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MEASURE INTEGRAL INCLUSIONS WITH FAST OSCILLATING DATA

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Abstract. We prove the existence of regulated or bounded variation solutions, via a nonlinear alternative of Leray-Schauder type, for the measure integral inclusion

$$x(t) \in \int_0^t F(s, x(s))\, du(s),$$

under the assumptions of regularity, respectively bounded variation, on the function u. Our approach is based on the properties of Kurzweil-Stieltjes integral that, unlike the classical integrals, can be used for fast oscillating multifunctions on the right hand side and the results allow one to study (by taking the function u of a particular form) continuous or discrete problems, as well as impulsive or retarded problems.

1. Introduction

Motivated by problems occurring in fields such as mechanics, electrical engineering, automatic control and biology (see [2,20,30]), an increasing attention has been given to measure-driven differential equations in the theory of differential equations: equations of the form

$$dx(t) = g(t, x(t))d\mu(t),$$

where μ is a positive regular Borel measure. An equal interest has been shown to the related integral problems and, more recently, for practical reasons (arising, e.g., from the theory of optimal control), to the set-valued associated problems. Such studies cover some classical cases like usual differential inclusions (when μ is absolutely continuous with respect to the Lebesgue measure), difference inclusions (for discrete measure μ) or impulsive problems (when the measure μ is a combination of the two types of measures).

We shall approach the matter of existence of solutions of measure integral inclusion

$$x(t) \in \int_0^t F(s, x(s))\, du(s) \tag{1.1}$$

via Stieltjes integration theory. As the function u will not be assumed absolutely continuous, we will not be able to find classical solutions. More precisely, we will

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work with regulated or bounded variation function u and the obtained solutions will be of the same kind.

Let us remark right from the start that the use of Riemann-Stieltjes integration theory is not possible when the function to integrate and the function u are both discontinuous. Also, the Lebesgue-Stieltjes integral will not be appropriate (as in [9, 17, 28, 8], to cite only a few) when the function under the integral sign is allowed to be highly oscillating. In the given situation, the most natural notion of integral is the Kuzweil-Stieltjes integral.

In the single-valued framework there is a series of existence results for Stieltjes integral equations using Kurzweil integral in the linear or nonlinear case (we refer to [29,25,16] or to the more recent [10,11,1]). As for the set-valued case, as far as the author knows, there are existence results via Lebesgue-Stieljes integral (such as [8]), but the problem has not been investigated yet in the setting of Kurzweil-Stieltjes integral.

In the first part, we will provide existence results for inclusion (1.1) under the assumption that u is regulated, using the notion of equi-regularity (introduced in [12]). In the second part, the function u will be assumed of bounded variation and the existence of solutions will be studied. Finally, in view of practical applications, the existence of bounded variation solutions will be obtained in a particular case: by considering Kurzweil-Stieltjes integrals of regulated functions with respect to a bounded variation function. As the theory of measure driven problems cover many well-known situations (see [1] for a discussion in this sense), for a particular function u, new existence results can be deduced for usual integral inclusions, difference inclusions, impulsive or retarded problems for systems with fast oscillating data.

2. Definitions and notation

Let $(X, \|\cdot\|)$ be a separable Banach space (the separability allows us to apply the classical measurable selection theorems, see [7]). A function $u : [0,1] \to X$ is said to be regulated if there exist the limits $u(t+)$ and $u(s-)$ for all points $t \in [0,1)$ and $s \in (0,1]$. It is well-known ([15]) that the set of discontinuities of a regulated function is at most countable, that regulated functions are bounded and the space $G([0,1], X)$ of regulated functions $u : [0,1] \to X$ is a Banach space when endowed with the sup-norm $\|u\|_C = \sup_{t\in[0,1]} \|u(t)\|$.

For a function $u : [0,1] \to X$ the total variation will be denoted by $\operatorname{var}_0^1(u)$ and if it is finite then u will be said to have bounded variation (or to be a bounded variation function). Any bounded variation function is regulated.

Let us now recall some basic facts from the theory of Kurzweil-Stieltjes integration in Banach spaces, which is a particular case of Kurzweil integration [16].

Let $u : [0,1] \to \mathbb{R}$. A partition of $[0,1]$ is a finite collection of pairs $\{(I_i, \xi_i) : i = 1, \dots, p\}$, where $I_1, \dots, I_p$ are non-overlapping subintervals of $[0,1]$, $\xi_i \in I_i$, $i = 1, \dots, p$ and $\cup_{i=1}^p I_i = [0,1]$. A gauge δ on $[0,1]$ is a positive function on $[0,1]$. For a given gauge δ we say that a partition $\{(I_i, \xi_i) : i = 1, \dots, p\}$ is δ-fine if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$, $i = 1, \dots, p$.

Definition 2.1. A function $f : [0,1] \to X$ is said to be Kurzweil-Stieltjes-integrable with respect to $u : [0,1] \to \mathbb{R}$ on $[0,1]$ (shortly, KS-integrable) if there exists a function denoted by $(KS)\int_0^{\cdot} f(s)du(s) : [0,1] \to X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε on $[0,1]$ with

$$\sum_{i=1}^{p} \big\| f(\xi_i)(u(t_i)-u(t_{i-1})) - \Big((\mathrm{KS})\int_0^{t_i} f(s)du(s) - (\mathrm{KS})\int_0^{t_{i-1}} f(s)du(s)\Big)\big\| < \varepsilon$$

for every δ_ε-fine partition $\{([t_{i-1},t_i],\xi_i):\ i=1,\dots,p\}$ of $[0,1]$.

The KS-integrability is preserved on all sub-intervals of $[0,1]$. The function $t\mapsto (KS)\int_0^t f(s)du(s)$ is called the KS-primitive of f with respect to u on $[0,1]$ (we refer to [29] or [25] for the case where X is finite dimensional).

Remark 2.2. When $u(s)=s$, this definition gives the concept of Henstock-Lebesgue-integrable function ([5]) or variational Henstock-integral [18]. If moreover X is finite dimensional, in the preceding definition the norm can be put outside the sum, giving the equivalent concept of Henstock integral (see [5, 18, 27] for a comparison between the two notions in general Banach spaces).

Definition 2.3. A collection $\mathcal{A}$ of KS-integrable functions is said to be KS equi-integrable if for every $\varepsilon>0$ there exists a gauge δ_ε (the same for all elements of $\mathcal{A}$) s uch that all $f\in\mathcal{A}$ satisfy the condition in Definition 2.1.

As the KS-integral satisfies the Saks-Henstock Lemma [25, Lemma 1.13], the proof of [25, Theorem 1.16] works in our setting and gives:

Proposition 2.4. *Let $u:[0,1]\to\mathbb{R}$ and $f:[0,1]\to X$ be KS-integrable with respect to u.*

(i) *If u is regulated, then so is the primitive $h:[0,1]\to X$,*

$$h(t)=(KS)\int_0^t f(s)\,du(s)$$

and for every $t\in[0,1]$,

$$h(t^+)-h(t)=f(t)[u(t^+)-u(t)],\quad h(t)-h(t^-)=f(t)[u(t)-u(t^-)].$$

(ii) *If u is of bounded variation and f is bounded, then h is of bounded variation.*

For the rest of this article, unless otherwise stated, the function u will be supposed to be regulated. The space of all functions that are KS-integrable with respect to u will be denoted by $\mathcal{KS}(u)$ and endowed with the supremum norm of the primitive (that is regulated, see Proposition 2.4 (i), namely the Alexiewicz norm with respect to u:

$$\|f\|_A^u=\sup_{t\in[0,1]}\|(\mathrm{KS})\int_0^t f(s)du(s)\|.$$

A compact convex-valued multifunction $\Gamma:[0,1]\to\mathcal{P}_{ck}(X)$ is said to be upper semi-continuous at a point $t_0\in[0,1]$ if for every $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that the excess of $\Gamma(t)$ over $\Gamma(t_0)$ (in the sense of Pompeiu-Hausdorff metric) is less than ε whenever $|t-t_0|<\delta_\varepsilon$. Otherwise stated,

$$\Gamma(t)\subset\Gamma(t_0)+\varepsilon B,$$

where B is the unit ball of X. A multifunction is upper semi-continuous when it is upper semi-continuous at each point $t_0\in[0,1]$. Moreover, it is completely continuous if it is totally bounded and upper semi-continuous. The symbol S_Γ stands for the family of measurable selections of Γ. We refer to [3, 7, 14, 23, 22] for any aspect (classical or not) related to multivalued analysis.

A technical result will be used (see [24]).

Lemma 2.5. *For any sequence $(\overline{y}_n)_n$ of measurable selections of a $\mathcal{P}_{ck}(X)$-valued measurable multifunction Γ, there exists $z_n \in \operatorname{conv}\{\overline{y}_m, m \geq n\}$ a.e. convergent to a measurable $\overline{y}$.*

3. Existence results - regulated case

In this section, we prove an existence result for measure integral inclusions considering Kurzweil-Stieltjes integrability with respect to a regulated function u, the main tool being the following concept:

Definition 3.1 ([12]). A set $\mathcal{A} \subset G([0,1], X)$ is said to be equi-regulated if for every $\varepsilon > 0$ and every $t_0 \in [0,1]$ there exists $\delta > 0$ such that:

(i) for any $t_0 - \delta < t' < t_0$: $\|x(t') - x(t_0^-)\| < \varepsilon$;
(ii) for any $t_0 < t'' < t_0 + \delta$: $\|x(t'') - x(t_0^+)\| < \varepsilon$ for all $x \in \mathcal{A}$.

A useful version of Ascoli's Theorem for regulated functions was proved in [19] (see also [12] in finite dimensional setting).

Lemma 3.2. *Let $\mathcal{A} \subset G([0,1], X)$ be equi-regulated and, for every $t \in [0,1]$, $\mathcal{A}(t) = \{x(t), x \in \mathcal{A}\}$ be relatively compact. Then $\mathcal{A}$ is relatively compact in $G([0,1], X)$.*

Moreover, as in the case of equi-continuous functions, one can prove the following result.

Lemma 3.3. *An equi-regulated family $\mathcal{A} \subset G([0,1], X)$ which is pointwise bounded is uniformly bounded.*

Proof. [19, Theorem 1.2] states that for every $\varepsilon > 0$ one can find a finite collection $0 = t_0 < t_1 < \cdots < t_{n_\varepsilon} = 1$ such that

$$\|x(t') - x(t'')\| \leq \varepsilon$$

for any $x \in \mathcal{A}$ and $[t', t''] \subset (t_{j-1}, t_j)$, $j = 1, \ldots, n_\varepsilon$. Take now $\varepsilon = 1$. There exist $0 = t_0 < t_1 < \cdots < t_{n_1} = 1$ such that

$$\|x(t') - x(t'')\| \leq 1$$

for any $x \in \mathcal{A}$ and $[t', t''] \subset (t_{j-1}, t_j)$, $j = 1, \ldots, n_1$. If we note by $M_j = \sup\{\|x\big(\frac{t_{j-1}+t_j}{2}\big)\|, x \in \mathcal{A}\}$ and by $N_j = \sup\{\|x(t_j)\|, x \in \mathcal{A}\}$, $j = 0, \ldots, n_1$, then for every $t \in [0,1]$ and any $x \in \mathcal{A}$ one gets

$$\|x(t)\| \leq \max\big(\{M_j + 1, j = 1, \ldots, n_1\} \cup \{N_j, j = 0, \ldots, n_1\}\big)$$

and the uniform boundedness property is achieved. $\square$

Bearing in mind the fact that the primitive of a function which is KS-integrable with respect to a regulated function is regulated as well, we prove the following result.

Proposition 3.4. *Let $u : [0,1] \to \mathbb{R}$ be regulated and $\mathcal{K}$ be pointwise bounded and KS equi-integrable with respect to u. Then the set $\{(KS)\int_0^{\cdot} f(s)du(s), f \in \mathcal{K}\}$ is equi-regulated.*

Proof. Fix $t_0 \in [0,1]$ and let $\varepsilon > 0$. There exists $M > 0$ such that $\|f(t_0)\| \le M$ for every $f \in \mathcal{K}$. One can also find a gauge δ_ε with

$$\sum_{i=1}^{n} \|f(\xi_i)(u(\tilde{t}_{i+1}) - u(\tilde{t}_i)) - (KS)\int_{\tilde{t}_i}^{\tilde{t}_{i+1}} f(s)du(s)\| \le \frac{\varepsilon}{2}, \quad \forall f \in \mathcal{K}$$

for any δ_ε-fine partition $\{(\tilde{t}_i, \tilde{t}_{i+1}), \xi_i, 0 = 1, \dots, n\}$. On the other hand, as u is regulated, there exist $\overline{\delta}_\varepsilon > 0$ such that

$$\|u(t') - u(t_0^-)\| \le \frac{\varepsilon}{2M}$$

whenever $t_0 - \overline{\delta}_\varepsilon < t' < t_0$ and the similar for the limit at the right.

We will prove that $\delta'_\varepsilon = \min(\delta_\varepsilon(t_0), \overline{\delta}_\varepsilon)$ satisfies that for every $t_0 - \delta'_\varepsilon < t' < t_0$:

$$\|(KS)\int_0^{t'} f(s)du(s) - (KS)\int_0^{t_0^-} f(s)du(s)\| < \varepsilon, \quad \forall f \in \mathcal{K}$$

(and, obviously, the same for the right limit). Indeed, as in the proof of [25, Theorem 1.16]:

$$\begin{aligned}
&(KS)\int_0^{t'} f(s)du(s) - (KS)\int_0^{t_0} f(s)du(s) \\
&= f(t_0)(u(t') - u(t_0)) + \Big((KS)\int_0^{t'} f(s)du(s) - (KS)\int_0^{t_0} f(s)du(s) \\
&\quad - f(t_0)(u(t') - u(t_0))\Big)
\end{aligned}$$

where the last term can be made, by Saks-Henstock Lemma, (in norm) less that $\varepsilon/2$ and, from here:

$$(KS)\int_0^{t_0^-} f(s)du(s) - (KS)\int_0^{t_0} f(s)du(s) = f(t_0)(u(t_0^-) - u(t_0)).$$

It follows that

$$\begin{aligned}
&(KS)\int_0^{t'} f(s)du(s) - (KS)\int_0^{t_0^-} f(s)du(s) \\
&= f(t_0)(u(t') - u(t_0^-)) + \Big((KS)\int_0^{t'} f(s)du(s) \\
&\quad - (KS)\int_0^{t_0} f(s)du(s) - f(t_0)(u(t') - u(t_0))\Big)
\end{aligned}$$

and so,

$$\|(KS)\int_0^{t'} f(s)du(s) - (KS)\int_0^{t_0^-} f(s)du(s)\| \le M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon$$

for any $f \in \mathcal{K}$ and t' with $t_0 - \delta'_\varepsilon < t' < t_0$. □

Let us recall a nonlinear alternative of Leray-Schauder type that will be applied below.

Theorem 3.5 ([21]). *Let D and $\overline{D}$ be open and closed subsets of a normed linear space E such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}_{ck}(E)$ be completely continuous. Then either*

(i) *the inclusion* $x \in T(x)$ *has a solution, or*
(ii) *there exists* $x \in \partial D$ *such that* $\lambda x \in T(x)$ *for some* $\lambda > 1$.

Applying this theorem will necessitate a convergence result, such as

Lemma 3.6 ([4, Theorem 6.1]). *Let* u *be* ACG^{**} *and* $(f_n)_n$ *a sequence KS equi-integrable with respect to* u *which pointwise converges to* f. *Then* f *is KS-integrable with respect to* u *and*

$$(KS)\int_0^1 f_n(s)du(s) \to (KS)\int_0^1 f(s)du(s).$$

It works for functions u that are more than regulated (but not necessarily of bounded variation), namely:

Definition 3.7 ([4]). (i) $u : [0,1] \to \mathbb{R}$ is said to be ACG^{**} if it is continuous and the unit interval can be written as a countable union of closed sets on each of which F is AC^{**};

(ii) A function $u : [0,1] \to \mathbb{R}$ is AC^{**} on $E \subset [0,1]$ if, for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ and a gauge $\delta : E \to \mathbb{R}_+$ such that, whenever $D1, D2$ are δ-fine partitions of E with $\sum_{D1\setminus D2} |t' - t''| < \eta_\varepsilon$, one has

$$\sum_{D1\setminus D2} |u(t') - u(t'')| < \varepsilon;$$

here $D1 \setminus D2$ denotes the collection of all connected components of $\cup D1 \setminus \cup D2$.

We give now the main result of this section. Notice that in [29] it was explained why the space of regulated functions is the best choice for the space of solutions.

Definition 3.8. A solution of measure driven inclusion (1.1) is a regulated function $x : [0,1] \to X$ for which there exists $g \in S_{F(\cdot,x(\cdot))}$ such that

$$x(t) = (KS)\int_0^t g(s)\,du(s), \quad \forall t \in [0,1].$$

Theorem 3.9. *Let* $u : [0,1] \to \mathbb{R}$ *be* ACG^{**} *and* $F : [0,1] \times X \to \mathcal{P}_{ck}(X)$ *satisfy:*
(i) *for every* $x \in X$, $F(\cdot, x)$ *is measurable;*
(ii) *for every* $R > 0$*:*
  (ii1) *the family*

$$\cup\{S_{F(\cdot,x(\cdot))}, x \in G([0,1], X), \|x\|_C \le R\}$$

  is pointwise bounded and KS equi-integrable with respect to u*;*
  (ii2) *the map* $x \in G([0,1], X), \|x\|_C \le R \to S_{F(\cdot,x(\cdot))}$ *is upper semi-continuous with respect to the* $\|\cdot\|_A^u$*-topology on the space* $\mathcal{KS}(u)$*;*
  (ii3) *for each* $t \in [0,1]$,

$$\big\{(KS)\int_0^t f(s)du(s), f \in S_{F(\cdot,x(\cdot))}\big\}$$

  is relatively compact for every $x \in G([0,1], X), \|x\|_C \le R$ *and*

$$\big\{(KS)\int_0^t f(s)du(s), f \in S_{F(\cdot,x(\cdot))}, x \in G([0,1], X), \|x\|_C \le R\big\}$$

  is bounded.

If moreover there exists R_0 such that $\|x\|_C \neq R_0$ for any regulated solution x of

$$x(t) \in \lambda\Big(x_0 + \int_0^t F(s, x(s))du(s)\Big)$$

for all $\lambda \in (0,1)$, then our integral inclusion possess regulated solutions with $\|x\|_C \leq R_0$.

Proof. Let $N : \overline{B_{R_0}} \to G([0,1], X)$ be the operator defined on the ball centered at the origin of radius R_0 of $G([0,1], X)$ by

$$N(x)(t) = \big\{(KS)\int_0^t f(s)du(s), f \in S_{F(\cdot,x(\cdot))}\big\}.$$

Obviously, the fixed points of this operator will be solutions to our inclusion.

We will check the hypothesis of Theorem 3.5. Let us note first that the values of N are convex and non-empty; indeed, hypothesis $ii2$) implies that for any $t \in [0,1]$ the map $F(t, \cdot)$ is upper semi-continuous and, thanks to hypothesis i), this yields the existence of measurable selections for the superpositional map $F(\cdot, x(\cdot))$.

Let us prove that the values are compact. We will get the relative compactness by Lemma 3.2. From hypotheses (ii1), we are able to apply Proposition 3.4 to obtain the equi-regularity, while the second condition in Lemma 3.2 is stated by hypotheses (ii3).

It remains thus to prove that the values are closed. Fix then x and consider a sequence $((KS)\int_0^\cdot f_n(s)du(s))_n \subset N(x)$ convergent to $g \in G([0,1], X)$ and show that there exists $f \in S_{F(\cdot,x(\cdot))}$ with $g(t) = (KS)\int_0^t f(s)du(s)$ for any $t \in [0,1]$. As F is compact convex-valued, one can find a sequence of convex combinations $\tilde{f}_n \in \operatorname{co}\{f_m, m \geq n\}$ that pointwise converges to some selection f of $F(\cdot, x(\cdot))$. Lemma 3.6 implies that

$$(KS)\int_0^t \tilde{f}_n(s)du(s) \to (KS)\int_0^t f(s)du(s)$$

and so, $g(t) = (KS)\int_0^t f(s)du(s)$ for any $t \in [0,1]$.

In the sequel, let us prove that N is completely continuous. The total boundedness comes from Proposition 3.4 and the pointwise boundedness hypothesis (ii3) since we can apply Lemma 3.3.

Let us now check that it is upper semi-continuous. To this aim, fix $x_0 \in \overline{B_{R_0}}$ and consider an arbitrary $\varepsilon > 0$. Hypothesis (ii2) says that there exists $\delta_{\varepsilon,x_0} > 0$ such that for any $x \in G([0,1], X)$ with $\|x - x_0\|_C < \delta_{\varepsilon,x_0}$:

$$S_{F(\cdot,x(\cdot))} \subset S_{F(\cdot,x_0(\cdot))} + \varepsilon B_A,$$

where B_A is the unit open ball of $\mathcal{KS}(u)$ endowed with the $\|\cdot\|_A^u$-topology. By the definition of $\|\cdot\|_A^u$, it follows that for every $f \in S_{F(\cdot,x(\cdot))}$ one can find $f_0 \in S_{F(\cdot,x_0(\cdot))}$ such that $\|(KS)\int_0^\cdot f(s)du(s) - (KS)\int_0^\cdot f_0(s)du(s)\|_C < \varepsilon$ which means that

$$N(x) \subset N(x_0) + \varepsilon B_G,$$

B_G being the open unit ball of $G([0,1], X)$ and thus, the upper semi-continuity of N is verified.

The conditions of Theorem 3.5 are satisfied and, as the alternative is excluded by hypothesis, it follows that the operator N has fixed points and our inclusion has solutions. $\square$

Another version of this result could be obtained in a similar manner.

Theorem 3.10. *Let $u : [0,1] \to \mathbb{R}$ be ACG^{**} and $F : [0,1] \times X \to \mathcal{P}_{ck}(X)$ satisfy:*

(i) *for every $x \in X$, $F(\cdot, x)$ is measurable;*

(ii) *for every $x \in G([0,1], X)$, the family $S_{F(\cdot,x(\cdot))}$ is KS equi-integrable with respect to u;*

(iii) *for every $R > 0$:*

  (iii1) *the map $x \in G([0,1], X), \|x\|_C \leq R \to S_{F(\cdot,x(\cdot))}$ is upper semi-continuous with respect to the $\|\cdot\|_A^u$-topology on the space $\mathcal{KS}(u)$;*

  (iii2) *for each $t \in [0,1]$,*

$$\{(KS)\int_0^t f(s)du(s), f \in S_{F(\cdot,x(\cdot))}\}$$

  is relatively compact for every $x \in G([0,1], X), \|x\|_C \leq R$ and

$$\big\{(KS)\int_0^t f(s)du(s), f \in S_{F(\cdot,x(\cdot))}, x \in G([0,1], X), \|x\|_C \leq R\big\}$$

  is uniformly bounded.

If there exists R_0 as in Theorem 3.9, then the integral inclusion possess regulated solutions.

Proof. Following the same line as in the preceding result, the operator N has relatively compact values: they are equi-regulated by hypothesis (ii) and Proposition 3.4 and they are pointwisely contained in a compact set by hypothesis (iii2). The values are closed (this can be proved as in Theorem 3.9) and convex. Besides, N is totally bounded by (iii2) and upper semi-continuous by (iii1). Thus, the conditions of fixed point theorem are checked and so, the existence of solutions is obtained. □

4. Existence results - bounded variation case

When u is of bounded variation, instead of [4, Theorem 6.1] we can use another convergence result.

Lemma 4.1. *Let $u : [0,1] \to \mathbb{R}$ be of bounded variation and $f_n : [0,1] \to X$ be a sequence of functions KS equi-integrable with respect to u that converges pointwise to $f : [0,1] \to X$. Then f is KS-integrable with respect to u and*

$$(KS)\int_0^1 f(s)du(s) = \lim_{n\to\infty} (KS)\int_0^1 f_n(s)du(s).$$

Proof. Let $\varepsilon > 0$. There exists a partition $\mathcal{P}_0 = \{(t_{i-1}, t_i), \xi_i\}_{i=1}^{n_0}$ of $[0,1]$ such that

$$\sum_{i=1}^{n_0} \Big\| f_n(\xi_i)(u(t_i) - u(t_{i-1})) - \Big((KS)\int_0^{t_i} f_n(s)du(s) - (KS)\int_0^{t_{i-1}} f_n(s)du(s)\Big)\Big\| < \varepsilon,$$

for all $n \in \mathbb{N}$. At the same time, one can find $n_\varepsilon \in \mathbb{N}$ such that $\|f_n(\xi_i) - f_m(\xi_i)\| < \frac{\varepsilon}{\operatorname{var}_0^1(u)}$ for every $i = 1, \ldots, n_0$ and every $m, n \geq n_\varepsilon$. It follows that

$$\sum_{i=1}^{n_0} \Big\| f_n(\xi_i)(u(t_i) - u(t_{i-1})) - \sum_{i=1}^{n_0} f_m(\xi_i)(u(t_i) - u(t_{i-1}))\Big\| < \varepsilon, \quad \forall m, n \geq n_\varepsilon,$$

whence

$$\Big\|(KS)\int_0^1 f_n(s)du(s) - (KS)\int_0^1 f_m(s)du(s)\Big\| < 3\varepsilon, \quad \forall m, n \geq n_\varepsilon$$

and the same for each $t \in [0,1]$: the sequence $((KS)\int_0^t f_n(s)du(s))_n$ is Cauchy. As in the proof of [27, Theorem 3.6.18] it can be proved that its limit $L(t)$ equals the KS-integral of f with respect to u on $[0,t]$. □

We thus obtain, this time for a bounded variation function u (instead of ACG^{**}):

Theorem 4.2. *Let $u : [0,1] \to \mathbb{R}$ be of bounded variation and $F : [0,1] \times X \to \mathcal{P}_{ck}(X)$ satisfy the hypothesis of Theorem 3.9, except (ii2), instead of which we impose:*

- (ii2') *the map $x \in G([0,1],X), \|x\|_C \le R \to F(t,x(t))$ is upper semi-continuous uniformly in t.*

Then our integral inclusion possess regulated solutions with $\|x\|_C \le R_0$.

Proof. Only the proof of the upper semi-continuity of N has to be changed. By (ii2') for each x_0 and $\varepsilon > 0$ there exists $\delta_{\varepsilon,x_0} > 0$ such that for any $x \in G([0,1],X)$ with $\|x - x_0\|_C < \delta_{\varepsilon,x_0}$:

$$F(t,x(t)) \subset F(t,x_0(t)) + \varepsilon B, \quad \forall\, t \in [0,1],$$

where B is the unit open ball of X. It follows that for every $f \in S^G_{F(\cdot,x(\cdot))}$ one can find $f_0 \in S^G_{F(\cdot,x_0(\cdot))}$ such that $\|f(t) - f_0(t)\| \le \varepsilon$ for every $t \in [0,1]$, whence (see [26])

$$\|(KS)\int_0^t f(s)du(s) - (KS)\int_0^t f_0(s)du(s)\| \le \|f - f_0\|_C \operatorname{var}_0^1(u) \le \varepsilon \operatorname{var}_0^1(u)$$

which means that

$$N(x) \subset N(x_0) + \varepsilon \operatorname{var}_0^1(u) B_G,$$

B_G being the open unit ball of $G([0,1],X)$ and thus, the upper semi-continuity of N is verified. □

For an alternative existence theorem, remark that in this setting a mean value result is available.

Lemma 4.3. *Let $u : [0,1] \to \mathbb{R}$ be of bounded variation and $f : [0,1] \to X$ be KS-integrable with respect to u.*

(i) *If u is nondecreasing, then*

$$(KS)\int_0^t f(s)du(s) \in (u(t) - u(0))\overline{\mathrm{co}}(f([0,t])), \quad \forall t \in [0,1].$$

(ii) *If u is of bounded variation, then*

$$(KS)\int_0^t f(s)du(s) \in \operatorname{var}_0^t(u)\overline{\mathrm{co}}(\{0\} \cup f([0,t])) - \operatorname{var}_0^t(u)\overline{\mathrm{co}}(\{0\} \cup f([0,t])),$$

for all $t \in [0,1]$.

Proof. When u is nondecreasing, the assertion is a consequence of the definition of KS-integral, since for any partition of $[0,t]$:

$$\sum_{i=1}^p f(\xi_i)(u(t_i) - u(t_{i-1})) = (u(t) - u(0)) \sum_{i=1}^p f(\xi_i) \frac{u(t_i) - u(t_{i-1})}{u(t) - u(0)}.$$

When u is of bounded variation, it can be written as the difference of two non-decreasing functions u_1 and u_2 and so, by the first step,

$$
\begin{aligned}
(KS)\int_0^t f(s)du(s) &\in (u_1(t)-u_1(0))\overline{\text{co}}(f([0,t])) - (u_2(t)-u_2(0))\overline{\text{co}}(f([0,t])) \\
&\subset \text{var}_0^t(u)\overline{\text{co}}(\{0\}\cup f([0,t])) - \text{var}_0^t(u)\overline{\text{co}}(\{0\}\cup f([0,t])).
\end{aligned}
$$

□

From Theorem 4.2 we then get the existence of bounded variation solutions.

Corollary 4.4. *Let $u : [0,1] \to \mathbb{R}$ be of bounded variation and $F : [0,1]\times X \to \mathcal{P}_{ck}(X)$ satisfy the hypothesis* (i) *and* (ii2') *of Theorem 4.2 together with:*

(ii1') *the family*

$$\cup\{S_{F(\cdot,x(\cdot))}, x \in G([0,1],X), \|x\|_C \le R\}$$

is KS equi-integrable with respect to u;

(ii3') *for each $t \in [0,1]$,*

$$\big\{(KS)\int_0^t f(s)du(s), f \in S_{F(\cdot,x(\cdot))}\big\}$$

is relatively compact for every $x \in G([0,1],X)$, $\|x\|_C \le R$ and for any bounded $A \subset X$,

$$F([0,1]\times A) \ \text{ is bounded.}$$

If there exists R_0 as in Theorem 3.9, then our integral inclusion possess bounded variation solutions with $\|x\|_C \le R_0$.

Proof. The only modification to be made in the proof of Theorem 4.2 is at the step where the total boundedness of the operator N must be verified, more precisely the pointwise boundedness of $N(\overline{B_{R_0}})$; at that point, under our assumptions, the property easily comes from Lemma 4.3 and hypothesis $ii3'$). Besides, as the found solution is the primitive of a bounded function with respect to a bounded variation function, by Proposition 2.4, it is of bounded variation. □

In concrete situations, the Kurzweil-Stieltjes integral is mostly used in the case where the integrand is regulated and the function with respect to one integrates is of bounded variation (or viceversa); therefore, it could be more convenient to have an existence result for this case. For this purpose, let us recall the following convergence result.

Lemma 4.5 ([26, Theorem I.4.17])**.** *Let $u : [0,1] \to \mathbb{R}$ be of bounded variation and $f_n : [0,1] \to X$ be KS-integrable with respect to u with $\|f_n - f\|_C \to 0$. Then f is KS-integrable with respect to u and $(KS)\int_0^1 f_n(s)du(s) \to (KS)\int_0^1 f(s)du(s)$.*

Applying it will be possible by using another result.

Lemma 4.6 ([19, Lemma 1.14])**.** *If an equi-regulated sequence of functions converges pointwise, then it converges uniformly towards the limit.*

Theorem 4.7. *Let $u : [0,1] \to \mathbb{R}$ be of bounded variation and $F : [0,1]\times X \to \mathcal{P}_{ck}(X)$ satisfy:*

(i) *for every $x \in G([0,1],X)$, the family $S^G_{F(\cdot,x(\cdot))}$ of regulated selections of $F(\cdot,x(\cdot))$ is non-empty;*

(ii) *for every* $R > 0$*:*

(ii1) *the family*

$$\cup\big\{S^G_{F(\cdot,x(\cdot))}, x \in G([0,1],X), \|x\|_C \le R\big\}$$

is equi-regulated;

(ii2') *the map* $x \in G([0,1],X), \|x\|_C \le R \to F(t,x(t))$ *is upper semi-continuous uniformly in* t*;*

(ii3) *for each* $t \in [0,1]$*,*

$$\big\{(KS)\int_0^t f(s)du(s), f \in S^G_{F(\cdot,x(\cdot))}\big\}$$

is relatively compact for every $x \in G([0,1],X), \|x\|_C \le R$ *and the family*

$$\big\{(KS)\int_0^{\cdot} f(s)du(s), f \in S^G_{F(\cdot,x(\cdot))}, x \in G([0,1],X), \|x\|_C \le R\big\}$$

is equi-regulated and pointwise bounded.

If moreover there exists R_0 *such that* $\|x\|_C \ne R_0$ *for any regulated solution* x *of*

$$x(t) \in \lambda\Big(x_0 + \int_0^t F(s,x(s))du(s)\Big)$$

for all $\lambda \in (0,1)$*, then our integral inclusion possess bounded variation solutions with* $\|x\|_C \le R_0$*.*

Proof. Consider now the modified operator $N : \overline{B_{R_0}} \to G([0,1],X)$ defined on the ball centered at the origin of radius R_0 of $G([0,1],X)$ by

$$N(x)(t) = \big\{(KS)\int_0^t f(s)du(s), f \in S^G_{F(\cdot,x(\cdot))}\big\}.$$

The proof of the fact that N has fixed points is essentially that of Theorem 3.9, except the point where it must be proved that the values of operator N are closed; here this comes from the fact that the sequence $\tilde{f}_n$ pointwise converges to f and it is equi-regulated so, by [19, Lemma 1.14], $\|f_n - f\|_C \to 0$. Moreover, [15, Corollary 3.2] states that f is regulated. Now applying Lemma 4.5 gives the convergence of the integrals of f_n towards the integral of f and thus the closedness of the values of N.

Let us now check that N is upper semi-continuous. Fix $x_0 \in \overline{B_{R_0}}$ and consider an arbitrary $\varepsilon > 0$. Hypothesis (ii2') yields that there exists $\delta_{\varepsilon,x_0} > 0$ such that for any $x \in G([0,1],X)$ with $\|x - x_0\|_C < \delta_{\varepsilon,x_0}$:

$$F(t,x(t)) \subset F(t,x_0(t)) + \varepsilon B, \quad \forall t \in [0,1],$$

where B is the unit open ball of X. It follows that for every $f \in S^G_{F(\cdot,x(\cdot))}$ one can find $f_0 \in S^G_{F(\cdot,x_0(\cdot))}$ such that $\|f(t) - f_0(t)\| \le \varepsilon$ for every $t \in [0,1]$, whence (see [26]):

$$\|(KS)\int_0^t f(s)du(s) - (KS)\int_0^t f_0(s)du(s)\| \le \|f - f_0\|_C \operatorname{var}_0^1(u) \le \varepsilon \operatorname{var}_0^1(u)$$

which means that

$$N(x) \subset N(x_0) + \varepsilon \operatorname{var}_0^1(u) B_G,$$

B_G being the open unit ball of $G([0,1], X)$ and thus, the upper semi-continuity of N is verified.

Finally, as any solution is the KS-primitive of a regulated function (therefore bounded) with respect to the bounded variation function u, Proposition 2.4. (ii) asserts that it is more than regulated: it is of bounded variation. $\square$

Remark 4.8. The imposition of assumption (ii3) (equi-regularity of primitives) together with (ii1) (equi-regularity of selections) might look artificial but, in fact, the equi-regularity of primitives follows from the equi-regularity of selections only if we impose a pointwise boundedness condition on the family of selections. This pointwise boundedness condition would be very strong since, by Lemma 3.3, it would imply the uniform boundedness and many of the properties given above would then be obtained in a much simpler manner.

Remark 4.9. New existence results can be deduced in particular cases, namely when u is absolutely continuous (leading to usual continuous problems), the sum of step functions (leading to discrete problems) or a sum between an absolutely continuous function and a sum of step functions (in which case one gets impulsive problems), as well as for retarded problems (see [1]).

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