} \int_0^t \phi_{n_k}(s) ds$$
$$= \lim_{k \to \infty} y_{n_k}(t) - y_0 = y_*(t) - y_0.$$

For an increasing sequence (s_n) convergent to t_* with $n < n'$ we obtain

$$\|y_*(s_n) - y_*(s_{n'})\| = \|\int_{s_n}^{s_{n'}} \phi(s) ds\| \le \int_{s_n}^{t_*} (\alpha(s)(2 + K) + 1) ds.$$

We have convergence of the right hand side of the inequality to 0 with $n \to \infty$, consequently $(y_*(s_n))$ is Cauchy sequence. It is convergent ($\mathbb{R}^N$ is complete) and we denote its limit $y_*(s_n) \to y_1$. We define $y_*(t_*) := y_1$ and we obtain continuous extension y_* on $[0, t_*]$. We will show, that $\tau(y_*(t_*)) - t_* = 0$, which means that t_* is time of jump for y_*.

Let $\epsilon > 0$. The function $\alpha(s)(2 + \bar{K}) + 1$ is integrable, so we can choose $t_\epsilon < t_*$ so that

$$\int_{t_\epsilon}^{t_*} 2[\alpha(s)(2 + \bar{K}) + 1] ds < \frac{\epsilon}{2}.$$

There exists $K_0 \in \mathbb{N}$ such that for $k > K_0$ we have $\|y_*(t_\epsilon) - y_{n_k}(t_\epsilon)\| < \frac{\epsilon}{2}$. We can estimate

$$\|y_{n_k}(t_{y_{n_k}}) - y_*(t_*)\|$$

$$\leq \|y_{n_k}(t_\epsilon) + \int_{t_\epsilon}^{t_{y_{n_k}}} \phi_{n_k}(s)ds - y_*(t_\epsilon) - \int_{t_\epsilon}^{t_*} \phi(s)ds\|$$

$$\leq \|y_{n_k}(t_\epsilon) - y_*(t_\epsilon)\| + \int_{t_\epsilon}^{t_{y_{n_k}}} \|\phi_{n_k}(s)\|ds + \int_{t_\epsilon}^{t_*} \|\phi(s)\|ds$$

$$\leq \|y_{n_k}(t_\epsilon) - y_*(t_\epsilon)\| + \int_{t_\epsilon}^{t_{y_{n_k}}} \left(\alpha(s)(2 + \bar{K}) + 1\right)ds + \int_{t_\epsilon}^{t_*} \left(\alpha(s)(2 + \bar{K}) + 1\right)ds$$

$$\leq \frac{\epsilon}{2} + \int_{t_\epsilon}^{t_*} 2(\alpha(s)(2 + \bar{K}) + 1)ds \leq \epsilon.$$

For $t_{y_{n_k}} > t_*$ we obtain

$$\|y_*(t_*) - y_{n_k}(t_{y_{n_k}})\| \leq \|y_*(t_*) - y_*(t_* - \varepsilon)\| + \|y_*(t_* - \varepsilon) - y_{n_k}(t_* - \varepsilon)\|$$
$$+ \|y_{n_k}(t_* - \varepsilon) - y_{n_k}(t_{y_{n_k}})\|,$$

but it is easy to see that

$$\|y_{n_k}(t_* - \varepsilon) - y_{n_k}(t_{y_{n_k}})\| = \|\int_{t_* - \varepsilon}^{t_{y_{n_k}}} \phi_{n_k}(s)ds\| \leq \int_{t_* - \varepsilon}^{t_{y_{n_k}}} \|\phi_{n_k}(s)\|ds$$

$$\leq \int_{t_* - \varepsilon}^{t_{y_{n_k}}} \left(\alpha(s)(2 + K) + 1\right)ds$$

$$= (2 + K)\int_{t_* - \varepsilon}^{t_{y_{n_k}}} \alpha(s)ds + t_{y_{n_k}} - t_* + \varepsilon,$$

so

$$\|y_*(t_*) - y_{n_k}(t_{y_{n_k}})\| \leq \|y_*(t_*) - y_*(t_* - \varepsilon)\| + \|y_*(t_* - \varepsilon) - y_{n_k}(t_* - \varepsilon)\|$$

$$+ (2 + K)\int_{t_* - \varepsilon}^{t_{y_{n_k}}} \alpha(s)ds + t_{y_{n_k}} - t_* + \varepsilon$$

$$\to \|y_*(t_*) - y_*(t_* - \varepsilon)\| + (2 + K)\int_{t_* - \varepsilon}^{t_*} \alpha(s)ds + \varepsilon,$$

as $k \to \infty$. From the arbitrariness of ϵ and ε we obtain $\|y_*(t_*) - y_{n_k}(t_{y_{n_k}})\| \to 0$.

Summarising, we have that $y_{n_k}(t_{y_{n_k}}) \to y_*(t_*)$. By the continuity of τ we obtain $\tau(y_*(t_*)) - t_* = \lim_{n_k \to \infty}(\tau(y_{n_k}(t_{y_{n_k}})) - t_{y_{n_k}}) = 0$ which means that t_* is the time of jump for y_*.

Step 2.b We make a similar reasoning with the part of segment $[0, a]$ after the jump. Form inequalities (3.6) and (3.8) we conclude that the family $\{y_{n_k}\}$ is equicontinuous and equibounded on $[t_* + \varepsilon, a]$. Therefore, from Arzela-Ascoli 2.3 theorem (passing to a subsequence if it is needed) we can assume that $y_{n_k} \to y^\varepsilon$, where y^ε is a continuous function and we extend it to a continuous function $y^* : (t_*, a] \to \mathbb{R}^N$ with y_{n_k} convergent pointwise to y^*. We know that:

- $\phi_{n_k}(t) \in \mathrm{cl\,conv}\, F\big(t, \mathrm{st}(y_{n_k}(t), \mathcal{U}_{n_k})\big) \subset \mathrm{cl}\, O_{1/n_k}\Big(F\big(t, B(y_{n_k}(t), \frac{1}{n_k})\big)\Big).$
- $y_{n_k}(t) \to y^\varepsilon(t)$ a.e. on $[t_* + \varepsilon, a]$,
- $\phi_{n_k} \in L^1([t_* + \varepsilon, a], \mathbb{R}^N),$

- by estimation (3.9) and the weak compactness of closed ball we obtain $\phi_{n_{k_l}} \rightharpoonup \phi$ on $[t_* + \varepsilon, a]$.

Again, by theorem 2.2 we obtain $\phi(t) \in F(t, y^\varepsilon(t))$, for a.e. $t \in [t_* + \varepsilon, a]$, so we obtain an information that $\phi(t) \in F(t, y^*(t))$ a.e. on $(t_*, a]$. By weak convergence $\phi_{n_{k_l}} \rightharpoonup \phi$ on $[t_* + \varepsilon, a]$ for $\Psi(\phi_n) := \int_t^a \phi_n(s)ds$ we have

$$\int_t^a \phi(s)ds = \Psi(\phi) = \lim_{k \to \infty} \Psi(\phi_{n_k}) = \lim_{k \to \infty} \int_t^a \phi_{n_k}(s)ds$$
$$= \lim_{k \to \infty} y_{n_k}(a) - y_*(t) = y_*(a) - y_*(t).$$

For decreasing sequence (s_n) convergent to t_* we obtain for $n < n'$:

$$\|y^*(s_n) - y^*(s_{n'})\| = \|\int_{s_{n'}}^{s_n} \phi(s)ds\| \le \int_{t_*}^{s_n} \alpha(s)(1 + \bar{K})ds.$$

By analogy to Step 2a, we obtain that $(y^*(s_n))$ is the Cauchy sequence, which is convergent in Banach space, so $y^*(s_n) \to y_2$ for some $y_2 \in \mathbb{R}^N$.

By continuity of I we have that $I(y_{n_k}(t_{y_{n_k}})) \to I(y_*(t_*))$, therefore $y_2 = y_*(t_*) + I(y_*(t_*))$.

We can define a function $\tilde{y} : [0, a] \to \mathbb{R}^N$ by concatenation y_* on $[0, t_*]$ with the function y^* on $(t_*, a]$. Obviously, for $t \le t_*$ the function $\tilde{y}$ is the solution of the problem (2.1). For $t > t_*$ we have

$$\tilde{y}(t) = \tilde{y}(t_*) + I(\tilde{y}(t_*)) + \int_{t_*}^t \phi(s)ds = y_0 + \int_0^{t_*} \phi(s)ds + I(\tilde{y}(t_*)) + \int_{t_*}^t \phi(s)ds,$$

where $\phi \in P_{G_n}(\tilde{y})$, so $\tilde{y}$ is the solution of the problem (2.1) for all $t \in [0, a]$, hence $\tilde{y} \in S$. The function $\tilde{y}$ is the limit of the sequence (y_{n_k}) in the space $CJ_1([0, a])$.

Step 3. We show, for every $n \in \mathbb{N}$, the contractibility of the set $\operatorname{cl} S_n$. Fix n such that we can define p' (see Step 1.) and take $\bar{y} \in \operatorname{cl} S_n$. We divide the interval $[0, 1]$ into two halves. Let $r \in [0, \frac{1}{2}]$. We consider the problem

$$\begin{aligned} \dot{y}(t) &= g_n(t, y(t)), \quad \text{for } t \in [a - 2r(a - t_{\bar{y}}), a], t \ne \tau(y(t)), \\ y(t) &= \bar{y}(t), \quad \text{for } t \in [0, a - 2r(a - t_{\bar{y}})], \\ y(t^+) &= y(t) + I(y(t)), \quad \text{for } t = \tau(y(t)). \end{aligned} \tag{3.10}$$

In the previous problem we denote by g_n selection of the map G_n. There exists exactly one solution of this problem; we denote it by $y^2_{\bar{y},r}$. Then $y^2_{\bar{y},r} \in \operatorname{cl} S_n$,

Next for $r \in (1/2, 1]$ we consider the problem

$$\begin{aligned} \dot{y}(t) &= g_n(t, y(t)), \quad \text{for } t \in [t^{\bar{y},r}, a], t \ne \tau(y(t)), \\ y(t) &= \bar{y}(t), \quad \text{for } t \in [0, t^{\bar{y},r}], \\ y(t^+) &= y(t) + I(y(t)), \quad \text{for } t = \tau(y(t)). \end{aligned} \tag{3.11}$$

where $t^{\bar{y},r} := t_{\bar{y}} - 2(r - \frac{1}{2})t_{\bar{y}}$. There exists exactly one solution of this problem, denoted by $y^1_{\bar{y},r}$, which also belongs to $\operatorname{cl} S_n$. Finally we consider the function $h : [0, 1] \times \operatorname{cl} S_n \to \operatorname{cl} S_n$ given by

$$h(r, \bar{y}) := \begin{cases} y^2_{\bar{y},r}, & r \in [0, \frac{1}{2}], \\ y^1_{\bar{y},r}, & r \in (\frac{1}{2}, 1]. \end{cases} \tag{3.12}$$

Now, we show that the function h is continuous. Due to the continuous dependence of solutions on initial conditions [13] we know that the function h is continuous on $[0, 1/2) \times \operatorname{cl} S_n$ and left continuous on $\{1/2\} \times \operatorname{cl} S_n$.

Let $((r_k, \bar{y}_k))_k$ be a sequence convergent to $(\frac{1}{2}^+, \bar{y})$. We know that if $t_{\bar{y}} < t_{h(r_k, \bar{y}_k)}$ then $\tau(h(r_k, \bar{y}_k)(t_{\bar{y}})) - t_{\bar{y}} > 0$ and we have

$$t_{\bar{y}} - \tau(h(r_k, \bar{y}_k)(t_{\bar{y}})) = \int_{t_{\bar{y}}}^{t_{h(r_k, \bar{y}_k)}} (\tau(h(r_k, \bar{y}_k)(\cdot)) - \cdot)'(\theta) d\theta < -p'(t_{h(r_k, \bar{y}_k)} - t_{\bar{y}}),$$

so

$$t_{h(r_k, \bar{y}_k)} < t_{\bar{y}} + (\tau(h(r_k, \bar{y}_k)(t_{\bar{y}})) - t_{\bar{y}})/p',$$

but for k such that $t_{\bar{y}} \le t_{h(r_k, \bar{y}_k)}$ one sees that

$$\left\| \bar{y}(t^{\bar{y}_k, r_k}) + \int_{t^{\bar{y}_k, r_k}}^{t_{\bar{y}}} \bar{\phi}_n(s) ds - \left(y^1_{\bar{y}_k, r_k}(t^{\bar{y}_k, r_k}) + \int_{t^{\bar{y}_k, r_k}}^{t_{\bar{y}}} g_n(s, y^1_{\bar{y}_k, r_k}(s)) ds \right) \right\|$$

$$= \left\| \int_{t^{\bar{y}_k, r_k}}^{t_{\bar{y}}} \bar{\phi}_n(s) - g_n(s, x^1_{\bar{y}_k, r_k}(s)) ds \right\|$$

$$\le \int_{t^{\bar{y}_k, r_k}}^{t_{\bar{y}}} \|\bar{\phi}_n(s)\| + \|g_n(s, y^1_{\bar{y}_k, r_k}(s))\| ds$$

$$\le \int_{t^{\bar{y}_k, r_k}}^{t_{\bar{y}}} 2\Big(\alpha(s)(1 + K) + \frac{1}{n}\Big) ds.$$

Hence, if $k \to \infty$ we obtain $h(r_k, \bar{y}_k)(t_{\bar{y}}) \to \bar{y}(t_{\bar{y}})$, so

$$\tau(h(r_k, \bar{y}_k)(t_{\bar{y}})) - t_{\bar{y}} \to 0.$$

For every k we have

$$t^{\bar{y}_k, r_k} < t_{h(r_k, \bar{y}_k)} \le t_{\bar{y}} + \mathbf{1}_{t_{\bar{y}} < t_{h(r_k, \bar{y}_k)}}(\tau(h(r_k, \bar{y}_k)(t_{\bar{y}})) - t_{\bar{y}})/p'$$

and

$$t^{\bar{y}_k, r_k} \to t_{\bar{y}} \leftarrow t_{\bar{y}} + \mathbf{1}_{t_{\bar{y}} < t_{h(r_k, \bar{y}_k)}}(\tau(h(r_k, \bar{y}_k)(t_{\bar{y}})) - t_{\bar{y}})/p',$$

therefore by the squeeze theorem

$$t_{h(r_k, \bar{y}_k)} \to t_{\bar{y}}. \tag{3.13}$$

Let us fix k for a while. By $\mathbf{1}$ we denote the function

$$\mathbf{1}_{t > t_0}(y(t)) = \begin{cases} y(t), & \text{for } t > t_0, \\ 0, & \text{for } t \le t_0. \end{cases} \tag{3.14}$$

We define function $\varrho_{\bar{y}_k, r_k} : [0, a] \to \mathbb{R}^N$

$$\varrho_{\bar{y}_k, r_k}(t) := y^1_{\bar{y}_k, r_k}(t) - \mathbf{1}_{t > t_{y^1_{\bar{y}_k, r_k}}}(I(y^1_{\bar{y}_k, r_k}(t_{y^1_{\bar{y}_k, r_k}})))$$

$$- (y^2_{\bar{y}, 1/2}(t) - \mathbf{1}_{t > t_{\bar{y}}}(I(y^2_{\bar{y}, 1/2}(t_{\bar{y}})))),$$

which is function of differences between $y^1_{\bar{y}_k, r_k}$, and $y^2_{\bar{y}, 1/2}$ with deleted changes caused by jumps. It is easy to see that $t_{\bar{y}} = t_{y^2_{\bar{y}, 1/2}}$. For $t^{\bar{y}_k, r_k} \le t \le t_{\bar{y}}$ we obtain

$$\|\varrho_{\bar{y}_k, r_k}(t)\| \le \|\bar{y} - \bar{y}_k\| + \int_{t^{\bar{y}_k, r_k}}^{t} \|g_n(s, y^1_{\bar{y}_k, r_k}(s)) - \bar{\phi}_n(s)\| ds$$

$$\le \|\bar{y} - \bar{y}_k\| + \int_{t^{\bar{y}_k, r_k}}^{t} \big(\|\bar{\phi}_n(s)\| + \|g_n(s, y^1_{\bar{y}_k, r_k}(s))\|\big) ds$$

$$\leq \|\bar{y} - \bar{y}_k\| + \int_{t^{\bar{y}_k,r_k}}^{t} 2\Big(\alpha(s)(1+\bar{K}) + \frac{1}{n}\Big)ds$$

$$=: z_{\bar{y}_k,r_k}(t)$$

and for $t > t_{\bar{y}}$ we have

$$\|\varrho_{\bar{y}_k,r_k}(t)\|$$

$$\leq z_{\bar{y}_k,r_k}(t_{\bar{y}}) + \int_{t_{\bar{y}}}^{t} \|g_n(s, y^1_{\bar{y}_k,r_k}(s)) - g_n(s, y^2_{\bar{y},1/2}(s))\|ds$$

$$\leq z_{\bar{y}_k,r_k}(t_{\bar{y}}) + \int_{t_{\bar{y}}}^{t} \Lambda(s)\|y^1_{\bar{y}_k,r_k}(s) - y^2_{\bar{y},1/2}(s)\|ds$$

$$= z_{\bar{y}_k,r_k}(t_{\bar{y}}) + \int_{t_{\bar{y}}}^{t} \Lambda(s)\|\check{y}^1_{\bar{y}_k,r_k}(s) + \mathbf{1}_{s > t_{y^1_{\bar{y}_k,r_k}}} I(y^1_{\bar{y}_k,r_k}(t_{y^1_{\bar{y}_k,r_k}})) - \check{y}^2_{\bar{y},1/2}(s)$$

$$- I(y^2_{\bar{y},1/2}(t_{\bar{y}}))\|ds$$

$$\leq z_{\bar{y}_k,r_k}(t_{\bar{y}}) + \Big| \int_{t_{\bar{y}}}^{t_{y^1_{\bar{y}_k,r_k}}} \Lambda(s)\|I(y^2_{\bar{y},1/2}(t_{\bar{y}})\|ds \Big|$$

$$+ \int_{\max\{t_{\bar{y}}, t_{y^1_{\bar{y}_k,r_k}}\}}^{t} \Lambda(s)\|I(y^1_{\bar{y}_k,r_k}(t_{y^1_{\bar{y}_k,r_k}})) - I(y^2_{\bar{y},1/2}(t_{\bar{y}}))\|ds$$

$$+ \int_{t_{\bar{y}}}^{t} \Lambda(s)\|\check{y}^1_{\bar{y}_k,r_k}(s) - \check{y}^2_{\bar{y},1/2}(s)\|ds,$$

hence by the Gronwall inequality we obtain

$$\|\varrho_{\bar{y}_k,r_k}(t)\| \leq \Big(z_{\bar{y}_k,r_k}(t_{\bar{y}}) + \Big| \int_{t_{\bar{y}}}^{t_{y^1_{\bar{y}_k,r_k}}} \Lambda(s)\|I(y^2_{\bar{y},1/2}(t_{\bar{y}})\|ds \Big|$$

$$+ \int_{\max\{t_{\bar{y}}, t_{y^1_{\bar{y}_k,r_k}}\}}^{t} \Lambda(s)\|I(y^1_{\bar{y}_k,r_k}(t_{y^1_{\bar{y}_k,r_k}})) - I(y^2_{\bar{y},1/2}(t_{\bar{y}}))\|ds \Big)$$

$$\times \exp \int_{t_{\bar{y}}}^{t} \Lambda(s)ds =: z_{\bar{y}_k,r_k}(t).$$

By previous convergences and continuity of I, we obtain

$$\|I(h(r_k,\bar{y}_k)(t_{h(r_k,\bar{y}_k)})) - I(h(1/2,\bar{y})(t_{h(1/2,\bar{y})}))\| \to 0, \qquad (3.15)$$

$$\sup_{t \in [0,a]} \|\varrho_{\bar{y}_k,r_k}(t)\| \leq z_{\bar{y}_k,r_k}(a) \to 0. \qquad (3.16)$$

Summing up, by (3.16), (3.13) and (3.15), if $(r_k,\bar{y}_k)$ converges to $(\frac{1}{2}^{+},\bar{y})$, then $y^1_{\bar{y}_k,r_k}$ converges to $y^2_{\bar{y},\frac{1}{2}}$ in norm in the space $CJ_1([0,a])$.

The function h, as continuous on $[0,1] \times \mathrm{cl}\, S_n$, is a homotopy. By definition of the function h we have $h(0,\bar{y}) = \bar{y}$ and $h(1,\bar{y}) = y^1_{\bar{y},1}$, so $\mathrm{cl}\, S_n$ is a contractible set.

Step 4. We show that properties needed to theorem 2.1 are fulfilled. The sets $\mathrm{cl}\, S_n$ are contractible in the power of Step 3.

If $x \in \cap_{n \in \mathbb{N}} \mathrm{cl}\, S_n$, then $x \in \mathrm{cl}\, S_n$ for every n. Therefore, there exists sequence $(d_n) \subset \mathbb{R}_+$ converges to 0 such that $B(x, d_n)$ (in $CJ_1([0,a])$) contains y_n $(y_n \in S_n)$.

Hence $y_n \to x$ in the space $CJ_1([0, a])$. Moreover, we know that subsequence y_{n_k} converges to a solution of problem (2.1), where $m = 1$, so $x \in S$. We obtain

$$S \subset \cap_{n \in \mathbb{N}} S_n \subset \cap_{n \in \mathbb{N}} \mathrm{cl}\, S_n \subset S,$$

so $S = \cap_{n \in \mathbb{N}} \mathrm{cl}\, S_n$.

We will show that $\sup\{d(z, S) | z \in S_n\} \to_{n \to \infty} 0$. Assume that there exists $\varepsilon > 0$ and a sequence (y_n) such that $y_n \in S_n$ and $d(y_n, S) \geq \varepsilon$. From the Step 3 we know that this sequence has subsequence (y_{n_k}) such that $y_{n_k} \to \tilde{y} \in S$, so $d(y_{n_k}, S) \to 0$. It is contrary to the choice of the sequence (y_n), hence $\sup\{d(z, S) | z \in S_n\} \to 0$. Therefore $\sup\{d(z, S) | z \in \mathrm{cl}\, S_n\} \to 0$. We obtain $S_n \subset S + B(0, p_n)$, where $p_n := \sup_{z \in S_n} d(z, S) \to 0$ with $n \to \infty$.

The compactness of S implies

$$\beta(\mathrm{cl}\ S_n) = \beta(S_n) \leq \beta(S) + p_n = p_n,$$

so $\beta(\mathrm{cl}\, S_n) \to 0$. Summing up, we can use theorem 2.1, which implies that the set S is an R_δ-set.   □

We will use this theorem to prove more general case.

Theorem 3.2. *Let the assumptions* (F0)–(F3), (H1)–(H3) *hold. Then every solution of the problem* (2.1) *for every* $j = 1, \ldots, m$ *meets* Σ_j *exactly once and the solution set* S *of this problem is an* R_δ-*set in the space* $CJ_m([0, a])$.

Proof. We show that we can divide the interval $[0, a]$ into m disjoint parts and any of them will have exactly one jump effect. Then we will be able to use the reasoning of theorem 3.1 on every such part, which will complete the proof.

By analogy to Step 1. in the proof of Theorem 3.1 we define a multivalued map $G_n : [0, a] \times \mathbb{R}^N \multimap \mathbb{R}^N$ and consider the problem

$$\dot{y}(t) \in G_n(t, y(t)), \quad \text{for } t \in [0, a], \ t \neq \tau_j(y(t)), \ j = 1, \ldots, m,$$

$$y(0) = y_0, \tag{3.17}$$

$$y(t^+) = y(t) + I_j(y(t)), \quad \text{for } t = \tau_j(y(t)), \ j = 1, \ldots, m.$$

We denote by $t_{y_n}^j$ the time of j-th jump for the function $y_n : [0, a] \to \mathbb{R}^N$. If the function y_n has less than j jumps we take $t_{y_n}^j := a$. Let y_n be an arbitrary solution of the problem (3.17) for $0 \leq t \leq t_{y_n}^2$. By analogy to theorem 3.1 in Step 1a. we show that there exists a constant $\bar{K}$ such that

$$\sup_{t \in [0, t_{y_n}^2]} \|y_n(t)\| \leq \bar{K}.$$

Next we will proceed similarly to the proof of theorem 3.1 (Step 1b), we show that there exists a constant $p' > 0$ such that

$$\tau_j'(y) \cdot \phi_n - 1 < -p'$$

for all $j = 1, \ldots, m$, and for enough big n, where $\bar{y}$ is fixed function with values in $\mathrm{cl}\ B(0, \bar{K})$, t is such that the assumption (H3) is satisfied, $\bar{y}(t) = y$, $\phi_n \in G_n(t, \bar{y}(t))$. We define the function $w_{n,j} : [0, a] \to \mathbb{R}$ in the following way:

$$w_{n,j}(t) := \tau_j(y_n(t)) - t, \quad j = 1, \ldots, m.$$

Let us fix a solution y_n of problem (3.17).

Now we prove by induction that the j-th time of jump is zero of the function $w_{n,j}$.

Basis. By assumption (H2) we have:

(a) $w_{n,j}(0) = \tau_j(y_0) > 0$,

(b) $w_{n,j}(a) = \tau_j(y_n(a)) - a < 0$,

(c) $w_{n,j}(t) = \tau_j(y_n(t)) - t < \tau_{j+1}(y_n(t)) - t = w_{n,j+1}(t)$ for all $t \in [0, a]$.

If there are no impulses then $w_{n,j}(t) \neq 0$ on $[0, a]$, but by the definition of $w_{n,j}$ for every j the function $w_{n,j}$ is continuous, which contradicts with a) and b). Hence there is at least one jump time.

Let $0 < t_{y_n}^1 < a$ be the first time of jump for the solution y_n. Then

$$w_{n,j}(t_{y_n}^1) = 0, \quad w_{n,j}(t) > 0, \quad \text{for } t \in [0, t_{y_n}^1).$$

By the assumption (H2) we obtain

$$w_{n,j}(t_{y_n}^1{}^+) = \tau_j(y_n(t_{y_n}^1{}^+)) - t_{y_n}^1 = \tau_j(y_n(t_{y_n}^1) + I_j(y_n(t_{y_n}^1))) - t_{y_n}^1 \leq 0.$$

For a.e. $t \geq t_{y_n}^1$ we have

$$w_{n,j}'(t) = \tau_j'(y_n(t)) \cdot y_n'(t) - 1 = \tau_j'(y_n(t)) \cdot \phi_n(t) - 1 < -p' < 0.$$

Hence $w_{n,j}$ is decreasing function at $[t_{y_n}^1, a]$, so y_n for every j meets Σ_j exactly once. By c) we know that the time $t_{y_n}^1$ of the first jump is zero of the function $w_{n,1}$.

Inductive step. We assume that the j-th time of jump is zero of $w_{n,j}$, $j < m$. Denote by $t_{y_n}^j$ the time for which we have $t = \tau_j(y_n(t))$. We know that

$$w_{n,l}(t_{y_n}^j{}^+) = \tau_l(y_n(t_{y_n}^j)) - t_{y_n}^j > \tau_j(y_n(t_{y_n}^j)) - t_{y_n}^j = 0$$

for $j > 1$, $l > j$. We consider interval $J := (t_{y_n}^j, a]$. By analogy to Step 1 in the proof of theorem 3.1 we have $w_{n,i}(t) \neq 0$, $t \in J$, $i \leq j$, as long as there is no jump caused by $w_{n,l}$ for $l > j$. There have to be at least one jump after $t_{y_n}^j$ and we denote it by $\tilde{t}$. By c) there are no jumps before the jump caused by $w_{n,j+1}$, so $\tilde{t} = t_{y_n}^{j+1}$.

Since both the basis and the inductive step have been proved, it has been proved by mathematical induction that there are exactly m jumps one for each Σ_j, $j = 1, \ldots, m$.

Next we will proceed similarly to the proof of theorem 3.1.

Let $(t_*^1, t_*^2, \ldots, t_*^m)$ be the limit of the sequence $((t_{y_n}^1, t_{y_n}^2, \ldots, t_{y_n}^m))$. We can show that on $[0, t_*^1]$ there exists the subsequence of (y_{n_k}) such that $I_1(y_{n_{k_l}}(t_{y_{n_{k_l}}})) \to I_1(y_*(t_*^1))$ and $\tau_1(y_*(t_*^1)) = t_*^1$, so

$$\tau_2(y_*((t_*^1)^+)) = \tau_2(y_*(t_*^1) + I_1(y_*(t_*^1))) > \tau_1(y_*(t_*^1)) = t_*^1.$$

By the assumptions (H1) and (H2) there exists $\varepsilon_1 > 0$ such that there is only one jump time of y_* on $[0, t_*^1 + 2\varepsilon_1]$ and we obtain that for all sufficiently big l there is only one jump time of $y_{n_{k_l}}$ on $[0, t_*^1 + \varepsilon_1]$. By analogy we find subsequence and $\varepsilon_2 > 0$ such that there is exactly one jump on $[t_*^1 + \varepsilon_1, t_*^2 + \varepsilon_2]$, and so on. On every such interval we proceed with reasoning from theorem 3.1. □

The following example shows the case of inclusion with exactly one jump, but can easily be rearranged to a multijump case.

Example 3.3. There are two trust funds with interest rates (dependent on time and amount of money) $\alpha(t, y_1) \in A(t, y_1)$ and $\beta(t, y_2) \in B(t, y_2)$ respectively where A and B are (multivalued) investment plans. It is available to transfer money once (in or out) without loosing interest.

We decided to start both trust funds with the same amount of money and transfer money from worse deposit to better one after clarifying which one is better. The amount of money that we wish to transfer would be proportional to difference in incomes.

This situation can be represented in the form of following differential inclusion:

$$(\dot{y_1, y_2})(t) \in F(t, (y_1, y_2)(t)), \quad \text{for } t \in [0,1], \ t \neq \tau((y_1, y_2)(t)), \tag{3.18}$$

$$(y_1, y_2)(0) = (y_0, y_0), \tag{3.19}$$

$$(y_1, y_2)(t^+) = (y_1, y_2)(t) + I((y_1, y_2)(t)), \quad \text{for } t = \tau((y_1, y_2)(t)) \tag{3.20}$$

with

$$F(t, y_1, y_2) = A(t, y_1) \times B(t, y_2),$$

where A and B fulfill assumptions (F1) and (F2) (for example continuous) and have interval values with $A(t, y), \ B(t, y) \in [-6y, 6y]$

$$I(y_1, y_2) = \begin{cases} (-y_1, y_1), & 0 \leq (1 + \frac{1}{\rho})y_1 < y_2, \\ (y_2, -y_2), & 0 \leq (1 + \frac{1}{\rho})y_2 < y_1, \\ (\rho(y_1 - y_2), \rho(y_2 - y_1)), & 0 \leq y_1 \leq (1 + \frac{1}{\rho})y_2 \leq (1 + \frac{1}{\rho})^2 y_1, \\ (0,0), & y_1 < 0 \text{ or } y_2 < 0 \end{cases}$$

and

$$\tau(y_1, y_2) := \operatorname{arc cot}(y_1 + y_2)/\pi.$$

For any $\alpha \in A(t, y)$ we have that

$$\alpha \geq -3(1/2 + 2y^2) = (\pi(p - 1))(1/2 + 2y^2)$$

where $1 > p = 1 - 3/\pi > 0$. We have analogous inequality for β therefore we obtain

$$\sup_{\varphi \in F(t, y_1, y_2)} \tau'_j(y_1, y_2) \cdot \varphi = \sup_{\alpha \in A(t, y_1)} \sup_{\beta \in B(t, y_2)} \frac{\alpha + \beta}{-\pi(1 + (y_1 + y_2)^2)}$$

$$\leq (1 - p) \frac{1 + 2y_1^2 + 2y_2^2}{1 + (y_1 + y_2)^2} \leq 1 - p < 1$$

All assumptions of Theorem 3.1 are satisfied, therefore the solution set of this problem is an R_δ-set.

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